e-Journal of Business Education & Scholarship of Teaching
Vol. 8, Iss. 2, 2014, pp: 76-99.
"http://www.ejbest.org"

Exponential Problems in Business Courses: The Translation of Time Units

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ABSTRACT

This paper reviews a few applications of the exponential function used primarily in business courses. It shows that translating time units in which queuing problems are measured can be used to overcome some apparent differences between queuing (arrival) problems and survival (decay) problems. There is a cautionary note concerning the limitations of altering time units after the fact, particularly for survival-type problems, as well in the initial assignment of time units in the research process.

Keywords: Exponential decay; discounting; force of mortality; queuing problems

JEL Classification: C10
PsycINFO Classification: 3530
FoR Code: 1303; 1502
ERA Journal ID#: 35696

Introduction

The exponential function and its discrete counterpart have all manner of applications in business texts—in finance, economics, accounting, and quantitative methods as well as others. It is utilized in compounding interest and in discounting. Discounting is a form of 'decay' equation which can also be applied to the depreciation of the value of an asset (constant rate of depreciation approach) and to the death of a population, represented by a survival function that exhibits a constant rate of decline over time. The latter has been used to illustrate the decay of a population of charged atoms. It has also been used to model the life of a light bulb as well as other applications.

Even before the student has entered a business course, he has very likely had some first-hand exposure to compound interest. From compound interest, it is a small step to understanding discounting, since it is simply the reverse of compounding. Discounting, the constant rate of depreciation model, and other applications of the 'decay' equation appear in different courses, and rarely, if ever, appear in a single business text. Nevertheless, it is a fairly easy transition from one application of the of 'decay' equation to others since all deal with the percentage decline in a stock or an asset, whether the stock or asset is measured in physical units or in monetary terms.

In quantitative methods, a seemingly different application of the exponential equation appears in the form of the queuing or 'arrival' problem. The so-called 'arrival' problem in which, say, so many customers arrive at a destination per time period appears to have little relation to the 'survival' problems in which a population (of assets or of atoms) or a light bulb has a given probability of decaying or dying per time period.

Some of the differences between 'arrival' problems and 'survival' problems can be illustrated by utilizing the discrete form of the exponential equation and comparing examples of both an arrival and a survival problem. However, these apparent differences can be overcome by a change in the time units in which the arrival problem is measured. I demonstrate this with a few examples. There appears to be a gap in the literature that examples such as those shown in this paper can fill, which is the purpose of this paper.

That is, the purpose here is to offer an example or two that may be used to bridge a gap, as I see it, that exists in the ways in which the exponential function is utilized in several contexts as a 'decay equation' and as it is used in quantitative methods, particularly as it relates to queuing or arrival problems.

Literature Review

In reviewing the literature on the exponential function and its discrete counterparts, I have limited the discussion to compound interest and discounting as they typically appears in business textbooks, then proceed to briefly review the discounting/decay function in a few of its incarnations.

If one dollar is set aside in a savings account, it will accumulate to equal $\$1(1 + r)$ at the end of one year compounded annually at interest rate r. The amount that one dollar will grow in t years if the interest rate is r percent compounded annually would be:

(1) $\$1(1 + r)^t$

For a 10 percent interest rate one dollar would grow to \$2.59 in 10 years. For an amount, say V, it will grow to $V(1 + r)^t$ by year t when compounded annually at r percent. An amount of \$1000 would equal \$2590 after 10 years at 10 percent interest.

The continuous counterpart to the discrete version shown in Equation (1) is the following equation:

(1.1) e^{rt}

The relationship between Equations (1) and (1.1) is that the amount $\$1e^{(1)(1)} = e = 2.71828$ is the value that one dollar will become by the end of year one if compounded continuously at an annual interest rate of 100 percent. Nominally, the interest rate would be 100 percent but in effect it would be 172 percent per year if interest were compounded continuously (Chiang 1974, 289).

There is a significant difference between a two-fold increase in one dollar and the 2.72-fold increase per Equation (1.1) under the assumption of 100 percent interest for year one. Even though the gap seems large, for small values of r, the two equations give similar values, especially for low values of t. For example, if r is replaced by i in Equation (1) and set equal to Equation (1.1) as follows:

$(1 + i)^t = e^{rt}$

then for i = 10 percent and t = 1: $1.10 = e^r$ taking logs, r = .0953.

That is, Equation (1) at 10 percent interest is equivalent to Equation (1.1) at 9.53 percent interest (Chiang 1974, 293). Whether someone is paying interest or receiving it, the difference between 9.53 percent and 10 percent can be significant. However, for purposes of illustration only, the assumption that i = r for low values of r is not that unrealistic.

The value per year of one dollar compounded continuously at r percent per year is $S_t = e^{rt}$. The change per time period in e^{rt} is the derivative of S_t:

(2) $dS_t/dt = e^{rt}/dt = r\,e^{rt}$.

The growth rate of $S_t = e^{rt}$ is rate of change as a percentage of S_t:

(2.1) $dS_t/dt\ 1/S_t = e^{rt}/dt\ 1/e^{rt} = r\,e^{rt}/e^{rt} = r$.

The rate r is the instantaneous rate of growth at the point in time t. This is interpreted to mean that if the rate r were sustained for a year, then e^{rt} will have increased by the amount re^{rt} the by year's end (Chiang 1974, 292).

Discounting

Discounting can be thought of as the opposite of compound growth (Chiang 1974, 293). The equation

(3) $\$1\ [1/(1 + r)^t]$

gives the value today of one dollar to be received t years from now, where r is the discount rate. The continuous counterpart to the discrete version shown in Equation (3) is:

(4) e^{-rt}

where the value of the r in (3) is not equal to the value of the r in (4).

Substituting i for r in Equation (3) and setting it equal to Equation (4) at t = 1 gives:

$1/(1 + i) = e^{-r}$ and at i = 10 percent

$1/(1.10) = .9091 = e^{-.0953}$, as previously shown.

Equation (4) is also called decay equation where (minus) r is the rate of decay or the negative growth rate (Chiang 1974, 293).

Equation (4), or its discrete counterpart Equation (3), has numerous applications, which include but are not limited to discount, depreciation, decay, and death. It can be used in discount to calculate the present value of a given dollar amount to be received in the future. It can also be used to discount a stream of income over time. It can be used in the calculation of a constant rate of depreciation of an asset over time. It can represent a decay equation for a group of charged atoms. It can also be thought of as a survival equation for a population or for an individual (for example, a light bulb). These are just a few of the applications of the so-called decay equation.

Discount: Present Value of an Amount to be received in the Future

Discount is a method by which an amount, usually expressed in monetary terms (e. g., dollars), to be received in the future is valued today. Present Value (PV) is a term used for the monetary value today of an amount to be received in the future. The dollar value of some amount to be received in the future is 'discounted' because, so the theory goes, a dollar to be received in the future does not have as much value as a dollar received today (Fisher 1930, 61-98; Frederick et al. 2002; Herbener 2011; Touchstone 2006, 66-67). One reason for this is because a dollar today if loaned to a borrower can yield a return in the future. Discount may also exist because of the risks, such as the uncertainties of death and default, that may intervene between now and the future that threaten future receipt.

To calculate the value today (PV) of the amount, say A, to be received in the future, the amount A is factored by $1/(1 + r)^t$, where r is the discount rate and t is the future time at which A is to be received. The value of the discount rate may vary depending upon the circumstances under consideration. To arrive at a suitable discount rate, it is common practice to start with a 'riskless' rate, such as the prevailing rate on government securities; for example, the six-month rate on U.S. Treasury Bills or the long-term government bond rate (Hirschey 2006, 578). This rate may be adjusted by other risk factors specific to the problem at hand (Hirschey 2006, 536).

The continuous version would be $PV = Ae^{-rt}$ (where r is not equal to r shown in the discrete equation).

Discount: Present Value of a Stream of Income over Time

For a constant stream of income of say Y per year, to be received for T years, the PV would be the sum of the yearly discounted income amounts for years one through T:

(5) $PV_t = \sum_1^T Y [1/(1 + r)^t] = Y/r [1 - (1/1 + r)^T]$.

The 'stream of income' is usually the income generated over time by an economic 'good' whether the good is a consumer good or is used in the production of other goods. All goods, particularly durable goods, can be thought of as providing 'services' over time that have a monetary equivalent, whether the value is explicit or imputed (I. N. Fisher [1906] (1965), 101, 106; I. N. Fisher 1930, 3). This could be the lifetime income to be received by a worker over the course of his working life. It could be the rent on an apartment unit that is received by the owner of an apartment complex. Similarly it could be an imputed dollar value of the services a homeowner receives from his home. (This can be based on its opportunity cost, that is, the income that is foregone by the homeowner from consuming the services of the house himself rather than employing it in its highest valued alternative.) Workers, apartments, and houses are all 'economic goods' in the sense that they produce services over time that have a monetary value or equivalent. For simplification purposes it can be assumed that the income, Y, is net of expenses.

The PV of a stream of income is calculated over a period of time. Typically, PV is calculated over the lifetime of the 'good' in question ending at time T. The time period is usually given in years. The lifetime in question (T) is the economic life not the physical life of the good—although the two may coincide. Some goods have 'scrap value' at the end of their working lives. Although not shown in Equation (5), that scrap value can be taken into account in the PV calculation. For instance, a car that is no longer serviceable to a taxi company may be sold for scrap or parts at the end of its working life. Humans, of course, have no scrap value since they cannot be bought or sold. Equation (5) assumes that Y is constant. This is a simplifying assumption. However, even if the income stream is uneven, it can be transformed into a constant stream via the calculation of permanent income. Permanent income may be expressed in terms of the asset's PV: $Y_p = r(PV)/(1 - e^{-rT})$ (Yaari 1964, 312). In discrete terms this would be $Y_p = r(PV)/[1 - 1/(1 + r)^T]$.

The continuous version of Equation (5) (with the qualification that r is not equal to r in the discrete version) is:

(5.1) $PV_t = \int_0^T Y e^{-rt} dt = Y/r (1 - e^{-rT})$ (Chiang 1974, 458).

Depreciation: Constant Rate of Decline (CRD) Method

The CRD method of depreciation begins with the value of the capital good in year zero. Utilizing the discrete version, the capital value in year zero is multiplied by $[1/(1 + r)^t]$ to obtain the asset values for years one through T, where T is the life of the capital good and r is the depreciation rate (in percentage terms). Depreciation is equal to the change in the capital value from one year to the next. With the CRD method, depreciation per year can also be calculated as the asset value in year zero multiplied by $r[1/(1 + r)^t]$, when the discrete version is employed.

The value of r is not given, however. It is calculated by solving the following equation for r:

(6) $[1/(1 + r)^T] = s/V$

where s is scrap value in year T, and V is the asset value in year zero. In order to solve for r, it is necessary to have a value for scrap in year T because the CRD equation is infinite.

The equation usually shown for determining r is: (6.1) $(1 - r)^T = s/V$ (Simpson et al. 1951, 268-69).

Both Equations (6) and (6.1) are discrete forms of the continuous equation:

(6.3) $e^{-rT} = s/V$ (Cramer 1958, 28; H. R. Fisher 1958, 49).

When Equation (6.1) is employed to solve for r, then in turn the asset value in year t is obtained by multiplying $(1 - r)^t$ by the asset value in year zero. Likewise, if Equation (6.3) is used to solve for r, then e^{-rt} is multiplied by the asset value in year zero to arrive at the asset value in year t.

Decay: Survival Function for a Constant Rate of Decline

The following equation

(7) $S_t = e^{-rt}$

can be thought of as a survival function that when multiplied by a given population, A, results in a constant rate of decline of that population over time.

The 'death rate' per time period is given by the derivative of S_t:

(8) $dS_t/dt = -re^{-rt}$.

The 'force of mortality' is given by the percentage rate of change of S_t:

(9) $dS_t/dt\ (1/S) = -r$.

That is, r is the constant instantaneous rate by which a given population declines over time. The negative signs for the 'death rate,' Equation 8, and the 'force of mortality,' Equation (9), will be ignored in discussion.[1] The survival function, S_t, is infinite. Even for very large values of t, there is a positive probability of survival. The 'death rate' per period falls over time, reflecting the decline in the survival rate over time. The force of mortality is constant, which indicates that the cause or causes of death is/are independent of age (time t) (Knight 2004, 1340).

A typical example of the survival function shown in Equation (7) is the rate of decay of charged atoms from an excited state to the ground state. Although this is not a business example, per se, similar problems appear in business-related texts (Ross 1997, 31). The duration of time for which an individual atom spends in the excited state varies from atom to atom. The time a particular atom remains in the excited state is not constant but is given by a range. The rate, r, is the probability that an excited atom will decay. The decay probability is independent of the time in which the atom has remained in the excited state (Knight 2004, 1340-41).

For a given population of excited atoms, the number falls exponentially over time. If A is the number of charged atoms at time zero, then the population at any given time will equal

Ae^{-rt}.

[1] I have enclosed the expression 'death rate' in quotation marks to distinguish it from the force of mortality. For some mortality functions, the force of mortality exceeds unity for some values of t; particularly for lower and upper ends of the life spectrum. At the upper end, as the survival rate (denominator) approaches zero, the force of mortality approaches infinity (Jordan 1967, 12-13). In Equation (9), the survival rate in the denominator and the numerator cancel out, leaving a constant force of morality.

Approximately, two-thirds will have decayed within the time interval ending at t equal to tau (τ). The time tau (τ) is referred to as the lifetime of the excited state and is equal to 1/r, the inverse of r, the rate of decay. At time τ, 63.2 percent of the atoms will have decayed; that is, will have undergone a quantum jump to a lower state. Time τ corresponds to $e^{-t/\tau} = e^{-1} = .368$; that is, a time at which the survival function has been reduced to 36.8 percent of its initial value (Knight 2004, 1339-1342). For an initial population of A, the population at time tau will equal $Ae^{-1} = .368A$.

In the decay process there is emission of a photon or a high-velocity particle that has mass. An experimental procedure to determine the rate of decay would examine a group of, say, Xenon atoms in the excited state and measure the number of photons emitted over time, since a photon would be emitted whenever an atom jumps to the ground state, that is, decays (Knight 2004, 1340). Once the time is measured at which 36.8 percent of atoms remain in the excited state, that is at $t = \tau$, the rate of decay is given as $r = 1/\tau$. The decay rate, r, can be verified by using the data on photon emissions to estimate the exponential decay equation Ae^{-rt}, where A corresponds to the initial quantity of excited atoms (Knight 2004, 1342).

In his Physics text, Randall D. Knight noted the following:

> If Δt is small, the probability of photon emission during time interval Δt is directly proportional to Δt. That is, if the emission probability in 1 ns is 1%, it will be 2% in 2 ns and 0.5% in 0.5 ns. (This logic fails if Δt gets too big. If the probability is 70% in 20 ns, we can *not* say that the probability would be 140% in 40 ns because a probability > 1 is meaningless.) We will be interested in the limit $\Delta t \rightarrow dt$, so the concept is valid and we can write
>
> $$\text{Prob(emission in } \Delta t \text{ at time t)} = r\Delta t$$
>
> where r is called the decay rate because the number of excited atoms decays with time. It is the probability per second with units of s^{-1}, and thus a rate. For example, if an atom has a 5% probability of emitting a photon during a 2 ns interval, its decay rate is $r = P/\Delta t = (0.05)/(2ns) = 0.025\ ns^{-1} = 2.5 \times 10^7\ s^{-1}$ (Knight 2004, 1341).

That is, if the rate of decay is 2.5 percent per nanosecond, it will be $2.52.5 \times 10^7$ per s^{-1}. The problems associated with the translation of time units are discussed below.

Method

A review of the literature gave several ways in which the 'decay' equation can be utilized in business texts and courses. The Results section will begin with a discussion of the decay rate r as it relates to Knight's quotation (2004, 1351) given in the previous section. The method used in this paper presents numerical examples of the discrete equations presented in the Literature Review along with a brief discussion of their interpretations. Thereafter is a cursory overview of the Poisson and exponential distributions followed by two applications of exponential problems commonly cited in the literature—a 'survival' problem and an 'arrival' problem. The apparent differences in these problems are discussed utilizing the discrete form of the exponential function. These apparent differences are reconciled by changing the time units in which the arrival problem is measured.

Results

The Decay Rate r

In the quote of Knight's given in the Literature Review, the decay rate is equal to the probability divided by Δt. If the force of mortality is .10 per year and Δt is equal to one year then: $r = P/\Delta t = .10 = .10/1$ or 10 percent.

For a time unit of one century r would equal 10. In accordance with Knight, 10 would be the rate per century.

As I see it, r may be changed mathematically by changing the time unit, but this may lead to problems of interpretation if the time unit becomes too large. To illustrate this, I will begin by showing two categories of examples. The first one considers the replacement of Supreme Court Justices, and the second is similar to the decay of atoms in that it hypothesizes a decline in a population (of assets).

First, suppose r measures the rate at which U. S. Supreme Court Justices are replaced over time, where time is measured in years. An example of a problem like this was given by Chou (1969, 210-211), and the rate, r, in that example was calculated as .5 per year. Now, if the time unit were changed to centuries instead of years, r would become 50 replacements per century. The decay rate, r, in this example is a count in units of Supreme Court (SC) justices.

Now suppose hypothetically that all of the Justices were replaced every year. In this case nine justices would be replaced per year at a rate r equal to nine per year. Even though 100 percent would be replaced yearly, the rate r would be in terms of the number of Justices, not the percentage. If the time unit were changed to replacements per century, r would equal 900.

Unlike the SC Justices example, the decay rate for energized atoms is not a count but a percentage. It can be estimated by finding the time at which approximately 62 percent of a population of energized atoms has decayed, that is time tau, and then setting r equal to 1/tau. For tau = 10, r equals ten percent.

For simplicity sake rather than atoms, I will assume a general case in which there are only, say, ten 'assets,' and all of the assets 'die' in the first year, the death rate would be 1/1 = 1 or 100 percent. In one year, 100 percent of the assets are assumed to die. The number that die in year one would be all ten assets. (In order to find the number that would die using the discrete form of the exponential equation, the rate, r, is multiplied by the survival function times the number of assets in year zero. This is discussed below).

Now, if the time unit were changed to one century then r would now equal 100. However, unlike the SC example, r = 100 is not a count; that is, it cannot be said that in one hundred years, 100 assets will have died. Unlike the SC example, whether r =1 (100 percent per year) or 100 (10,000 percent per century), it is a percentage, not a count per time period. Also, unlike the second SC example in which all of the Justices are renewed every year, the stock of assets is not renewed at all. (That is, it cannot be said that 1000 assets would have died in one century.) Once the stock of ten assets has died off, there are no more assets to die. That is, there are limits to changing the time units for decay or 'survival' problems. This will be discussed later within the context of two examples—one 'survival' problem and an 'arrival' problem.

Before ending this section, I need to clarify two points. One concerns the asset example in which I assumed that all of the ten assets would die in the first year. If all of the assets die within the first year, then r is equal to 100 percent; however, the discrete form of the exponential equation would not show a 'death rate' of 100 percent. The discrete survival function as well as the 'death' function (that is, r[1/(1+r)]) would equal: 1/(1+1) = 1/2.

That is, with r = 100 percent, the survival rate for year one would be .5 (50 percent) and the 'death rate' would also equal .5 (50 percent). (Even though 100 percent of the assets do not die in year one according to the discrete form of the equation, when r = 100 percent, the sum of the values $1/(2)^t$ approach one (100 percent) fairly quickly. The sum equals almost 88 percent by year three. The sum equals one as t approaches infinity.) For the discrete case $e^{-1(1)}$ = .3678.

(That is, returning to a population of excited atoms when r = 100 percent, period one would correspond to the lifetime of the excited state.)

So, even though r is 100 percent in year one, not all of the population dies off in year one for the exponential case. This seems contradictory and could be seen as a limitation of the exponential equation; however, it may be more likely that if 100 percent of a population dies within the first year, then perhaps the time unit of a year is not a sensitive enough measure. A time unit less than one year may show a pattern of exponential decay.

For the SC case in which it was assumed for illustrative purposes that all Justices were replaced every year, it may be that the exponential distribution does not apply. Both the sensitivity of the time measure and testing the appropriateness of the exponential distribution are considered later in the paper.

The second point I would like to clarify concerns the rate of decay of 100 percent on a yearly basis becoming 10,000 percent when the time unit is changed to one century. The rate of decay on a yearly basis would still be 100 percent. In other words, when the rate is 10,000 percent when the time unit is one century, it is still the case that it is (10,000/100) percent on a yearly basis.

Discount: Present Value of an Amount to be received in the Future

Equation (3) gives the PV of $1 to be received in t years. In general, for any amount A to be received t years from now discounted at rate r, the PV today would be $A[1/(1 + r)^t]$.

For example in the discrete case, for a discount rate of, say, 10 percent, $1000 to be received one year from now would be worth today, or have a PV of, $1000 [1/(1.10)], or $909.91. If $1000 were to be received two years hence, its PV today would be worth $1000 $[(1/1.10)^2]$ or $826.40. The PV of an amount of $1000 to be received t years from now would be equal to $1000 $[1/(1.10)^t]$.

Discount: Present Value of a Stream of Income over Time

Entries in Table 1 can be used to provide numerical content for Equation (5). Table 1 Column 2 gives the discounted values of $1 for years 1 through 10 for a discount rate of 10 percent. For a 'good' that yields $1 per year and has a life of 10 years, the PV would be $6.14 if the discount rate is assumed to be 10 percent. The PV is shown as the bottom line sum of Table 1 Column 2. To arrive at the PV of any constant stream of income, Y, discounted at 10 percent over 10 years, simply multiply Y by 6.1445.

Thus, the PV of $1000 per year would be $6144.50. (Table 1 Column 8 gives the discounted values for $1000 for years 1 through 10 and the sum, PV of $6144.50.)

Depreciation: Constant Rate of Decline (CRD) Method

Returning to the CRD Equation (6), for T = 10, an asset value in year zero of $6144, and a scrap value in year 10 of $2368 gives: $1/(1+r)^{10} = 2368/6144$, the solution of which gives a relative rate of depreciation, r, equal to .10 or 10 percent.

Table 1 Column 7 gives the asset values for years zero through ten. Yearly depreciation using the CRD approach is shown in Table 1 Column 5. If the objective of the asset owner is to replace the asset at the end of its working life, a fund would need to be set aside for that purpose. Table 1 Column 6 gives the depreciation fund, which is simply the accumulated amount of per year depreciation. In the CRD approach to depreciation, it is assumed that interest is not earned on the fund. As can be seen, the sum of depreciation set aside for years one through ten ($3776) plus the scrap value in year ten ($2368) totals $6144, which is the value of the asset in year zero, and the amount that would be needed to replace the asset, assuming there was no change in asset price.

Decay: Survival Function for a Constant Rate of Decline

Table 1 Column 2 gives purely hypothetical values for $1/(1.10)^t$ (which is the discrete form of Equation (7) for $e^{-.0953t}$). Use of the discrete form for exposition purposes has the advantage of allowing the creation of tables that are easily interpreted. Table 1 Column 2 is based on the assumption that r, the probability of emitting a photon atom within the next nanosecond, is equal to 10 percent; therefore at t = 1/.10, or tau = 10. At time tau, the value of the discrete survival function $1/(1.10)^{10}$ should be equal or close to .368. As can be seen, the value for period 10 shown in Table 1 Column 2 is .3855. The value of .3855 can be interpreted as the 'cumulative' probability of survival in period 10 (Sachko 1975, 28-31). That is, 38.55 percent of the original atoms remain in the excited state. Because this example is based on the discrete version using r = 10 percent, this percentage is only approximately equal to 36.8 percent. It is equivalent to the value for $e^{-.0953(10)}$, which is .3855. The value for $e^{-.10(1)}$ is .3678. Table 1 Column 2 only shows survival rates for periods one through ten, but of course, the survival function is infinite so values beyond period ten would exist but are not shown.

At time $\tau = 10$, approximately 63.2 percent of the atoms will have decayed. Column 3 of Table 1 gives the values for $.10[1/(1.10)^t]$, which when multiplied by A gives the number of decayed atoms for period t. The sum of Table 1 Column 3 is .61445, indicating that 61.445 percent of the atoms have jumped to the ground state by t = 10. (The value of 61.445 is only approximately equal to 63.2 because the values in Table 1 are based on the discrete formulae.)

Death: Survival and Mortality Functions for a Single Economic Asset and the Calculation of Average Age

The analysis for the decay of atoms can also be applied to the life of an economic asset in some instances. For example, there may be some assets for whom chance or exogenous factors are of primary, if not sole, importance in the cause of early 'death.' By early death I mean the destruction of the asset before the anticipated end of economic life, assuming the asset's life is finite. If death is limited to 'chance' then the survival function for a single economic asset would be the same as that shown for atomic decay: (7) $S_t = e^{-rt}$.

And the force of mortality, r, reflecting death due to chance, would be the (absolute value of the) percentage change in the survival function.

Now, even though there may be a probability of early death due to 'chance' factors, for most economic assets it is also the case that their lives are finite. That is, if an asset managed to survive factors related to early death, it would still die at some point, denoted by T.

If the focus was limited to the life of a single economic asset that has a 10 percent probability of early death due to 'chance' and a maximum economic life of T = 10 years, then Table 1 Column 2 would reflect the discrete counterpart of Equation (7) for years one through ten. That is, the values in Table 1 Column 2 would reflect the 'cumulative' survival probabilities for an asset subject to a probability of early death, r, and a maximum economic life, T. Table 1 Column 3 gives the yearly 'death rates' associated with early death of the asset due to 'chance.'

The average or expected age of the asset is calculated in the same manner as the Present Value of an income stream. The average age is the sum of the per year survival rates for years one through T. For the asset represented by Table 1, this would be obtained by summing the entries in Column 2, the total of which is 6.145 years. This average age equation is:

(10) $1/r\ [1 - (1/(1 + r)^T]$, which in this case equals: $1/.10\ [1 - (1/1.10)^{10}] = 6.145$.[2]

In the special case for which the maximum life of the asset, T, is equal to tau, where tau was previously defined as the time period at which approximately two-thirds of a population has decayed or died off, the average age formula becomes:

(11) $T[1 - (1/(1 + r)^T]$.

Since in the case shown in Table 1 Column 2, T = 10 is also equal to 1/r = 1/.10, the average age can be calculated using Equation (11). (It should be noted that because life is finite at age T, time tau no longer represents the time at which the asset has approximately 37 percent of its life remaining.)

Table 1 Columns 2 and 3 and give the survival and 'death' probabilities due to 'chance' for the asset. In year one, the sum of the survival probability in Table 1 Column 2 and the (early) 'death' probability shown in Table 1 Column 3 equal one (.9091 + .0909 = 1), as they must since: $1/(1+r) + r[1/(1+r)] = 1$.

Table 1 Column 4 gives the cumulative probabilities of dying due to early death (chance), which are the cumulative values per year of the entries shown in Table 1 Column 3. For each year of the asset's life, the per year sum of the cumulative probability of survival (Table 1 Column 2) plus the cumulative probability of death due to chance (Table 1 Column 4) equals one. For example, for year 2 the sum of .8264 plus .1736 equals one. It is also the case that sum of the bottom line figures for Table 1 Column 2 (6.1445) and Column 4 (3.8555) equals the maximum life of 10 years.

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[2] My usage of Present Value in year one (PV_1) as average age corresponds to the general definition given to the term 'average life' used by Preinreich (1938) as well as Kurtz (1930). For years subsequent to year one (PV_t, $t > 1$), I use the expression 'life expectancy,' which corresponds to Preinreich's (1938, 220) and Kurtz's (1930, 142) general definition of that term.

Discount and Negative Growth

The examples I have presented thus far have considerable overlap. I have used the same (or nearly the same) numerical values in order to emphasize the similarities. However, there are also marked differences as well which I will highlight below. One of the differences hinges on the difference between discount and decay.

In a section of his text on Mathematical Economics entitled 'discounting and negative growth,' Alpha C. Chiang refers to the continuous-discounting formula as an exponential growth function, where minus r is the instantaneous rate of growth. 'Being negative, this rate is sometimes referred to as a rate of decay. Just as interest compounding exemplifies the process of growth, *discounting illustrates negative growth'* (Chiang 1974, 293; italics added).

Decay: Force of Mortality, Depreciation Rate, and Discount Rate

What decay means depends on context. In the case of the decay of energized atoms, it can be thought of as the rate of reduction or 'death' of charged atoms over time. For illustrative purposes, I have used the discrete form of the decay equation. As a result of multiplying a given population at time period zero by the values in Table 1 Column 2 (which end at t = 10, but would continue indefinitely), it can be seen that the population dwindles over time by the amounts equal to the entries in Table 1 Column 3 multiplied the population at time zero. The rate r is the rate of decay or the rate of death (force of mortality) of the population.

The decay equation can reflect the decline in a population due to death, but when it is applied to a single asset that declines in value over time using the CRD method, the diminution is depreciation not death, where depreciation is the reduction in the value of the asset as it ages. The values in Table 1 Column 2 multiplied by the capital value in year zero gives the asset value per year based on the CRD approach. The entries in Table 1 Column 3 multiplied by the initial capital value gives the depreciation per year. The rate r is now the rate of depreciation. Unlike a population of atoms, an asset is assumed to have a finite life, T. Also, with the CRD method a scrap value at time T must be given in order to solve for the deprecation rate, r. Using depreciation of an asset as an example of decay, an amount of $6144 will have depreciated to a value of $6144 $(1/1.10)^{10}$ = $2369 (rounding error) by year ten (see Table 1 Column 7 Years 0 and 10, respectively).

In contrast to the CRD approach to depreciation, the discounting equation is used to reflect what a value to be received in the future is worth today. So an amount of, say $6144, to be received in 10 years would be worth $2368 today at a 10 percent discount rate; that is, 6144(1/1.10)^{10}$ = $2368. As a depreciation rate, r indicates the percentage by which $6144 decays over time using the CRD method. As a rate of discount, r is used to calculate how much an amount of $6144 to be received in the future is valued today. The discounted amount of $6144 to be received in the future does not decay over time. The amount of $6144 is discounted because it has less value today since it will not be received until 10 years hence.

In the discounting case, $6144 is to be received in 10 years and is valued at $2368 today, and in the depreciation case an asset valued today at $6144 will have a value of $2368 by year 10. The discount of an amount equal to $6144 to be received in the future is a very different problem from the depreciation of an asset valued at $6144 even though the quote from Chiang above does not make any distinction between discounting and decay.

Asset Value: CRD and PV Approaches

An asset today may be worth \$6144.50. What the Present Value formulation relates is how that asset may have acquired a value equal to \$6144.50. In economic theory, it is assumed that an economic asset's value is based on its earning power—how much income it generates over time, for instance. This 'income' may not be explicit but imputed based on the value of the services provided by the good over time to its owner. If an asset has a (net) income earning potential of \$1000 per year for 10 years, then at a 10 percent discount rate, it will be worth \$6144.50 today. (This may not be the purchase price of the asset, but the purchase price will tend to reflect the income earning potential based on its PV.)

Here r is a discount rate, the basis for which is an interest rate. The PV per year uses the sum of $1/(1.10)^t$ from year 1 through year T (where T now varies from 10 to 1) multiplied by the income of \$1000 (assuming income is constant), where r = 10 percent.[3]

In contrast with the PV approach to asset determination over time, the CRD approach multiplies the value of the asset at time zero by $1/(1+r)^t$. With both approaches to asset value, the PV method and the CRD approach, depreciation is defined as the difference between the values of the asset from one year to the next. However, with the CRD approach, annual depreciation is $r(1/1+r)^t$ multiplied by the asset value in year zero, where r is the depreciation rate. With the PV approach, r is the discount rate and yearly depreciation is:

$dep = PV(r)/[(1 + r)^T - 1]$ where T varies from T to 1.[4]

With the PV approach, r is the interest rate and the interest earned on PV per year is rPVt (shown in Table 1 Column 11). The sum per year of interest (Column 11) plus depreciation (Column 10) equals the return or income (\$1000) (I. N. Fisher [1906] (1965), 237). In comparison, using the CRD approach the only 'return' per year is depreciation (Column 5). No interest is earned on the asset (Simpson et al. 1951, 269).

It is also assumed under the CRD approach that no interest is earned on the depreciation fund if a fund is set aside. In contrast, the PV approach usually assumes that interest can be earned on a fund, called a Sinking Fund (SF), to replace the value of the asset at time T. The SF amount is a constant per year figure that is equal in value to depreciation in year one. If the interest earned on the SF is the same as that used to discount PV, then the yearly SF amount plus accumulated interest will be equal to depreciation in year t. Consequently, the accumulated SF with interest by year T will equal the Present Value of the asset in year one. For the PV given in Table 1 Column 8 (total), the SF amount would be \$385.55 per year, which at 10 percent interest would equal \$6144.50 by year T = 10 (Allen 1967, 240).

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[3] Present value (PV) of \$1 is the sum of $[1/(1+r)^t]$ from t = 1 through t = T. For values of PV_t, T varies from T to 1. In other words, if T = 10, then for PV_1, the summation is from t = 1 through T = 10; for PV_2, the summation is from t = 1 through T = 9; and so on up to PV_{10}, where the 'summation' is for T = 1 only.

[4] For example, for T = 10 for a given Present Value, say PV_1, the depreciation in year one would be calculated as $dep_1 = PV_1 (r)/[(1 + r)^{10} - 1]$; for year two depreciation would be $dep_2 = PV_2 (r)/[(1 + r)^9 - 1]$; and so on until year ten when depreciation would equal $dep_{10} = PV_{10}(r)/[(1 + r)^1 - 1]$.

Table 1.

Tabular Content for Examples of the Discrete Form of the Exponential Equation

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	
t	$1/(1.10)^t$	$(.10)$x $1/(1.10)^t$	$\Sigma(.10)$x $1/(1.10)^t$	Dep_t 6144 x Col 3	Dep Fund	Value= 6144 x Col 2	1000 x Col 2	PV_t	Dep_t ΔCol 9	Int .10 x Col 9	
0						6144					
1	.9091	.09091	.0909	558	558	5586	909.1	6144.5	385.5	614.5	
2	.8264	.08264	.1736	508	1066	5078	826.4	5759.0	424.1	575.9	
3	.7513	.07513	.2487	462	1528	4616	751.3	5334.9	466.5	533.5	
4	.6830	.06830	.3170	420	1948	4196	683.0	4868.4	513.2	486.8	
5	.6209	.06290	.3791	381	2329	3815	629.0	4355.3	564.5	435.5	
6	.5645	.05645	.4355	347	2676	3468	564.5	3790.8	620.9	379.1	
7	.5132	.05132	.4868	315	2991	3153	513.2	3169.9	683.0	317.0	
8	.4665	.04665	.5335	287	3278	2866	466.5	2486.9	751.3	248.7	
9	.4241	.04241	.5759	261	3539	2605	424.1	1735.5	826.4	173.6	
10	.3855	.03855	.6145	237	3776	2369	385.5	909.1	909.1	90.9	
	6.1445	.61445	3.8555	3776	+2368		6144.5	6144.5	6144.5	3855.5	3855.5
				+2368 = 6144	6144						

Brief Overview of the Poisson Distribution

The discussion thus far has focused on the exponential equation or its discrete counterpart. There has been no mention of the exponential distribution per se. The next section will give a brief overview of the exponential distribution. First the Poisson distribution will be introduced since it is from the Poisson process that the exponential model is often generated (Chou 1969, 215).

In discussing probability a starting point is to hypothesize an experiment in which there are trials for which there is random selection and more than one outcome is possible. For example, a binomial experiment measures the outcome of a specified number of randomized trials in which two outcomes are possible. The binomial case takes into account the number of occurrences as well as non-occurrences of an event.

In contrast, for the Poisson model, the focus is on simply a count of the number of occurrences of an event for a specified unit of time, not its non-occurrences (Chou 1969, 205). For instance, for a gas station the interest may be in the number of customers who arrive per time period, but not the number who do not arrive.

For the Poisson distribution the interest is in a random variable, X, that represents the number of occurrences of an event within a fixed time. It is a discrete distribution in which X can take the values 0, 1, 2, 3, . . . The mean of the distribution, λ, is constant. The average number of occurrences for a specified unit of time t is λt (Chou 1969, 206; Hillier and Lieberman 1967, 294).

The probability of exactly x occurrences for time period t is

(12) $p(x; \lambda t) = e^{-\lambda t} (\lambda t)^x / x!$ (Chou 1969, 205).

Connection of the Poisson to the Exponential Distribution

The Poisson distribution is discrete and is used to calculate probabilities related to the number of occurrences of an event within a fixed unit of time. The exponential distribution is continuous and is used to examine probabilities related to amount of time between successive Poisson occurrences (Anderson et al. 1996, 84; Chou 1969, 215).

The mean number of occurrences in a Poisson process is the constant rate, λ, per unit of time. The mean for the exponential distribution is the inter-arrival time 1/λ (Chu 2003, 179). For the Poisson distribution, the mean and variance have the same value. For the exponential, the mean and the standard deviation are the same. The mean of the exponential distribution is the average amount of time between Poisson occurrences or '*the expected time until the first occurrence of the event*' (Chou 1969, 216; italics added). In the previous sentence, the first interpretation of the mean of the exponential distribution can be viewed as applying more to arrival problems, and the second to survival problems.

The exponential model is said to have the property of being memoryless. As this relates to problems of survival, this means that given that some entity has survived until t, the probability that it will survive s additional periods or more (that is, that it will survive at least t plus s periods) is the same as the initial probability that the entity will survive for s or more periods. That is to say that given that an entity is alive at t, the distribution for its remaining life is no different from the original distribution for its life; it has no memory of having been alive until time t (Ross 1997, 237). As it so happens the exponential distribution is the only distribution with the memoryless property. That is,

$e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$ (Ross 1997, 237).

In the exponential distribution, T is the random variable representing the time between events or equivalently the time of the first event (Hiller and Lieberman 1967, 295) for a given λ. For T>t for a specific amount of time t, the expression T>t means an event has not yet happened within the time period (0,t) (Chou 1969, 215). The probability that there is no occurrence of an event between time zero and time t is given by the probability that T is greater than t:

(13) $P(T > t) = P(\text{zero occurrence in } (0,t)) = e^{-\lambda t} (\lambda t)^0/0! = e^{-\lambda t}$

Equation (13) is the mathematical link between the exponential and Poisson distributions. Equation (13) is the Poisson Equation (12) with the number of occurrences, x, set equal to zero. For the survival model, Equation (13) gives the probability that the remaining life exceeds the interval 0 to t.

The probability that the first event (arrival or death) will take place within the interval 0 to t is given by the cumulative distribution function of the exponential variable T:

(14) $T(t) = P(T \le t) = 1 - P(T > t) = 1 - e^{-\lambda t}$, (where $\int_0^t \lambda e^{-\lambda t} = 1 - e^{-\lambda t}$) (Ross 1997, 34, 239; Hiller and Lieberman 1967, 295).

That is, the probability that the random variable T is less than or equal to a specified time t is given by the area for the exponential density function from zero through t; where the density function of the exponential variable is the derivative of Equation (14) and is given as follows:

(15) $t(t) = \lambda e^{-\lambda t}$ (Ross 1997, 238; Chou 1969, 216; Anderson et al. 1996, 82-83).

For the survival model, the failure rate or 'hazard rate' of the exponential distribution further demonstrates its memoryless property. The failure rate, r(t), is the conditional probability density that an entity will not survive an additional time dt given that it has already survived to time t (from 0 to t) and is equal to the instantaneous rate of change, shown below as Equation (15) divided by Equation (13), that is:

(16) $r(t) = \lambda e^{-\lambda t} / e^{-\lambda t} = \lambda$ (Ross 1997, 240).

The failure rate is the probability than at entity that is t periods of age will 'fail,' and it is identical to the probability that a new entity will fail. That is, r(t) is constant and independent of time and is equal to λ.

The failure rate, λ, is the reciprocal of the mean of the exponential distribution which is given by:

(17) $E(T) = \int_0^\infty t(\lambda)\, e^{-\lambda t}\, dt = 1/\lambda$.

The mean, 1/λ, is the expected time between arrivals. When Equation (13) is interpreted as the 'survival equation,' 1/λ is the expected length of life; that is the expected time between 'deaths' is the average age 1/λ (Ross 1997, 240).

Light Bulb Failure: A Survival Problem

A typical problem in a quantitative methods or decision theory text might be as follows:

Suppose that the amount of time that a light bulb works before burning itself out is exponentially distributed with mean ten hours. Suppose that a person enters a room in which a light bulb is burning. If this person desires to work for five hours, then what is the probability that he will be able to complete his work without the bulb burning out? (Ross 1997, 239)

This problem is about the survival of a light bulb, so it is analogous to the atomic decay problem as well as the survival of a given population based on a constant rate of decline. The mean time the light bulb will burn before extinguishing is 10 hours, which equates to an average life of 10 hours, where $\lambda = .10$ is the mean of the Poisson distribution representing .10 'arrivals' per hour. That is, the mean of the exponential distribution is $1/\lambda = 10$ hours between 'arrivals,' where an arrival is the burning out of a light bulb.

The probability that a light bulb will 'live' more than t hours is given by Equation (13) for $\lambda = .10$:

(13) $P(T > t) = P(\text{zero occurrence in } (0,t)) = e^{-\lambda t} = e^{-.10t}$

Therefore the probability that a light bulb will remain unextinguished for 5 hours or more would be

$P(T > 5) = P(\text{zero occurrence in } (0,5)) = e^{-.10(5)} = .6065$, or 60.65 percent.

The probability that the light bulb will fail between 0 and 5 hours is given by equation (14) for $t = 5$:

(14) $P(T \leq 5) = 1 - P(T > 5) = 1 - e^{-.10(5)} = .3935$.

In other words, given that the average life of a bulb is 10 hours, the probability is 39.35 percent that the time between deaths is between 0 and 5 hours.

The probability of dying (extinguishing) between the fifth and sixth hour is given by Equation (15) for $t = 5$ and $\lambda = .10$:

(15) $.10e^{-.10(5)} = .06065$, or 6.065 percent.

The instantaneous rate of failure, r(t), or the 'hazard rate' is given by Equation (16):

$r(t) = .10e^{-.10t} / e^{-.10t} = .10 = \lambda$.

An Arrival Problem

The exponential distribution can also be applied to problems in which customers arrive at a service destination at random times (Hillier, and Lieberman 1967, 38). For instance, suppose that the average number of customers arriving at a gas station is 3.2 per hour. That is, the mean of the Poisson distribution would be $\lambda = 3.2$. This translates into an arrival time between customers of one every $1/\lambda = .3125$ hours, the mean of the exponential distribution.

The probability that there will be no arrivals within the first hour is given by

$P(T > 1) = P(\text{zero occurrence in } (0,t)) = e^{-3.2(1)} = .04076$, or 4.076 percent,

which is the probability that there will be an arrival after the first hour.

The probability that there will be an arrival in t = one hour or less would be

$P(T \le 1) = 1 - P(T > 1) = 1 - e^{-3.2(1)} = (1 - .04076) = .95924$, or 95.924 percent.

The density function for this problem is: $3.2\ e^{-3.2t}$

The probability that there will be an arrival between the first and second hour is:

$3.2\ e^{-3.2(1)} = 3.2(.04076) = .1304$, or 13.04 percent.

The value for $3.2\ e^{-3.2t}$ for t = one is typically interpreted as the extending over the first period. It has just been shown that the 'change' during the first hour is equal to the area $[1 - e^{-3.2(1)}] = .95924$, not .1304. The problem with this interpretation is that the equation applies for very small changes in t (or around t). Whether the change is from zero to one or one to two, in either case, the resulting value will be off. I have chosen here (and for consistency elsewhere as well) to assume that the change is forward. This is because the survival probability on which $\lambda e^{-\lambda t}$ is based, that is, $e^{-\lambda t}$, is defined for $(T > t)$.

The Survival and Arrival Problems: The Intercept

The intercept of the density function for the arrival problem is the value for $3.2\ e^{-3.2t}$ at t = zero, $3.2\ e^{-3.2(0)} = 3.2$.

The area under the density function is equal to one:

$\int_0^T \lambda\ e^{-\lambda t}\, dt = (1 - e^{-\lambda T})$ and $\lim_{T\to\infty} (1 - e^{-\lambda t}) = 1$ (Ross 1997, 34).

The discrete counterpart to integration is summation. It may seem curious that the 'sum' of $\lambda e^{-\lambda t}$ is equal to one, with an intercept equal to 3.2. Of course, for the integral the value at t = 0 is deducted in the 'summation.' (In fact, the value of $3.2\ e^{-3.2t}$ doesn't equal one until t = .36348 hours or approximately 22 minutes.)

The idea that the integral of $\lambda e^{-\lambda t}$ is equal to 1 as t tends to infinity seems more intuitively understandable when $e^{-\lambda t}$ is interpreted as a survival function. When $e^{-\lambda t}$ is seen as a survival function, then λ is the force of mortality. The value of the constant, λ, must be less than one since a thing cannot be more than 100 percent dead. It is also the case that the sum of all the death rates over time for a given population must equal one. The exponential equation is infinite. It would seem to follow that the 'sum' of the death rates, $\lambda e^{-\lambda t}$, would equal one as t tends to infinity. This is more easily visualized using the discrete form of the equation. For example, it can be seen in Table 1 that the sum of the 'death rates' shown in Column 3 sum to .61445 by year 10. In this instance, year 10 is the same as tau = 1/.10. Therefore, 38.55 percent of the population remains alive at that point. At year 15, 76.06 percent of the population has died off and 23.94 percent of the population would remain alive. Extending the values beyond year 15 would show 'death rates' that continue to become smaller and smaller.

When λ is interpreted as a 'force of mortality' it is typically less than one and viewed as a percentage. In the arrival problem considered here, the average number of arrivals is 3.2 units.

Sum of the 'Survival Rate' and 'Death Rate' in Period One

As was seen earlier for the discrete version: $1/(1 + r) + r[1/(1+r)] = 1$.

For the continuous case, it can be seen that when $t = 1$: $e^{-\lambda(1)} + \lambda e^{-\lambda(1)} = 1$ when

$e^{-\lambda(1)} = 1/(1+\lambda)$,

and that is only approximately the case for low values of λ.

For example, for the continuous case for, say, the light bulb example (a survival problem), in time period one the sum is very close to one:

$e^{-\lambda t} + \lambda e^{-\lambda t} = e^{-.10} + .10e^{-.10} = .9953 \approx 1$.

However for the arrival problem, with $\lambda = 3.2$ customers per hour, summation for the t = one is $e^{-\lambda t} + \lambda e^{-\lambda t} = e^{-3.2} + 3.2e^{-3.2} = .1712$.

The numerical disparity between the sums (that is, .9953 and .1712) can in part be lessened by a change in the units of time measurement for the arrival problem. For instance, a customer arrival rate of $\lambda = 3.2$ per hour translates into $\lambda = .053$ arrivals per minute, for an exponential mean time between arrivals of $1/.053 = 18.87$ minutes. With time measured in minutes, the sum of $e^{-\lambda t} + \lambda e^{-\lambda t}$ at $t = 1$ is very close to one:

$e^{-\lambda t} + \lambda e^{-\lambda t} = e^{-.053} + .053e^{-.053} = .9983$.

For the discrete version of the survival equation, it is also the case that the 'death rate' at period one is equal to one minus the survival function at $t = 1$. That is,

$r[1/(1+r)] = 1 - [1/(1 + r)]$.

But again for the continuous case for $t = 1$: $\lambda e^{-\lambda} = (1 - e^{-\lambda})$ when $e^{-\lambda} = 1/(1 + \lambda)$,

which is only approximately true when λ is small in value.

(For the discrete case, $\sum_1^t r[1/(1+r)^t] = 1 - [1/(1 + r)^t]$, and for the continuous case: $\int_0^t \lambda e^{-\lambda t} = 1 - e^{-\lambda t}$.)

As a point of comparison, for the arrival problem when time was expressed in hours the two values, that is, $(1 - e^{-\lambda})$ and $\lambda e^{-\lambda}$, were not even close as shown below:

$P(T \leq 1) = 1 - P(T > 1) = 1 - e^{-3.2(1)} = (1 - .04076) = .95924$, and

$3.2\, e^{-3.2(1)} = 3.2(.04076) = .1304$.

However, once the arrival problem is expressed in terms of minutes, the probability of no arrival within the first minute would be given by

$P(T > 1) = P(\text{zero occurrence in } (0,t)) = e^{-.053(1)} = .9481$,

or a 94.81 percent probability of an arrival after the first minute.

The probability that there will be an arrival in t = one minute or less would be

$P(T \leq 1) = 1 - P(T > 1) = 1 - e^{-.053(1)} = (1 - .9481) = .0519.$

The value of .0519 is very close to $\lambda e^{-\lambda} = .053e^{-.053} = .0502$.

Discussion

The Survival Problem and a Change in Time Units

In the Results section, some of the apparent differences between the arrival problem and the survival problem were highlighted by using the discrete form of the exponential function. These apparent differences were shown to be overcome by a change in the time units in which the arrival problem was measured.

The units of time measurement can also be changed for the survival problem. To illustrate this, I will return to an example given earlier in which the survival function was assumed to have a force of mortality, r (which to be consistent with the notation used for the exponential distribution discussion will be replaced by λ) equal to 10 percent. In the earlier example, the discrete version was used (see Table 1 Column 2). If the continuous version of the survival function is used, then assuming an infinite life, the exponential mean time between deaths would be 10 years, or more appropriately, the average life would be 10 years. The sum of $e^{-\lambda t} + \lambda e^{-\lambda}$ is equal to .9953 (which incidentally is equal in value to the sum for the light bulb problem).

Now suppose instead of years, months were used for the unit of time. The value for λ would now be 1/120 = .0083. The survival probability of not dying the first month (that is, the probability of living more than one month) would be:

$e^{-.0083(1)} = .9917$, and the 'death rate' between month one and month two would be:

$.0083e^{-.0083(1)} = .00823.$

Thus, the sum: $e^{-.0083} + .0083e^{-.0083} = .99993.$

Note that the 'death rate' for t= month one is .823 percent, which is lower than the 'death rate' for t = one year (equal to 9.091 percent) when a year is used as the time unit. This is because the force of mortality based a time unit of a month (.0083) is lower than the force of mortality when a year is used (.10). (The survival rate is higher when time is reckoned monthly than when time is reckoned on a yearly basis, but the lower force of mortality outweighs the higher survival rate.)

If instead of months or years, a century was used for the time unit, then λ = 10 and 1/λ = .10. Now the probability of surviving the first period (century) would be:

$e^{-10(1)} = .0000454,$

and the probability of dying ('death rate') between the 100^{th} year and 200^{th} year would be: $10e^{-10(1)} = 10(.0000454) = .000454.$

As in the case in which time was reckoned monthly instead of yearly, once again the death rate for t = 1 is lower when a century is used (.0454 percent) than when time is reckoned yearly (9.091 percent). This is the case even though the force of mortality is high (equal to 10 in the case in which t is equal to one century). One might think that the death rate at 100 years of age would be very high, but this is a probability of dying out of those who will probably survive beyond 100 years (that is, it is a joint probability reflecting the force of mortality multiplied by the survival probability for period one). Because the probability of surviving beyond year 100 is very low (.00454 percent), the death rate for the next 100 years is also very low. Thus, for t = one period (century) the sum of the two rates, ($e^{-10(1)} + 10e^{-10(1)}$), equals .000499.

Caution in Translating Time Units

As has been shown, for the survival problem with the century as the time unit instead of years, the survival equation becomes

e^{-10t} and the density function becomes 10 e^{-10t}.

For year one, t would now be equal to 1/100, so the death rate for year one would be:

$10\ e^{-10(1/100)} = 10(.9048) = 9.048$,

that is, the probability of dying from the first year to year 100 would be 904.8 percent. This is 100 times the probability of dying for year one when time is expressed in years:

$.10\ e^{-.10(1)} = .10(.9048) = .09048$, or 9.048 percent.

The density function assumes that the survival rate at time t holds throughout the period, whether the period is one year or one century. When the time unit is changed, λ also changes. If the change in the time unit results in λ greater than one, the interpretation of λ becomes problematic for survival problems. (When the time unit is changed to one century, λ becomes 904.8 percent; of course, on a yearly basis, λ is still 9.048 percent.)

In the arrival problem, λ is expressed as a count. For the survival problem λ is a percentage. (This is not to say that the rate in survival problem cannot be expressed in units; it is just not the case under consideration here.) It must be multiplied by the survival function to arrive at a 'death rate,' which in turn must be multiplied by a population figure to obtain a count (that is, a count of the 'death rate' in units at t).

When the survival problem concerns a population of energized atoms, λ is the proportion of the population that is expected to decay per period (which is interpreted to mean the probability that a member of the population taken at random will decay). If the survival problem is about the life of a light bulb, then λ is the probability that the bulb will extinguish per period. It is not the proportion of the bulb that will die, since for a single unit, it will either die or survive.

If the survival equation is applied to a single entity (a light bulb, for example) a value of λ that exceeds 1 is nonsensical because a single entity cannot be more than 100 percent dead. Similarly, if the survival equation (times λ) is multiplied by a given population, λ greater than one is meaningless because a population can only die off once; it cannot die by a multiple of itself.

Unlike the survival problem, for the arrival problem the value of $\lambda e^{-\lambda t}$ is not multiplied by a population at time zero (whether that population is a single light bulb or a population of energized atoms) to reflect a dwindling population over time, therefore λ may assume values greater than 1.

The Choice of the Time Unit

Earlier, the value of tau = 1/r was defined as the lifetime of the excited state of an atom. In general tau = 1/r is the time at which 63.22 percent of an entity or a population has decayed or died off. In terms of years, for r = .10, tau is equal to 10 years. In terms of centuries, for r = 10, tau is .10 centuries. In either case, the value of the survival function at t = tau is

in years: $e^{-.10(10)}$ = .3678; and in centuries: $e^{-10(.10)}$ = .3678; indicating 36.78 percent of the life is remaining.

In this case, the time interval should be measured in units of years or in units that are 'divisibles' of 10 years in order to result in probabilities that would be meaningful. That is, using years results in r (or λ) less than one. If the average life is 10 years, then the death rate is .10 deaths per year. Mathematically, the time interval for the survival problem can be changed to give a value of r greater than one, but that would be outside the range of probability for a given population.

Of course, the problem of determining the appropriate time unit is not unique to survival problems. Usually the time unit is fairly obvious based on the nature of the problem. A smaller (more sensitive) measurement can be aggregated. However, if a larger interval is used initially, it cannot subsequently be subdivided into smaller units.

The previous statement seems at odds with the examples shown earlier involving the translation of time units; for example, subdividing hours to minutes and years to months. This needs clarification. If in undertaking a study, an objective is to test for the accuracy of the assumption of an exponential distribution, and the time unit that is initially chosen is too large, it may not be possible to perform certain tests; the Chi-square for instance.

For example, suppose in a study examining the time between soccer goals that the number of hours between goals was used instead of, say, the number of minutes. Then in a test on the accuracy of the assumption of the exponential distribution, the researcher would run into a problem since games are 90 minutes long. One hour would not be a sensitive enough measure of time.

One such study was undertaken by Singfat Chu (2003). Chu's objective was to provide a real life case to test the assumption of the Poisson and the exponential distributions. This assumption, according to Chu, is seldom justified in textbooks. The author noted that for most 'arrival' data, only numerical counts of events are available. Information on the sequential timing of events is required in order to perform these tests. Chu used data on World Cup tournaments for soccer to calculate time between goals for 232 games from 1990-2002. Chu calculated 2.4784 goals per 90 minute game (or .2754 per 10-minute interval for a given game), and 36.25 minutes between goals. A number of tests, including the Chi-square, were used to check the exponential and Poisson fit. A ten-minute interval was used as the time unit to test for the appropriateness of the exponential distribution (Chu 2003).

Once a unit of time is selected, it still may be necessary to combine time periods for testing purposes. For instance, the Chi-square method requires cells of five observations or more. If some cells are found to be insufficient then aggregation of time units after the fact may be necessary.

Conclusion

This paper began by reviewing a few applications for the exponential equation used primarily in business courses. The discrete version was discussed first, mainly because it is not uncommon for the business student to be introduced to the discrete version prior to the continuous equation. Some similarities and differences between the various applications of the exponential function were noted.

After a brief overview of the exponential distribution, two applications were presented similar to those typically covered in quantitative methods or decision theory texts: one 'survival' problem and an 'arrival' problem. A person accustomed to seeing the exponential equation applied to 'decay' problems, cannot help but be a bit dazzled by its use in queuing theory and by those who pioneered its use; in particular, A. K. Erland (Saaty 1957). Death, decay, and depreciation seem to have little in common with customers arriving at, say, a gas station; that is, until it is noted that death is a type of arrival (limited to one per customer).

The properties associated with the discrete version were found useful in highlighting some of the apparent dissimilarities between the arrival and survival problems. Most of these can be overcome by a change in the time unit in which the arrival problem is measured. There was also a cautionary reminder about the limitations of assigning and altering the time units in which the problems are measured.

As mentioned previously, most students have very likely had some experience with compound interest prior to entering a business course or reading a business-related textbook. This informal exposure may aid the student when a more formal presentation of the concept is encountered. Once compound interest is understood, discounting is more easily grasped since it is simply compounding in reverse. Similarly, other 'decay' applications share commonalities with discounting. Unlike compound interest, it is less likely that most students have encountered an 'arrival' problem informally; that is, prior to having taken a course or reading a text. The examples presented here show that some of the apparent differences between decay/survival problems and queuing/arrival problems are lessened by changing the time units for the arrival-type problems. The examples fill a gap that I have found lacking in the business textbook literature. They may help the student in making the transition from the understanding of survival problems to arrival problems. This is conjecture; however, I think there is merit in considering supplementary approaches to the presentation of material if for no other reason than it may open discourse among educators to other possible approaches.

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