Stimulating mathematical reasoning
with simple open-ended tasks

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The importance of mathematical reasoning is unquestioned and providing opportunities for students to become involved in mathematical reasoning is paramount. The open-ended tasks presented incorporate mathematical content explored through the contexts of problem solving and reasoning.

Introduction

This article presents a number of simple tasks that may be suitable for stimulating students' mathematical reasoning. While each task is presented as a possible context in which mathematical reasoning can be applied, solutions may also be obtained by non-mathematical means. This allows all students to be engaged, although the teacher may wish to guide discussion towards the salient mathematical aspect(s) of the task. The open-ended nature of the tasks also allows various levels of mathematical reasoning to be applied.

According to Stacey (2012), opportunities to reason should feature in mathematics classes at all year levels, and students' reasoning should become more sophisticated over time. Students need to understand the distinctive ways in which we reason and justify assertions in mathematics (Haylock & Manning, 2014). As students formulate and test mathematical conjectures, they develop a foundation for later studies in reasoning and proof.

It can be difficult for teachers to introduce challenging tasks in such a way that makes them accessible, rather than daunting, to students (Cheeseman, Clarke, Roche, & Walker, 2016). According to Boaler (2016), effective mathematical tasks should be open-ended and provide an appropriate degree of challenge for a range of students. One of the ways such tasks can be developed is by locating dimensions of possible variation and changing them. In this way, it is left to the learner to select the appropriate level of challenge at any given moment (Mason, Burton, & Stacey, 2010).

Features of tasks that foster the development of mathematical reasoning include open-ended questions, encouraging multiple conjectures, and are appropriate for students at multiple year levels (Richardson, Carter, & Berenson, 2010). Open-ended questions generate discourse, allow flexibility in thinking, and encourage various solutions at different ability levels. Earlier, Sullivan and Lilburn (2005) identified three characteristics of 'good' questions. Firstly, good questions require more than simply remembering a fact or reproducing a skill. Secondly, good questions may have several acceptable answers. Finally, both students and teachers learn from good questions; students learn by answering the questions and teachers learn about student thinking from the answers that they provide.

The tasks presented here incorporate concepts from the mathematics content strands including perimeter and area, symmetry, number properties, and fractions. This content is explored through the contexts of problem solving and reasoning described in *The Australian Curriculum: Mathematics* (Australian Curriculum and Reporting Authority [ACARA], 2016). For the purposes of consistency, all tasks have been presented using a common format.

Shapes

In this task, students are presented with three 2-D shapes (see Figure 1) and asked to select one shape that is unique based on their understanding of its properties. Since this task permits multiple solutions, students' responses can be evaluated on the basis of their mathematical reasoning. For example, students could select the circle because it is the only shape with no corners. The square is an equally valid choice because it is the only shape with four sides, the only example of a regular polygon, and the only shape which contains right angles.

Alternatively, the triangle could be selected because it is the only shape with three sides, the only example of an irregular polygon, and the only shape that contains acute angles.

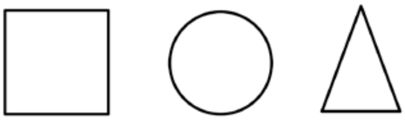

Figure 1. Which of these shapes is the odd one out?

Each of the shapes has certain characteristics in common with one or both of the others, allowing students to apply various lines of mathematical and non-mathematical reasoning to the task. For example, students could compare the reflective and rotational symmetry of the three shapes. The triangle has one line of reflective symmetry, and the square has four lines; the circle is the only shape with an infinite degree of symmetry. The circle and square possess rotational symmetry, while the triangle does not. Furthermore, while both the square and triangle tessellate, the circle does not. The square and triangle can be constructed from triangular pieces while the circle cannot.

It is also conceivable that subjective factors such as prior knowledge could impact on students' reasoning. For example, students who have learned to calculate perimeter but not circumference might select the circle as the odd one out based on their personal understanding. In a similar manner, a student might know how to partition squares and circles (but not triangles) into equal-sized pieces. A less mathematical (but equally valid) line of reasoning could be that square and round windows both feature on Play School, while triangular windows do not.

This task has been designed to elicit discussion of mathematical concepts such as area and perimeter, regular and irregular polygons, and line and rotational symmetry. Such discussions provide opportunities for teachers to introduce and scaffold the use of age-appropriate mathematical vocabulary. Students could also be asked to create their own examples that are illustrative (or not) of the concept being discussed, such as regular polygons or shapes with rotational symmetry.

Numbers

In this example, drawn from Cotton (2016), students are presented with a set of three numbers (3, 4 and 9) and asked to identify the 'odd' one out. Once again there is no incorrect answer to the problem. For this reason, the task is equally appropriate for a diverse range of learners. Rather than assessing the correctness of a particular response, the teacher should focus on listening to the students' explanations to gauge the sophistication of the mathematical reasoning that is being applied.

For example, three could be selected as the 'odd' one out because it is the only prime number. In addition, three is the only one of the numbers that is not itself either a perfect square and/or the square of a prime (since $4 = 2^2$ and $9 = 3^2$). Four could be selected because it is the only one of the three numbers that is even. Four is also the only multiple of four or, alternatively, the only number that is not a multiple of three. Nine could be selected because it is the only number that is larger than 5. It is also $\frac{3}{4}$ of 3×4.

It is also possible to justify the selection of the 'odd' one out for non-mathematical reasons. For example, the numeral four is usually written using straight pen strokes, while three and nine are not. Lateral thinkers might conclude that three is the 'odd' one out after counting the number of letters used to write three, four, and nine (i.e., 5, 4, and 4), respectively.

Ultimately the teacher is in the best position to decide on the appropriate degree of scaffolding to provide for any given task. Here it is suggested that emphasising the word 'odd' when presenting the task ensures that the task retains a low floor by drawing attention to one of the mathematically significant aspects of the problem. Such clues can be provided either as part of the task description or used to prompt students if necessary.

3D objects

We live in a three-dimensional world. Any objects that can be seen or touched (i.e., solids) are three-dimensional. The stimulus for this task consists of three solids, and students are asked to select one that is unique (see Figure 2). The three solids are a rectangular-based pyramid, a hexagonal prism, and a solid with uniform cross section. While the latter object is a right cylinder (albeit with non-circular cross section), referring to it as such belies the common usage of the term. This is an example of a situation in which mathematical terminology should be used judiciously in order to prevent confusion. For convenience the objects are henceforth referred to as a prism, pyramid, and 'other' solid, respectively.

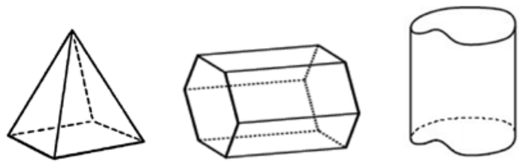

Figure 2. Which of these 3-D objects is the odd one out?

Note that there are various reasons why each of the three objects could be considered unique. Based on its geometric properties, the pyramid is unique in that it is the only object which has an apex. Since it tapers to a point, the pyramid is also the only object which does not have a uniform cross section. The pyramid is also the only object which has a number of triangular faces or, equivalently, whose net includes triangles.

The hexagonal prism is unique because it is the only object that contains regular polygons. It is also the only object which has hexagonal faces. Both the pyramid and 'other' solid have horizontal cross sections and so the prism is the only object with a vertical cross section. Put another way, the hexagonal prism is the only object that is not sitting on its 'base' (which is necessarily hexagonal).

The right cylinder is unique because it is the only object with a non-polygonal base. It is also the only object that does not have one or more rectangular faces, although its net does contain a rectangle. Since the cylinder has no polygonal faces, it is the only one of the objects that is not a polyhedron.

A similar procedure is employed to find the volume of the prism, pyramid, and cylinder. Calculating the volume of each of the three objects involves multiplying the area of the base by the height. In the case of the pyramid, the volume is one-third of the resulting product. This makes the pyramid unique in yet another sense.

Fractions

The purpose of this task is to stimulate discussion of the various models used to represent fractions, since students encounter fractions in a range of familiar contexts. This task was designed for students in the middle and upper primary years. Figure 3 shows three representations of three-quarters: the regional (or area) model, the set (or discrete) model, and the number line (or linear) model. While the three representations are numerically equivalent, each of the models has unique features that contribute to the understanding of the broader fraction concept.

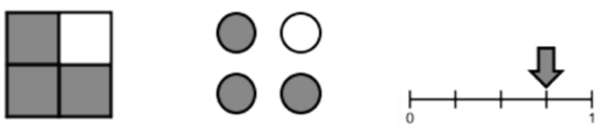

Figure 3. Which of these is the odd one out?

For example, it could be argued that the number line is better suited to representing improper fractions than either the area or set models. Representing fractions greater than one whole is significantly challenging using the set model, while it can be done by adding one or more regions to the area model. The number line is also the only model that readily allows negative numbers to be represented. Complementary fractions such as $\frac{1}{4}$ and $\frac{3}{4}$ are also much less likely to be confused when represented on a number line than when using the set or area models.

The set model is unique in that the unit is composed of discrete parts. Equivalent fractions therefore have clearly distinct representations in the set model. For example, $\frac{3}{4}$ is represented using a set of four items while $\frac{6}{8}$ is represented using a set of eight items. This is not the case for the area model, in which the fractions such as $\frac{3}{4}$ and $\frac{6}{8}$ are merely distinguished by the number of regions into which the whole is divided. Equivalent fractions are also more difficult to distinguish when represented on a number line since they occupy the same position.

The area model is unique in that various representations of the same (or equivalent) fractions may appear quite distinct. While an essential aspect of this represent-tion is all regions have equivalent areas, the shapes themselves do not have to be congruent. Students must therefore understand the relationship between the area of the component parts and the whole.

Area and perimeter

In this task, students are presented with three shapes that have the same area (see Figure 4). Students are asked to identify similarities and differences between the three shapes. For example, students might reason that the rectangle is unique because it consists of two rows and three columns, while the other figures each occupy three rows and three columns. Alternatively, while all of the shapes consist of six squares, the arrangement of squares in the rectangle means that there are seven common edges, while each of the other figures has five. The rectangle is also the only example of a convex polygon, while the other shapes are examples of concave dodecagons.

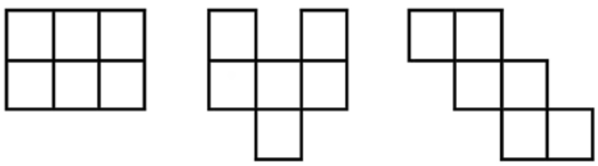

Figure 4. Which of these shapes is the odd one out?

In Figure 5, students are provided with three shapes that each have a perimeter of 12 units. Students are asked to identify the similarities and differences between the shapes. For example, students might reason that the square is unique as it is the only quadrilateral (or indeed the only regular polygon). In addition, the square is convex, while the other two shapes are concave. The square also has 12 common edges, while each of the other shapes has just four. Finally, the square has an area of nine square units, compared to five square units for each of the other two shapes. The second shape may be considered unique as it is the only one that is a net for an open cube. The third shape does not have a central square and does not have three rows and three columns. Unlike the other two shapes, it possesses a single axis of symmetry and does not have rotational symmetry.

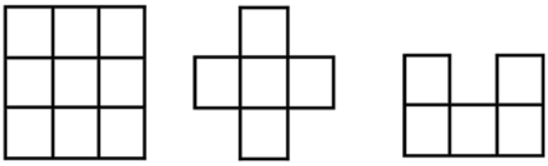

Figure 5. Which of these shapes is the odd one out?

Conclusion

Mathematical reasoning should feature in mathematics classes at all levels (Stacey, 2012). According to Boaler (2016), effective tasks should be open-ended, allowing students to demonstrate diverse levels of achievement. Such tasks need not be complex, since reasoning tasks are often much richer than they might initially appear (Clarke, Clarke, & Sullivan, 2012). It can be genuinely difficult for teachers to imagine how tasks can be introduced in such a way that makes them accessible to students (Cheeseman et al., 2016). The tasks presented here attempt to stimulate reasoning about various mathematical ideas such as perimeter and area, numbers, symmetry, and fractions. Each task is open-ended in that it permits a variety of solutions that can be arrived at through mathematical (or non-mathematical) reasoning. Teachers are encouraged to modify the tasks to suit their individual contexts.

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