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THE PROCESS OF STUDENT COGNITION IN CONSTRUCTING MATHEMATICAL CONJECTURE

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Abstract

This research aims to describe the process of student cognition in constructing mathematical conjecture. Many researchers have studied this process but without giving a detailed explanation of how students understand the information to construct a mathematical conjecture. The researchers focus their analysis on how to construct and prove the conjecture. This article discusses the process of student cognition in constructing mathematical conjecture from the very beginning of the process. The process is studied through qualitative research involving six students from the Mathematics Education Department in the Ganesha University of Education. The process of student cognition in constructing mathematical conjecture is grouped into five different stages. The stages consist of understanding the problem, exploring the problem, formulating conjecture, justifying conjecture, and proving conjecture. In addition, details of the process of the students' cognition in each stage are also discussed.

Keywords: Process of Student Cognition, Mathematical Conjecture, qualitative research

Abstrak

Penelitian ini bertujuan untuk mendeskripsikan proses kognisi mahasiswa dalam mengonstruksi konjektur matematika. Beberapa peneliti mengkaji proses-proses ini tanpa menjelaskan bagaimana siswa memahami informasi untuk mengonstruksi konjektur. Para peneliti dalam analisisnya menekankan bagaimana mengonstruksi dan membuktikan konjektur. Artikel ini membahas proses kognisi mahasiswa dalam mengonstruksi konjektur matematika dari tahap paling awal. Proses tersebut dikaji melalui penelitian kualitatif yang melibatkan enam orang mahasiswa pada Jurusan Pendidikan Matematika Universitas Pendidikan Ganesha. Proses kognisi mahasiswa dalam mengonstruksi konjektur matematika dikelompokkan ke dalam lima tahap yaitu memahami masalah, mengeksplorasi masalah, merumuskan konjektur, menjustifikasi konjektur, dan membuktikan konjektur. Proses kognisi mahasiswa pada setiap tahap juga dibahas secara detail.

Kata kunci: Proses Kognisi Mahasiswa, Konjektur Matematika, Penelitian Kualitatif

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Mathematical conjecture is important in mathematics. It plays a vital role in mathematical development as formalization of conjecture is good and inevitable for mathematics, as large mathematical theories get bigger (Mazur, 1997). Constructing mathematical conjecture involves abstraction and generalization processes related to ideas that are initially hypothetical in nature (Norton, 2000; Nurhasanah, Kusumah, & Sabandar, 2017). In addition, constructing mathematical conjecture and developing proofs are two fundamental aspects of professional mathematical work (Alibert & Thomas, 2002) and is the first step in invention (National Council of Teacher of Mathematics-NCTM, 2000).

Beside Mathematics, mathematical conjecture also plays an important role in mathematics instruction. NCTM (2000) stated that a program in mathematics instruction should enable all students to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof. Many researchers such as Boero, Garuti, Lemut, and Mariotti (as cited in Manizade & Lundquist, 2009) argue that the student must work through internal arguments and sort through solutions that are plausible, similar to ones that a mathematician goes through when building a proof during the process of constructing a conjecture. Boero, Garuti, Lemut, and Mariotti propose that the process of constructing or building conjecture should be emphasized more in mathematics instruction. Besides, constructing mathematical conjecture or making a prediction has three benefits in the mathematical classroom since it can reveal students' conception, plays an important role in reasoning, and fosters learning (Lim et al., 2010). Indonesian current national curriculum known as *Kurikulum 2013*, stipulates a scientific approach as the common learning approach for all subjects taught in all school levels where building or constructing conjecture is one of the activities in reasoning (Kemendikbud, 2013).

Different researchers give different definitions for mathematical conjecture. Ponte et al. (1998) state that a mathematical conjecture is a statement that answers a certain question and is considered to be true. Pedemonte (2001) states that a conjecture is a statement that is strictly connected to an argument and a set of conceptions where the statement is potentially true because some conceptions allow the construction of an argument that justifies it. However, Norton (2004) states that conjectures are ideas formed by a person (the learner) that satisfies the following properties: the idea is conscious (though not necessarily explicitly stated), uncertain, and the conjecture is concerned with its validity. Conjecture in this paper is synthesized from these researchers. Mathematical conjecture is a mathematical statement that is hypothetical in nature, where the statement is potentially true and is constructed by the students using their own knowledge based on the information provided or the given problem.

The truth or falsity of conjecture is proven through a reasoning process by using logical rules or a counter example. Once a conjecture has been proved, then it becomes a valid statement (Pedemonte, 2001). Proving a conjecture for a mathematician or a novice, such as a student, is generally different (Fiallo & Gutierres, 2007). In general, students prove conjecture empirically, narratively, visually, and algebraically (Healy & Hoyles, 2000).

Most research on the process of student cognition in constructing mathematical conjecture focuses on how students construct mathematical conjecture. In addition, testing conjecture constructed by other students is also a topic of interest (Jiang, 2002). Conjecturing in mathematics teaching and the learning process has become an important topic of research in many mathematical fields. Constructing mathematical conjecture in Calculus was studied by Morrow (2004), in Geometry by Gillis (2005) and Yevdokimov (2006), in Number Theory by Morselli (2006), in Trigonometry by Fiallo dan Gutierres (2007), in Statistics by Liu Dan Ho (2008), and in Differential Equation by Burtch (2012), to mention a few.

Constructing mathematical conjecture involves a lot of processes of cognition. Some of these processes have been studied by many researchers (Ponte et al., 1998; Pedemonte, 2001; Norton, 2004; Morseli, 2006; Canadas et al., 2007), however, none of these researchers discusses in detail the process of cognition from the beginning, rather, they put more focus on the stages of constructing and proving the conjecture. In this paper, the process of student cognition in constructing mathematical conjecture is discussed in detail from understanding a problem to proving the conjecture. Understanding the problem is a crucial stage since students start their planning to find a conjecture when they understand the problem. Based on synthesizing the process of cognition in constructing mathematical conjecture proposed by Ponte, et.al (1998) and Morseli (2006) and in combination with Polya's (1945) first step in the problem-solving process, we study the process of student cognition in constructing mathematical conjecture using five different stages. The stages consist of understanding the problem, exploring the problem, formulating the conjecture, justifying the conjecture, and proving the conjecture. Table 1 shows the relationship between the process of student cognition studied here and that studied by Ponte (1998) and Morseli (2006).

Table 1. Comparison of steps in constructing mathematical conjecture

Ponte et al. (1998)	Morseli (2006)	This paper
• Proposing questions and establishing conjectures	• Exploring the problem to find out a property	• Understanding the problem • Exploring the problem
• Testing and refining the conjectures	• Formulating and communicating the conjecture	• Formulating a conjecture
• Arguing and proving the conjectures	• Exploring the conjecture and discovering theoretical arguments that validate it	• Justifying the conjecture
	• Constructing a proof that must be acceptable to the community of mathematicians	• Proving the conjecture

Table 2 shows the focus of analyses on the process of student cognition in constructing mathematical conjecture in the five stages.

Table 2. Stages in constructing mathematical conjecture and its analysis focus

No.	Stage	Focus of analysis
1	Understanding the problem	What is the process of student cognition in understanding a given problem, such as the process of understanding the information in the problem?
2	Exploring the problem	What is the process of student cognition in exploring a problem, such as translating a problem, manipulating a problem, and identifying properties to be constructed as a conjecture?

3	Formulating a conjecture	What is the process of student cognition in formulating conjecture such as types of conjecture, writing a sentence of conjecture, and finding a consideration used in writing conjecture?
4	Justifying the conjecture	What is the process of student cognition in reasoning and generalizing conjecture?
5	Proving the conjecture	What is the process of student cognition in proving conjecture?

METHOD

A qualitative research method was used in this study. In the academic year 2013/2014, six students from the Mathematics Department of Ganesha University of Education were chosen as research subjects. The students were chosen based on their mathematical ability and gender. One male and one female student were chosen respectively from high, moderate, and low mathematical ability. The subjects were coded as S1, S2, S3, S4, S5, and S6.

Data on the process of cognition was collected from each subject through a task-based interview. Each subject was given two mathematics problems from which he or she constructed a mathematical conjecture. After constructing mathematical conjectures from the problems, subjects were interviewed based on their work. The interview was carried out twice for triangulation purposes with the second interview taking place two months after the first. The following problems were used in the first interview.

Problem 1

Given a parallelogram ABCD with length AB as *a* unit and length BC as *b* unit, draw a bisector line from each point A, B, C, and D. The bisector line from point A intersects the bisector line from point B at point N, from point B and point C at point M, from point C and point D at point L, and from point D and point A at point K. Construct a conjecture about the quadrilateral KLMN such as its form, position, side length, area, or the like.

Problem 2

For any natural number *n*, the function $f_n(x)$ from R to R is defined as:

$$f_n(x) = \begin{cases} x^n, & x \in [0,1] \\ 0, & else \end{cases}$$

Construct a conjecture about $f_n(x)$ related to *n* such as its value, area, sequence, series, or arch length.

The problems used in the second interview were similar to those in the first interview but with different point names or variables. The interviews were recorded, transcripted, and then analyzed using Miles' and Huberman's (1994) method for analyzing qualitative data.

RESULT AND DISCUSSION

The process of student cognition in constructing mathematical conjecture was explained in five different stages, namely understanding the problem, exploring the problem, formulating conjecture, justifying conjecture, and proving the conjecture.

The results of the data analysis for all subjects were abstracted to obtain a general idea about the students' process of cognition in constructing mathematical conjecture. Abstraction as a process of cognition for all subjects in constructing mathematical conjecture was done by determining common processes for all subjects and eliminating the condition of the problem used. The process of student cognition in constructing mathematical conjecture is specified in the five stages mentioned previously. The detailed process of cognition in each stage is given in Table 3.

Table 3. Detailed process of student cognition in constructing mathematical conjecture

Stage	Process of cognition
Understanding the problem	• Reading the problem to determine what is given and what is asked. • Determining what is given in the problem by using his/her own words. • Determining what is asked in the problem by using his/her own words.
Exploring the problem	• Translating or transforming the problem into figures or graphs. • Manipulating a problem by using various figures or graphs that reflect special cases. • Finding invariant properties or patterns by observing changes in the figures or graphs. • Connecting relevant mathematical knowledge by identifying properties or patterns observed in the changes on the figures or graphs. • Stipulating the properties or patterns observed to be constructed as conjecture.
Formulating a conjecture	• Remembering the kinds of conjectures obtained from exploring a problem to be formulated as conjecture. • Writing conjecture by referring to the result from an exploration of a problem, mathematical language, and sentence type as a point of reference. • Believing the formulation of conjectures can be understood by other people.
Justifying the conjecture	• Explaining the reasons for conjecture by using a picture or graph, measuring or counting, or a mathematical connection by relating relevant mathematical knowledge to its corresponding conjecture. • Generalizing the conjecture is done by observing some cases to find a property or pattern and then visualize it so it is valid for all other cases. • Being aware of the deficiency or mistake underlying the formulation of a conjecture. Identifying the deficiency or mistake underlying the formulation of conjecture or its reason enables students to correct the conjecture.
Proving the	• Being aware that the truth of conjecture must be proved and giving expression to this in steps.

conjecture	• Choosing kinds of proof according to the constructed conjecture. • Organizing the proof. Proving the conjecture is done by showing figures or graphs, writing mathematical sentences, counting, connecting relevant mathematical knowledge, and concluding that the conjecture has been proved.

Some of these processes have been discusses in Canadas, et.al (2007).

An interesting finding was obtained in stage 2. To find conjecture, students drew some figures or graphs. The following dialogue shows Student S1's thoughts on finding properties as conjecture for Problem 1.

Researcher	:	What are you doing after understanding the information
Student S1	:	I drew some parallelograms, ABCD. Here I drew ten parallelograms [All parallelograms marked as a, b,..., j drawn by student S1 are given in Figure 1]. I drew them with different lengths of a and b. The figures were drawn with the same length of a and different length of b.
Researcher	:	What is the purpose of drawing?
Student S1	:	I drew a parallelogram and its bisector lines to get their intersection and to get the quadrilateral KLMN
Researcher	:	Why do you make a lot of figures?
Student S1	:	In order to see the shape, position, and area of the quadrilateral if the length of the parallelogram is changed.

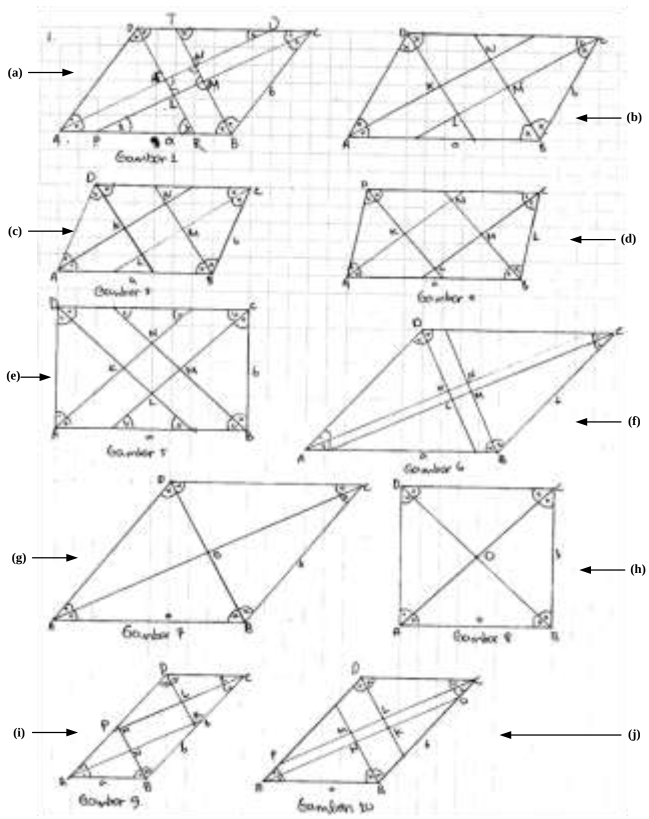

Figure 1. Student S1's work

A similar process of cognition is used to find a mathematical conjecture in the exploring problem stage for Problem 2. The following dialogue shows Student S2's thoughts on finding properties as conjecture for the problem.

Researcher	:	After understanding the information, what are you going to do?
Student S2	:	I drew some graphs as these. [Student S2 points to graphs a, b, c, and d in Figure 2].
Researcher	:	Can you explain them?
Student S2	:	These graphs are families of power function graph with natural numbers as its power. For x = 1, the graph is a straight line, for x = 2, the graph is part of a parabola, and so on.
Researcher	:	Why did you make four graphs?
Student S2	:	For ease of seeing the relationship between them. By drawing some graphs, I can see its value for a given x in its support area enclosed by the arch and x-axis, and arch length when n changes from 1, 2, 3 and 4.

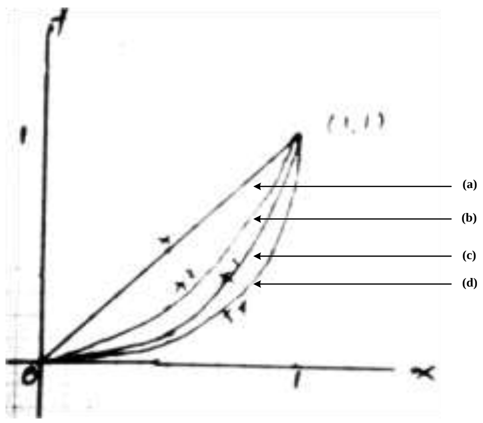

Figure 2. Student S2's work

Another interesting finding in the process of student cognition in constructing mathematical conjecture was seen in the proving stage. All subjects are aware that their conjecture must be proved in order to make it a true statement. This process of cognition can be inferred from student S3's dialogue

Researcher	:	How do you convince someone about your conjecture?
Student S3	:	I will show him the conjecture is true.
Researcher	:	How do you show it is true?
Student S3	:	I will give him a proof.

A similar dialogue was obtained from other subjects in this stage when they were asked how they would make someone believe their conjecture.

The proof of conjecture organized by students was consistent with Healy and Hoyles' (2000) findings. Almost all students use narrative and visual proof for their conjecture in problem 1, whereas they used empirical and algebraic proof for their conjecture in problem 2. An example of narrative proof was given by Student S1 as depicted in Figure 3. He made a conjecture about the form of KLMN by stating, "The form of KLMN is always rectangle". To prove his conjecture, he drew a parallelogram, ABCD, and its bisector lines. Then he showed that both KL and NM and KN and LM were parallel and its angle is 90° respectively by using the congruence property of the triangle.

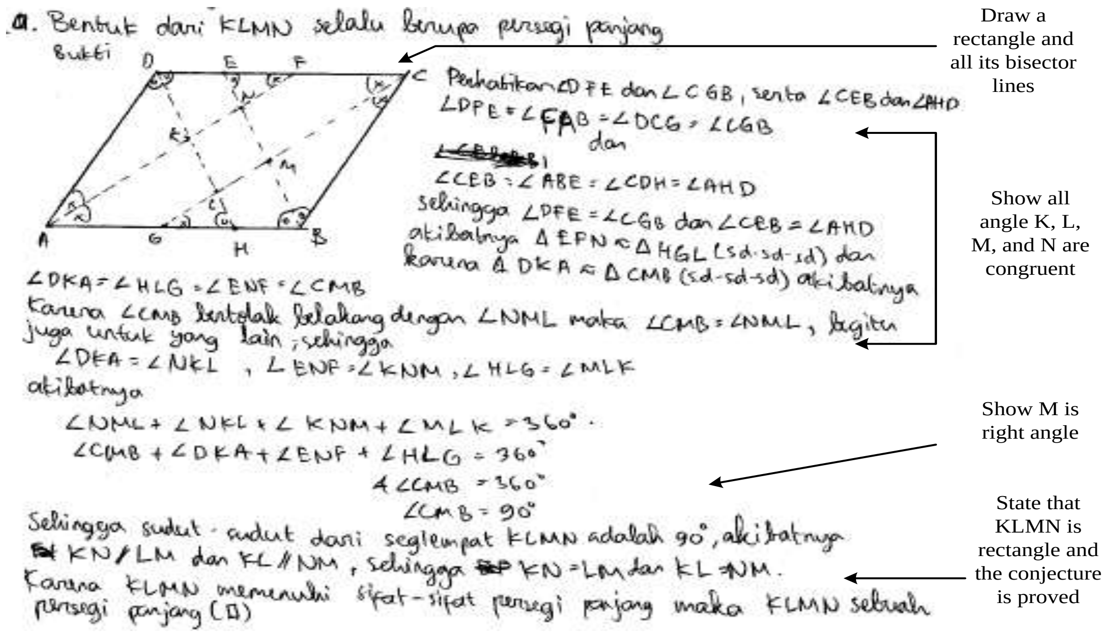

Figure 3. A proof by Student S1

An example of visual proof was given by Student S4 as depicted in Figure 4. She made a conjecture about the arch length by stating, “If *n* increases then the arch length increases”. To prove her conjecture, she drew a lot of arches and said that from the graphs, the arch length for *n*=2 is longer than for *n*=1 because the two graphs have two common terminal points but the points are connected by a straight line for *n*=1 and part of a parabola for *n*=2. A similar reason was used when comparing the length of arch for *n*=2 and *n*=3.

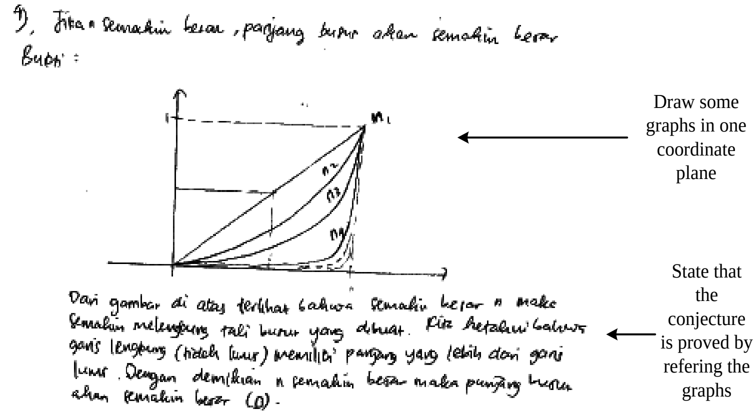

Figure 4. A proof by Student S4

An example of algebraic proof was given by student S5 as depicted in Figure 5. She made a conjecture about integral value of *f*(*x*) by stating, “Integral values of *f*(*x*) decreases as *n* increases”. She proved her conjecture by evaluating $\int x dx = \frac{1}{2}x^2 + C$, $\int x^2 dx = \frac{1}{3}x^3$, $\int x^3 dx = \frac{1}{4}x^4 + C$ and identified their results as sequence of $\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}$ since $x \in [0,1]$. She said the values of the sequence were decreasing as *n* increased. It is known from the interview that she concluded the values of the sequence were decreasing by identifying it as a harmonic sequence that she already knew from Calculus.

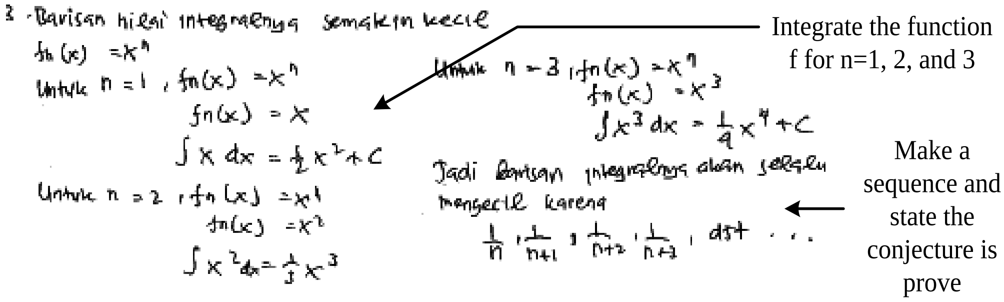

Figure 5. A Proof by Student S5

CONCLUSION

The process of student cognition in constructing mathematical conjecture can be classified into five different stages, namely understanding the problem, exploring the problem, formulating a conjecture, justifying the conjecture, and proving the conjecture. The process of student cognition in understanding the problem stage are reading the problem, determining what is given and what is asked, and restating this with his/her own words. The process of student cognition in exploring the problem stage are translating or transforming the problem into figures or graphs, manipulating the problem by using various figures or graphs, finding invariant properties or patterns by observing the changes in the figures or graphs, stipulating the properties or patterns observed from the figures or graphs that would be constructed as conjectures, and connecting relevant mathematical knowledge in identifying properties or patterns observed from the changes in the figures or graphs. The process of student cognition in formulating the conjecture stage are remembering kinds of conjecture obtained from exploring the problem to be formulated as conjecture, writing down the conjecture by referring to the result from exploring the problem and by using language and sentence types as points of reference in writing the conjecture, and believing that the formulation of conjectures can be understood by other people. The process of cognition in justifying the conjecture stage explains the reasons for the conjecture, generalizing the conjecture, and being aware of the deficiency or mistake underlying the formulation of conjectures or their reasons. The process of cognition in the proving conjecture stage are being aware that the truth of conjecture must be proved, choosing the type of proof according to the constructed conjecture, and organizing the proof. Proving the conjecture is done by showing figures or graphs, writing mathematical sentences, doing some counting or algebra manipulation, connecting relevant mathematical knowledge, and concluding that the conjecture had been proved.

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