A tale of two kiddies: A Dickensian slant on multiplicative thinking

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The work of four primary students is discussed in terms of their contextual understanding of multiplicative concepts. The difference between teaching for understanding and procedural teaching is highlighted.

Evidence suggests that some students have learned procedures with little or no underpinning understanding while others have a much more connected and conceptual levels of understanding. An analogy is drawn with Charles Dickens' character, Mr Thomas Gradgrind, who would have endorsed procedural teaching and abhorred teaching that encouraged understanding.

Introduction

> Now, what I want is, Facts. Teach these boys and girls nothing else but Facts. Facts alone are wanted in life. Plant nothing else, and root out everything else. (Dickens, 1854, p. 1)

> . . . [the children are] the inclined plane of little vessels then and there arranged in order, ready to have imperial gallons of facts poured into them until they were full to the brim. (Dickens, 1854, p. 2)

Charles Dickens may have created the fictional character of Thomas Gradgrind over 160 years ago but one suspects that some of his traits have endured to the present day. The basis for saying that is some of the evidence presented in this article, the subject of which is the work of four students from one primary school—three Year 6 and one Year 5—who were interviewed about aspects of multiplicative thinking. The interviews are part of a larger study into multiplicative thinking. Initially, two students from the same class—Thomas and Oliver, the 'main protagonists'—are compared. Further data are provided from the interviews with Sissy and Lucas. The four students demonstrate vastly different skills and levels of conceptual understanding with evidence strongly suggesting that some had learned in a procedural way (Thomas and Lucas) while others had learned conceptually (Oliver and Sissy). One wonders how Mr Thomas Gradgrind, ". . . with a rule and a pair of scales, and the multiplication table always in his pocket" (Dickens, 1854, p. 2) would regard a conceptual approach to teaching mathematics, the use of manipulative materials, and having children explain their thinking!

Background

Conceptual understanding

Since Skemp's (1976) seminal article espousing the benefits of relational understanding over instrumental understanding, others (Hiebert, 1999; Pesek & Kirshner, 2000) have stated a similar view about the value of teaching conceptually (developing relational understanding) compared to teaching procedurally (developing instrumental understanding). Skemp (1976, p. 86) described instrumental knowledge as akin to "rules without reasons" whereas relational understanding is "knowing both what to do and why". He provided an example of instrumental/procedural learning in describing a boy who had learned to remove and later replace the decimal point when multiplying and noted that this was fraught with danger as students inevitably try to apply such 'rules' to other situations such as division, and even addition and subtraction. Also, Skemp pointed out that relational/conceptual learning had many advantages—it is more adaptable to a range of situations, it connects different concepts, and it motivates children. Pesek and Kirshner (2000) found that students who were taught conceptually outperformed those who were taught procedurally, then conceptually. Both Pesek and Kirshner (2000) and Hiebert (1999) noted that once students had learned a procedure, they were reluctant to then learn the mathematics underpinning it. Similarly, Anthony and Walshaw (2002) found that students in their study generally could not reason multiplicatively and were unable to explain concepts such as commutativity. They attributed this, as did Warren and English (2000), to the predominance of teaching computational procedures at the expense of concepts, relationships, and connections, which severely limit students' understanding of mathematical structure.

Multiplicative thinking and the use of manipulatives

Multiplicative thinking involves a complex set of ideas which underpin most of the mathematics learned beyond the middle primary years. It involves working flexibly with a range of numbers including very large and very small numbers, relationships between quantities including ratio and proportion, and a variety of situations involving multiplication and division (Siemon, Dole, Breed, Izzard & Virgona, 2006). Various researchers (Carbonneau, Marley, & Selig, 2013; Delaney, 2010) have described the benefits of using manipulatives to teach mathematical concepts, particularly when used in combination with teacher demonstration, instructional guidance, focused teacher questioning, and conversation. Van de Walle, Karp, and Bay-Williams (2013) described how tiles, cubes and counters can be used to construct arrays to help develop an understanding of multiplication and division. The four interviews on which this paper focuses are based on assessing students' understanding of multiplication and division and whether or not they can use bundling sticks to help them explain what is happening. It is interesting to note that Swan and Marshall (2010) asked teachers to list the manipulatives they used most often. The eight most commonly used manipulatives ranged from Pattern Blocks (84.3% of teachers) and Base Ten Blocks (81.9%) to 35.1% for Cuisenaire Rods. Bundling sticks were not among the most commonly used manipulatives yet they are useful for demonstrating the notion of sharing into equal groups. Pseudonyms are used for the children who were interviewed.

Discussion of interviews

Thomas – a Year 6 student

Thomas presented at the interview as a very confident student. When given the first task, "Show me and tell me how you would work out the answer to 7 × 15", he instantaneously gave the answer of 105. Thomas then explained his working:

Thomas: First I timesed seven by five , which is 35, and I leave that five in its place—the ones place—then I times seven times one equals seven. Then plus 3 is ten.

When asked how he would do it if he wrote it down, Thomas used the standard vertical setting out but did not show the 'carrying of the three tens' (see Figure 1). He was then asked to use materials to show what happened when seven was multiplied by 15 to make 105.

Thomas: [pause] I don't know . . . I've never worked with sticks before.

The interviewer then showed him a group of 15 [a bundle of ten and five singles] and said "What would you need to do to show me that? [indicated the written algorithm]" After some further pause, Thomas said, "Can I work in my own way?" He took some time but did make a group of seven bundles of ten which he put together to make 70 and then seven groups of five which he put together to make 35.

Thomas clearly had a good grasp of the vertical algorithm for multiplication so was asked to show and explain how he would work out 29 × 37. He calculated the correct answer quickly (see Figure 1). The following conversation between Thomas and the interviewer ensued:

Int: [indicating the gap in the ones place on the second line of working] What's this little gap here?

Thomas: Nothing . . . [pause] . . . I'll say zero.

Int: Why did you put that there?

Thomas: [hesitatingly] I don't know really, my maths teacher tells me that's how it works.

Int: But you wouldn't put that 87 under there [indicated the 10 and 1 column—see sample]. Is there any reason why?

Thomas: No, I don't know a reason why.

Int: Do you think it might be something to do with that you're multiplying that by 30 so it is ten times bigger?

Thomas: [No response]

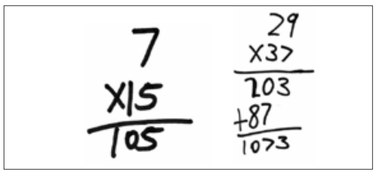

Figure 1. Work samples from Thomas.

When asked to show and explain how he would work out the answer for 90 ÷ 7, Thomas immediately wrote '12 R6'—he worked quickly again. However, he again struggled to show the process with the bundling sticks. He started to 'think aloud' and after a long pause, said, "Ahh, I can't explain . . . I really don't know". The interviewer then suggested to Thomas that the exercise required him to put 90 into seven groups and with quite a lot of assistance, he started to use the bundling sticks. However, he struggled to split the 90 sticks evenly and when he eventually did (with assistance), he left out the six 'leftovers'. Thomas was then given the task of finding the answer to 200 ÷ 13. He promptly used a mental strategy and wrote the answer as 15 R5.

Thomas was also asked to solve 23 × 4 and 400 × 23. For the first example, he immediately wrote the correct answer of 92. He took some time to answer 400 × 23 and explained as follows:

Thomas: It's 9200. I did four times 23 which is 92 and added zeros.
Int: If you added a zero, how many times bigger does it make the answer?
Thomas: [asked for the question to be repeated] Ten times.
Int: If you added two zeros . . .
Thomas: A hundred.

What does this suggest about Thomas?

Thomas appears to have a very good command of number facts, and knows the vertical algorithm for multiplication. However, it is of concern that he had considerable difficulty when working with the bundling sticks to try to explain what happened in the multiplication and division processes. The way in which he explained the working for 7 × 15 suggests that he is working with digits rather than digit values (Anghileri. 2000) and his explanation of working out for 29 × 37 was very procedural. This concern is heightened by his comment that he has not worked with bundling sticks, does not know why he placed digits in certain positions in the algorithm, and that his reason for doing so was that his teacher told him that it works that way. For the last two examples, Thomas did not employ his knowledge of the product of 23 and four and the commutative property to derive the product of four and 23. Further, he was hesitant in expressing an understanding of 'times bigger' even though he procedurally added zeros. Thomas seems to be a good example of a student who may have taught "rules without reasons" (Skemp, 1976).

Oliver—a Year 6 student

Oliver began impressively and explained his thinking for 7 × 15. He said "I would do 70 plus 35, and that would be 105, because seven times five is 35 and seven times ten is 70". Here, he demonstrated and applied an understanding of standard place value partitioning. When asked to show it with bundling sticks, he immediately selected seven bundles of ten and then made three groups of ten and a group of five. For 29 × 37, he did not use the vertical multiplication algorithm and explained his thinking as follows:

Oliver: There's 20 times 30 plus 20 times seven, then nine times 30 and nine times seven.

He wrote the partial products and arrived at the correct answer (see Figure 2). Although he did not name it, Oliver demonstrated an understanding of the distributive property and how to apply it to multi-digit multiplication. On this basis he appears to be well placed to learn the vertical algorithm and furthermore, to understand why it works.

When asked about the division example (90 ÷ 7), he explained that, "Seven times ten is 70 and seven times two is 14, and there would have to be six left over. It would be 12 . . . but there are six left over . . . I don't know how to write that". When using the bundling sticks, Oliver took seven bundles of ten and explained it as "Seven times ten equals 70, wait . . . there would be nine tens . . . Seven times ten is 70, then of those they're seven times two which is 14 . . . that would make twelve [sevens] . . . and there would be the six left over".

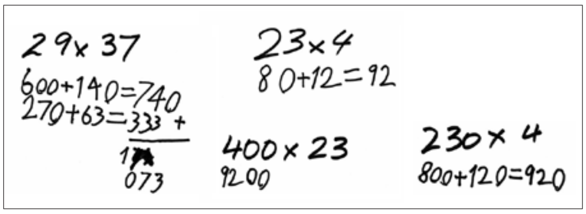

Figure 2. Work samples from Oliver.

Oliver was also asked about 23 × 4, 400 × 23, and 230 × 4 (see Figure 2). Again, he demonstrated an understanding of the distributive property based on standard place value partitioning as well as some understanding of multiplication by powers of ten. For the question 400 × 23, the following conversation developed:

Oliver: That would be 23 times 4 which is 92 and because it is a hundred, add on two zeros.
Int: OK, so what are you doing to the 92 when you add on two zeros?
Oliver: You are changing it from 92 to 920, then to 9200.
Int: So when you add a zero onto the end, how many times bigger does it make it?
Oliver: It makes it ten times bigger.
Int: What about 230 × 4?
Oliver: 200 times 4 is 800 and 30 times 4 is 120, add them together and it's 920. It's exactly the same as 23 times 4 only it is in the hundreds . . . and that one [indicated 400 × 23] was the same only it went into the thousands.

What does this suggest about Oliver?

It is worth reiterating that Thomas and Oliver are in the same class and that Thomas is a Year 6 student while Oliver is a Year 5. However, they seem to hold dramatically different levels of understanding about multiplication and division. Oliver demonstrated a strong understanding of the processes and was able to explain his thinking confidently and competently. Even when he used 'a rule' such as 'adding a zero', he was immediately able to explain why it worked. Oliver was comfortable working with the bundling sticks and used them effectively to demonstrate numbers of equal groups in multiplication and division. As well, he demonstrated an understanding of the distributive property of multiplication based on place value partitioning and he used this to calculate answers using partial products. Also, Oliver's explanation for 23 × 400 suggests that he is aware of the multiplicative relationship between places.

Sissy—a Year 6 student

Sissy described herself as "not very good at multiplying". However, in contrast to Thomas, Sissy was able to explain the multiplication and division processes and demonstrate them confidently with bundling sticks. For both of the multiplication examples (7 × 15 and 23 × 4) and the division example (90 ÷ 7), she demonstrated an understanding of the concept of numbers of equal groups. Her use of the bundling sticks for 7 × 15 is shown in Figure 3 and the conversation with the interviewer follows here.

Sissy: You'd get one bundle of ten and five individuals and you'd do that seven times.
Int: What would you do with that now?
Sissy: You add up all the tens and then add up the fives.
Int: And what do you do with all of the fives when you've added them together?
Sissy: Make them into tens and add them onto the tens.

What does this suggest about Sissy?

Sissy did not rate her ability at multiplying very highly yet she demonstrated on each occasion, a conceptual understanding of multiplication and division in terms of numbers of equal groups. When using the bundling sticks for the division example, she allocated one group of ten to each of seven 'people' and then split the remaining 20 sticks, quickly realising that there were insufficient for each person to have an equal share of 13 and so there had to be a remainder of six. Sissy is not yet using a written multiplication algorithm but was able to explain the process in a conceptual way whereas Thomas could not. As a consequence of Sissy's understanding, she seems well placed to understand and effectively use written algorithms when they are introduced. Her teacher would also be better placed to assist her in the knowledge that Sissy is comfortable with using manipulatives which can be used at any time to reinforce her understanding of aspects of the algorithm.

Lucas—some intriguing explanations

Lucas began well enough, calculating the answer for 7 × 15 and demonstrating it with bundling sticks. However, he struggled with 29 × 37 as is shown by the following conversation and his work sample (Figure 4). He began by writing 20 × 30 and said, "You would do that the other way around [i.e., 30 × 20]". When asked if it would make any difference, he confidently said it would. Lucas then went on to attempt to explain his method:

Lucas: [He counted on by 20s] With five of them you get 100, which is saying that twenty 30s would be 400.Then you've got 9 times 7 so you add a zero on the seven. That gives you 63. You add them together and it's 463.
Int: Are you sure about that?
Lucas: I did it and switched them around.

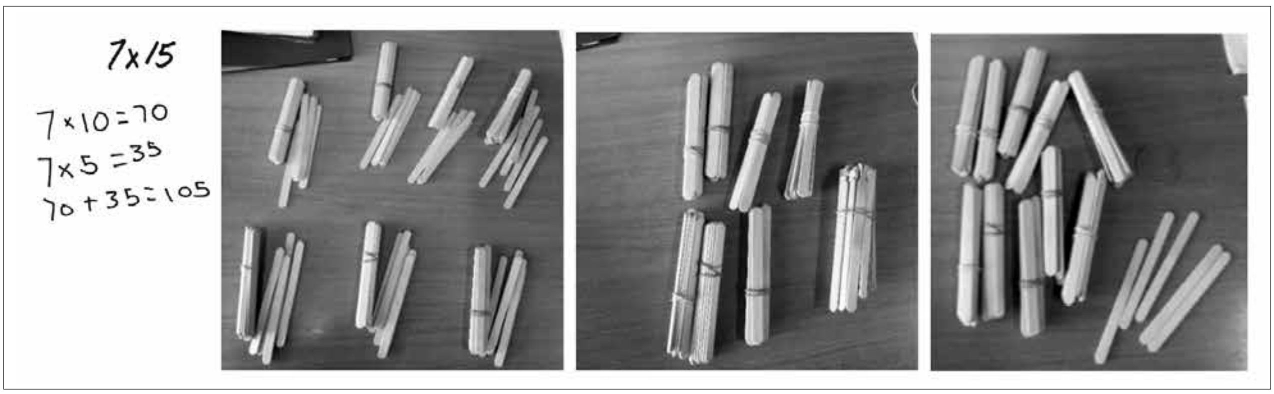

Figure 3. Work samples from Sissy.

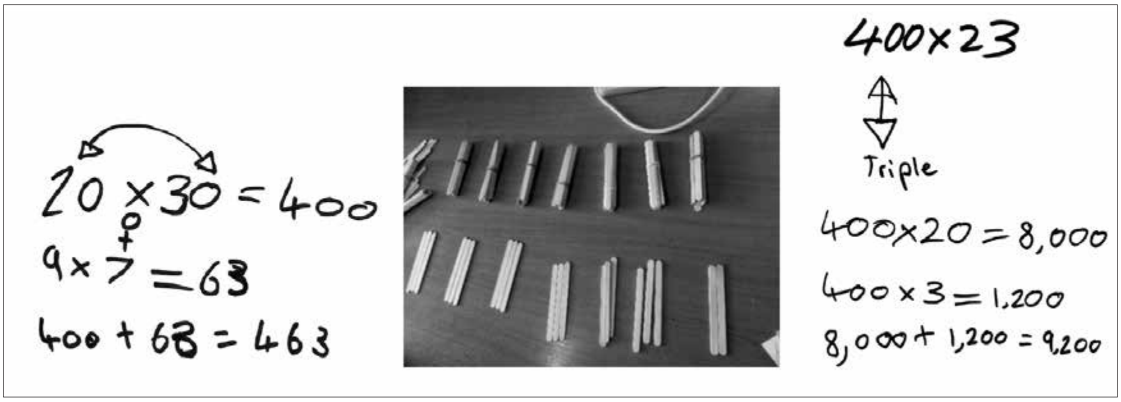

Figure 4. Work samples from Lucas.

Lucas also experienced difficulty with 90 ÷ 7. He started by saying that he had to put the 90 into seven groups but his explanation did not instil confidence that he understood it.

Lucas: You do 90 divided by ten . . . there's two numbers added on so you have to double, and what I do is then 90 divided by seven, so how can you divide up 20 . . . you have to divide up 20 into basically nine groups . . . 20 into ten groups is two, so into nine is three . . . or maybe a decimal? . . . yeah, it's quite hard".

Lucas was able to partially solve it with the bundling sticks but only with considerable prompting. He seemed to know that the groups had to be equal but was confounded about what to do when he could not share the last 20 sticks evenly. As shown in Figure 4, he left the groups as unequal.

The final piece of work from Lucas (Figure 4, right hand sample) indicates some confusion around terminology. He appears to have learned a 'rule' of sorts for multiplying by powers of ten but it is not mathematically sound. He was explaining how to do 400 × 23.

Lucas: What you have to do is triple the number that was there [indicated the 400].

Int: Why?

Lucas: Because this is that number tripled [pointed to the 400] . . . if four goes to 40, it is doubled, if it is 400, it is tripled.

Int: Are you sure about that; tripled and doubled?

Lucas: [continued unabated] What I would do is triple four to make 400, or what you could do is . . . 400 × 20 . . . 400 times ten is 4000, which means that 4000 times 20 is 8000. Then do 400 times three . . . which is 1200, then what I would do is say 8000 plus 1200 equals 9200.

What does this suggest about Lucas?

Lucas has arrived at the correct answer for 400 × 23 but one can only wonder why he has equated multiplying by powers of ten with 'doubling' and 'tripling'. Following a procedure or rule without understanding why it works is unhelpful but more problematical when the procedure or rule is based on a confusion of concepts. Lucas appears to know something about the commutative property but does not understand it as shown by his comment that turning around 20 × 30 would make a difference. Also, he is following a commonly used procedure where children write the larger number first in an algorithm.

Lucas's understanding of splitting into equal groups does not seem to be robust. He did know that he had to split the 90 sticks but when it could not be done without a remainder, he became confused. Perhaps he is accustomed to working with examples that always divide evenly. If so, that is not helpful and does not reflect division in real contexts.

Conclusions

There are clear differences in the levels of knowledge and understanding shown by the four students. Some were comfortable using bundling sticks and could competently explain the processes of multiplication and division, yet had not developed the use of algorithms. Another had an excellent knowledge of procedures but was unable to explain why they worked, and another appeared to be using procedures and rules without reason. While the sample in this article is small, it does serve to indicate situations that teachers are likely to encounter. The question is, "What do they do about it?"

First, the purpose of this article is not to diminish the role of procedural knowledge. It is an important aspect of the proficiency of fluency (Australian Curriculum Assessment and Reporting Authority, 2017) which acknowledges that students need to choose appropriate procedures and apply them with accuracy in flexible ways.

Another aspect is the ability to use procedures to efficiently calculate answers. The issue at stake here is that the use of procedures or rules without reason can constitute a problem for students when the mathematics becomes more complex. It is advocated that a knowledge of connections that exist within multiplicative concepts such as numbers of equal groups, factors and multiples, the inverse relationship between multiplication and division, and properties of multiplication, provide a sound foundation upon which to build an understanding of why procedures work. Rittle-Johnson, Schneider, and Star (2015) described the ongoing debate about the relationships between conceptual and procedural knowledge, noting the oft-held belief that conceptual knowledge supports procedural knowledge and that the connection between them is unidirectional. Given the absence of empirical evidence to specifically support that, Rittle-Johnson et al. suggest that the relationship is bidirectional, and that for deep mathematical understanding to develop, there needs to be a focus on both conceptual and procedural knowledge. For instance, one could suggest that Oliver and Sissy are well placed to learn to use, and understand the multiplication algorithm, but they certainly need to do so in order to handle calculations with numbers larger than two digits.

Second, Thomas's knowledge of procedures is a strength but it would be much stronger if it were underpinned by some understanding of the processes involved. He appears to be a 'procedures first' student as indicated by his explanation of 'adding zeros'. Teachers commonly encounter students like Thomas who come to them with previous knowledge of procedures but with uncertainty around the use of manipulatives. Even though manipulatives like bundling sticks may not help students like Thomas do a calculation that he can already do with a procedure, it is important to expose him to them in order to strengthen his understanding. Also, the use of a sliding strip device to indicate movement of digits when multiplying by powers of ten would show students like Thomas that it is more than just 'adding a zero'.

The other key point to emerge is the power of manipulatives, in this case bundling sticks, to identify conceptual weaknesses in students. Thomas knew the procedures but could not explain the underlying concepts—the bundling sticks demonstrated that. Conversely, the bundling sticks could be said to have facilitated Oliver and Sissy to demonstrate their understanding of the concepts. As was noted earlier, it has been established that the careful use of manipulatives accompanied by teacher demonstration, discussion, and astute focused teacher questioning, can develop an understanding of the mathematical structure that underpins procedures. To return to the Dickensian analogy, it is not the teaching of procedures that is of concern, but rather the use of "rules without reasons" and teaching "nothing else but facts" (as Gradgrind would have it). As Skemp so ably said over forty years ago, it should be about "knowing what to do and why", a combination of procedural and conceptual knowledge.

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