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Mathematics in finance and economics: importance of teaching higher order mathematical thinking skills in finance

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ABSTRACT

This paper addresses the importance of teaching mathematics in business and finance schools of tertiary institutions of Australia. The paper explores the nature of thinking and reasoning required for advancement financial or economic studies involves the use of higher order thinking and creativity skills (HOTS) for teaching in mathematics classes. The paper demonstrates how various skills required in finance are related to the mathematical way of thinking and reasoning. The 2007 global crisis has focused attention on financial thinking generally and mathematics teaching of finance in particular for the lack of skills by finance personnel. This could be addressed through student learning and teaching by reshaping business schools to include well designed financial mathematics courses that are compulsory, in degree programs. The mathematical reasoning patterns, thinking, explanations, simplifications required in transfer of knowledge to students is effectively taught if experts in the field are explicating or presenting proofs. The essence of the conceptual development of models and their assumptions and solutions requires training beyond those of the non-mathematics personnel, mainly due to their lack of years of time spent on such mathematically work. If the methodology described is adhered to our future business and finance students could be afforded a better educational experience in mathematical thinking and reasoning as well as higher order thinking and creative skills in their courses and degree programs.

Keywords: Financial mathematics; applied mathematics, higher order thinking (HOTS), mathematical reasoning, finance education

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Introduction

Years of lecturing mathematics and statistics and researching student learning allow lecturers at university to gain some knowledge about the many difficulties students freshly coming into universities and adult learners coming in later years, experience while learning mathematics. The non-routine task of lecturing first year students to supervising to their doctorate completions especially from widely different disciplines allows lecturers to comprehend more deeply student learning and understanding of procedures and concepts in their respective fields. In this way, university lecturers are researchers on the job and thus gain a deeper understanding of student prior knowledge as well as their ongoing university learning experiences. It seems that mathematics lecturers see greater numbers of students from a wide variety of course programs in universities for mathematics is a compulsory course for many programs, although this point may be debatable. In the author's experience this large pool of students allow math lecturers opportunities to observe more critically different student learning behaviours as many clearly find mathematics difficult and in many cases students believe mathematics to be a useless field of study and not applicable to their immediate life or their careers more generally. The "difficulty feature of mathematics" provides lecturers opportunities to learn about student prior math experiences, their understanding of concepts, their self-beliefs; the author claims perhaps more so than other disciplines since students complain more about their mathematics learning than many other courses (Tularam, 2013). The many types and expression of concerns with mathematics often exposes students' inadequacies. The author believes that from such a rich data base of experiences, mathematics lecturers may be able to provide significant insights in the progress of tertiary students in different disciplines especially those that require mathematical insights in their applications (Roca and Tularam, 2010; Tularam, 1997, 1998, 2002, 2011, 2013; Tularam et al, 2010).

Although a mathematician, the author's case is special in that he has had added opportunities to teach mathematics to finance students for a number of years; teaching courses such as risk management, time series, forecasting and financial modelling and stochastics and Ito calculus. In addition, the supervising of higher degree postgraduate students in finance, including those graduating with doctorates in mathematical finance have allowed the author significant opportunities to observe student learning throughout their university life till achievement of their highest education. The rich experiences gained over time has allowed the author access to deeper understating of students of finance in terms of their mathematical needs; that is, in terms of their capabilities and as well as their gaps or misconceptions in learning and understanding. This research paper is a reflective analysis from a rich data base of research experiences that may provide possible ways in which gaps in student learning may be filled in finance disciplines so that students may with understanding apply various mathematically based financial tools. Although such an analysis can be undertaken for other disciplines, this paper is focused on finance, and the analysis and research are qualitative, and based on a reflective analysis of student interviews and documented comments, A critical analysis is also presented regarding higher order mathematical thinking and its importance in financial studies.

The first part of this paper presents incoming student reflections from those thinking of taking up finance as a degree course followed by analysis of student experiences regarding their learning at the undergraduate and higher degree levels. A brief historical perspective is then presented on the influence of mathematics both in procedural and the more-deeper critical finance issues that itself has initiated a live and productive research area for mathematicians and in some cases propagated real advancements in higher mathematics itself. In addition to others, the case of Black-Scholes is presented as an example for the use of higher order critical mathematical thinking in finance. In the end, the in depth analysis shows that mathematics is such in grained in finance that finance students ought to learn the mathematics in a manner that shows deeper understanding of mathematics including assumptions and simplification issues considered in them.

Mathematics in finance should be understood by the students rather than be taught in an ad hoc “patch up routine exercise” manner that is highlighted in the reflections of many active workers in the field even when they do make a valid point. The analyses show that the manner of teaching and approach generally does not lead to the learning of the higher order thinking skills (HOTS) so desperately needed in advanced financial thinking particularly in the times of global crises. Using mathematical procedures in finance, this paper shows the nature and levels of thinking skills students require to cope with higher financial learning and decision making in financial studies; all of which can be exposed and taught in a facilitating manner to students only in “well-designed financial mathematics courses” that is taught by mathematicians who are trained in both pure mathematics and applications rather than by solely finance personnel who may have taken some aspects of mathematics as application tools in some courses and in higher studies in finance or related studies.

Har’s (2011) definition is a good start to understanding the nature of mathematical thinking - “A good critical thinker (a) raises vital questions and problems, formulates them clearly and precisely; (b) gathers and assesses relevant information, using abstract ideas to interpret it effectively comes to well-reasoned conclusions and solutions, testing them against relevant criteria and standards; (c) thinks open-mindedly within alternative systems of thought, recognizing and assessing, as need be, their assumptions, implications, and practical consequences; and (d) communicates effectively figuring out solutions to complex problems” with supporting evidence and proofs (Har, 2011; Tularam, 1997, 1998; Tularam and Kelson, 1998)

The question of finance education

The importance of financial knowledge in everyday living let alone advanced complex applications in the real world of business and accounting cannot be doubted. As a student at the University of Wisconsin stated that “the most useful major that a young person can choose today for a career in the future is a finance degree. A finance degree will allow a person to have the knowledge on how to effectively control their finances, give them vast employment opportunities, and allow them to contribute to individuals and the economy”. One may argue that the situation is now changed because of the Global Financial Crisis (GFC), yet the author would argue otherwise since crises are the points at which revolutions in thinking take place and new way of thinking is often generated or proposed. The knowledge to make decisions about debt, investment, and future retirement plans are all important basic tools necessarily in real life. Finance degree students will be able to make critical decisions about stock market, mortgages, and investment opportunities. Clearly, the financial knowledge and tools acquired at universities will help bring real life examples to workforce making Australian workplace a smarter one.

Although the new finance majors may be worried about employment it is still noted that the field of finance is one of the fastest growing occupations. There is a demand for the best with higher starting salary and excellent benefits as noted in the CEO salaries around the world. Stock markets will not go away any time soon and people with money will want to improve their wealth positions so finance majors will have an edge because they work with different aspects of the economy on a daily basis. It is important to realize the responsibility placed on finance majors in real life is great when they are given the opportunity to guide individuals and businesses to possible investment decisions with goal to provide for their best possible wealth positions. The recent crisis is the impetus and an opportunity for financial departments to reshape the teaching of incoming students by making certain they leave universities with the best tools in finance. There is a need for our students to forge a new way of thinking in mathematical finance that is desperately needed rather than being amused by debates about blame.

The main aim of this paper is to highlight the ingrained nature of mathematics in finance. A second aim is to show how various skills needed in finance are related to mathematical way of thinking and as such have to be taught by those in with a high level of mathematics training such as mathematicians who have undergone years of problem solving studies. Finally, it is proposed that student learning and teaching needs to be reshaped to include well designed financial mathematics courses that are compulsory yearly in finance and business degree programs. The first section of this paper presents an analysis of prospective students in search of information regarding finance degrees. The prospective students inquiring about possible fields of study they may undertake at universities. Out of many only some examples are presented. The replies by members of finance community who are “actually working in the field” are then analysed. Such an analysis provides a perspective that is often missed in other quantitative based research. This section is followed by a brief view on mathematicians and history of mathematical finance thus providing a background to the importance of mathematics to finance. The definition and examples of various higher order mathematical thinking and their link to mathematics and financial studies is then presented. The paper concludes by demonstrating the importance of mathematical higher order thinking skills with examples followed by a discussion of the future of mathematical thinking in finance.
A real life qualitative analysis

Case Study 1: Information gathering by a prospective finance student asking for advice from field workers in finance

I am starting school as early as this Fall. I'd like to make a decision on a major before I even sign up. I am trying to think of things I'm good at and enjoy. I'm basically on the fence with majoring in Social Work, where I would be working either in a VA, a general hospital, community clinic or counseling center, etc. Pay would be around $40k a year after about two years (on average in this area), at which point I could get my Master in just a year. Then there's Finance major. I enjoy working with my finances. I think I'm pretty good at it. I'm pretty sure I could do it all day, five days a week, for the next 30 years. However...I dropped out of High School, having only completed Algebra 1.5. I immediately got my GED after dropping out, and scored a 540 of 800 on my math. The lowest of all my scores...I checked Boise State University's registrar, and it appears that only two math courses are required for the Bachelors (not sure about Masters), and you have a choice of two for each of the two slots. Math 143 College Algebra and Math 160 look like I may be able to grasp them well. However, it still scares me a bit. Math is and always has been my weakest subject. However, I wasn't one to study in High School. I didn't apply myself well. And I even took my GED without studying. So maybe I'm not as weak with math as I think? Maybe it's simply one of the two subjects that challenge me (the other being science, mainly because I don't believe a lot of it). What do you guys think? Other than speaking with my university's advisors, how else could I figure out if this is a good field for me to pursue? Thanks so much everyone!

Out of many responses, one of the answers by a member of the finance field:
You really ought to be good at math to do finance. Being strong at math made finance pretty easy for me when I got my MBA. However, it really depends on what you are doing with your finance degree. A lot of the jobs are pretty routine. Run the same spreadsheet, plug in new numbers and analyze the output. Mortgages are like that. Almost everything is automated and there really isn't much new. Also anything to do with investing other people's money doesn't require you to be Einstein. There are plenty of quantitative people in the background that crunch the numbers. Lots of finance positions are really just sales. If it is only two course, I'd look into what they cover and just brush up on them before you take the classes. Having said all of this, if

you are really good at math, it's easy to do many different jobs. You never know might have been a bad high school teacher. Good luck.
Case Study 2: “I am a high school graduate. I am thinking of majoring in finance but do I need to do well in math, mainly calculus? …and do I need to know economics as well? ...accounting skills is a must in finance I presume? …and does it pay good?”

One of the many answers chosen as the best by the candidate was:

I've been where you've been so I want you to take this to heart and know that it's the best answer you will receive! The only kinds of Math you will ever need in your lifetime no matter what career you go into will be just the basics! PERIOD! Unless you're going into some kind of design field, then you'll need a lot more Math than that. Other than that, you'll only need to know how to add, subtract, multiply, and divide! That's it! The rest of the Math classes that other institutions of higher learning decide to give you for a degree are irrevelant and they will be once you graduate! For a career in Finance, you'll need to take a Corporate Finance course. It takes you over all of the types of Math and equations that you'll need for a career in Finance. Since this type of Math points towards your career, it's good to take. You'll also need Accounting, which will be great for you once you master (or pass) it. You'll be taught about different types of accounting for the three types of business offered: sole proprietorships, partnerships, and corporations. Do not worry about classes like calculus, physics, or trigonometry. They won't matter to you years from now and you'll never use them in the real world. These classes are bound to give you headaches and plenty of fits, especially if you don't know what you're doing and you're forced to find someone to help you who doesn't know what they're doing either! I never needed calculus, but trigonometry gave me so much hell I was forced to take it at another college that was real lenient on me when it came to this type of Math. That's what you want to always look for: leniency. Not everyone is a genius when it comes to Math. Some need a lot more help than others. If you are forced to take any high-end Math classes like the ones I mentioned, then this is something you want to keep in mind: You don't need to know everything about that type of Math, all you need is a moderate amount, just enough to get you by!

A deep qualitative analysis of the two replies can be undertaken using interview analysis techniques that involve categorising of comments that can be done either in a word by word or sentence-wise manner. In the end however such an analysis can be revealing and can help generate hypotheses that can be tested. Essentially, in both cases, there is a search for knowledge by the inquirer. The first reply provides sound information of how important an in depth understanding of mathematics is in finance generally when the person says, “Having said all of this, if you are really good at math, it's easy to do many different jobs.”. Clearly mathematics is critically important when the goal is to have an overall understanding in the finance area. Yet in the second case, the replier places the mathematics in a different context. The reply makes a little mockery of much of mathematics to be learned in general. An important comment was: “The rest of the Math classes that other institutions of higher learning decide to give you for a degree are irrevelant and they will be once you graduate!” that shows that the advice is indeed incorrect even when this person perceives it to be true in real life in the finance field. Another interesting comment is also important and one to be considered seriously by finance departments is: “You don't need to know everything about that type of Math, all you need is a moderate amount, just enough to get you by!” This is a rather disturbing even though there are many more similar comments made in the reply. “Just enough to get you by!” is important and there is no doubt that this is true in many real cases but the danger is also “very” real. The inquirer or replier are persons who may indeed one day become a CEO; the manager or director or a head of a major company in charge of making critical finance decisions. This is not an unlikely scenario as the readers can provide examples of similar individuals in the field during the crises. Often such individuals feel the same way about the mathematics as the comments made by the replier quoted. As to how much mathematics the CEO's or managers really do understand one cannot say but the author would be fairly certain to say many would be lacking in their understanding of critically important aspects of models etc. What is more important is such prospective director or CEO will then go ahead and make the critical decisions about the activities of large fund management institutions and their investment possibilities (see Wall Street Greats or high flyers of the Past or read any risk management books on crashes of public companies).

The more important issue that is not highlighted in the above exchange is the processes involved in the learning of mathematics. In the cases quoted, the acquisition of mathematical procedures or low order skills has been highlighted but more critically important mathematics thinking skills has been ignored. The acquiring of higher order thinking skills can be readily gained through a "proper" mathematics learning program. The pursuit of mathematical knowledge forces individuals to acquire what is termed higher order thinking creative skills that are largely reflective and abstract thinking based. The processes of learning such skills will be "by passed" if a routine learning or teaching approach is undertaken; that is, if Case 2 guidance quoted earlier is taken then while student being proficient in procedural work he or she will generally lack application and general problem solving ability. This can attributed to the learning and teaching style undertaken; this is mainly because of critical mathematical skills referred to as the higher order thinking skills are not acquired in the context of mathematical finance learning. If such skills are not practiced routinely then students will show a lack of their use while problem solving. The important lesson to be learnt is that when financial mathematics is taught by those who have not undertaken rigorous mathematical learning over long periods of time, they will themselves lack such critical context based skills base and during their teaching processes such critical practicalities will not be passed on or highlighted. The critical mathematical problem solving skills will not be passed onto those who are in the learning phase in financial studies. Clearly, if Case 2 ill advice is taken, then the student will indeed lack the learning or practice of such skills mainly because their mathematics learning will be only to "pass the course" rather than undertaken seriously to become a "financial problem solver".

A question posed by a student a few years ago was to "explain the importance of Financial Mathematics" and the best voted answer was: "the most important use of financial math can be described in 2 words: minimum and maximum…minimize cost and maximize profits. Also, financial math will give you concrete numbers for inventory, best use of storage costs, compound interest and future values. I'm sure others could add to this list". While the answer appears appropriate its focus is mainly on routine concepts taught from "straight financial mathematics texts" that explains maximums and minimums and mathematics of forward and futures pricing including storage costs etc. The mathematics of risk analysis, critical assumptions in the models used or modeling of financial stock pricing are not noted; and neither is stochastical nature of price changes over time. It is however interesting that "others could add to this list" was mentioned. Essentially, the teaching of mathematics in finance appears to be focused on some specific tools rather than conceptual mathematical thinking of finance and modeling or financial problem solving. The critical concern is that financial mathematics teaching may be focused on routine procedural mathematical thinking rather the higher order creative thinking that is required for decision making and in complex analyses.

Mathematicians working in finance

It is interesting to note that often mathematicians are seen to be different from normal persons. It is believed that mathematicians are rather serious individuals with possibly "glasses" sitting at their computers day long working on abstract and rather incomprehensible or impractical math. This is true to a great extent about pure mathematicians as indeed they do study prime numbers, zeta functions and so on but if pure mathematicians did not exist much of applications that are in this world would not be possible; such as understanding atomic behaviour, explaining complex weather systems, space and inter-planet travel and so on. The applied mathematicians place a higher focus on the application of their pure mathematical knowledge as tools and models to solve real life problems. In the field of finance one of the first major applications of mathematics was shown by Black and Scholes. This application of mathematical knowledge in finance may have lost some favour over time due to various reasons but the leaders in the field realize the some major conceptual leaps were made in finance due to this new way of thinking and approach. In the past, the calculations such as the net value or a theoretical study of inflation and money supply and application of probability and statistics in risk analysis, as well as Ito calculus in stochastics of market pricing have all contributed majorly in the finance industry. New or exotic products have been developed based on the advancement and more systematic and accurate risk-management options. The advancements made by mathematicians have led to high levels of profits over the years for the business and finance industry generally.

In the recent global crisis (GFC) many have disputed the use of models and mathematics related to them. There are a number of analogies we can make but the earthquake example appears useful there. The Richter magnitude scale was developed in 1935 by Charles F. Richter of the California Institute of Technology as a mathematical device to compare the size of earthquakes. One must note that the occurrence of devastating earthquakes do not make seismic mathematics or the Richter scale useless but rather much more learning must occur concerning outliers or extreme incidents so that more knowledge about earthquake systems can be gained. Only then the new knowledge can be utilized by engineers to build stronger and earthquake resistant buildings and bridges; or even render areas uninhabitable; that is, to build away from high risk areas. This analogy is not to far-fetched and clearly may be applied to finance because the GFC may have been caused by financiers not making appropriate decisions about risks related to exotic options and allowing tranches to be given credit rating for investors to use before the recent Global Financial Crisis (GFC). Therefore, history shows that mathematicians have been working on financial problems over a long period of time providing much help to finance personnel particularly in providing proofs for their insights or hunches so that the same tools may be applied with confidence in the field.

The ingrained nature of mathematics in finance

Loius Bachelier was the first to place applied mathematics into the forefront of finance. In his dissertation he used some sophisticated mathematics in finance relating to the theory of speculation (1900). It seems that Bachelier was one of the first to study continuous-time stochastic processes discussing Brownian motion and its application to option-pricing. Option trading is now an immense financial activity and some "exotica" options in recent times that led us to the path of the GFC. Bachelier also developed the solution to a particular diffusion equation, some five years before Einstein that shows the extent and nature of his mathematical knowledge he had; The diffusion equations changed physics and chemistry learning forever and of course option pricing did as well but after the work done much later by Black and Scholes. Bachelier's work was not noted in the finance area till the mid 1950's. Up to this time it would seem the most sophisticated mathematics had been elementary discounting to calculate present values of future cash-flows in finance.

In the mid 1950's, Markowitz studied portfolio selection to maximize return by holding certain amounts of desirable stocks. Harry Markowitz studied the moments of stocks and devised a mean-variance optimisation problem in which the expected return of the portfolio was studied subject to variability of that return. Thus grew the modern portfolio theory - MPT. Linear regression was used to study risk and return of an entire portfolio. An optimization strategy was used to choose a portfolio with largest mean return subject to acceptable levels of variance in the return. William Sharpe at the same time used mathematics in determining the correlation between each stock and the market. Eventually Harry received a Nobel Prize with William Sharpe for the development of the famous capital asset pricing model in 1990. This has been the basis of much institutional investing but many have criticized the descriptive aspects of the model in recent times.

Later, Robert Merton and Paul Samuelson replaced the one-period models by continuous time, Brownian-motion models, and the quadratic utility function implicit in mean–variance optimization was replaced by more general increasing, concave utility functions. However, the great leap in sophistication came with the development of option pricing the Fisher Black and Myron Scholes who applied stochastic calculus to options. The high leverage of an option allows a buyer the right, but not the obligation, to buy something in the future at a price which is set now. In fact the CBOE (Chicago Board Options Exchange) began trading listed options in April 1973- a month before the publication of the Black and Scholes paper. By 1975, virtually all traders were valuing and hedging their option portfolios, using the Black and Scholes model and the rest is history; although today highly criticized by those who do not understand the assumptions of the mathematical model it is still used widely. In fact the authors themselves rather that attempting to make the model more real life like, desired to make money out of their success at the first level of development and as such were doomed to failure.

By the 1980s, the development of academic theory and its application was at an advanced stage with applications. Now it seems that much of the leading edge work is occurring mostly in commercial organizations. This is not unusual as links with industry often lead to research funds and then of course the dealings with the funding companies are enhanced. Harry had dealings with DAIWA, while Sharpe became a consultant. A US investment Bank employed Black while Scholes worked for a competing firm. There is presently many PhDs in number theory, quantum relativity, control theory (best from Cambridge to Stanford) working in full time research positions in major financial institutions.

More sophisticated mathematical models and derivative pricing strategies have been developing although credibility of some has been damaged. A number of authors of books such as the Black Swan (Taleb, 2010) have had much to do with this notwithstanding the crashing of many financial companies. However the most telling blow has been caused by the fall out caused by the major financial crisis of 2007 (GFC). Nonetheless there are now bodies similar to INET (Institute of New Economic Thinking) examining and helping establish more effective theories and methods to move forward with.
A brief background including the historical analysis shows that finance appears to be ingrained with mathematics. Even from the beginnings of this century mathematicians have made major contributions to the development of finance more generally. Important conceptual leaps and advances can be attributed to the use of mathematics. Given this high status of mathematics in finance, it is appropriate to comment on the teaching of the mathematics in finance and such a comment can be well explained by someone well versed in all of the above. Clearly, mathematics needs to be taught in finance degree courses using a well development “situated” program if only to understand the historical growth of financial ideas and the related calculations. Judging from the brief historical analysis it is appropriate to state that a standard to advanced level of pure and applied mathematics is required and ought to be taught in the finance degree programs.

What went wrong if the "best" mathematicians developed the models?

While the accuracy of model development and application is a lengthy process in today's computer response time framework the demands on fast delivery are being imposed on the best workers in the field. There are many examples but one of the recent rather simple but important example is examined here. In particular, one can examine case of the person who developed and designed the tranches (exotic options) in US. A highly trained young mathematician developed the so called tranches for a major company. However, he assumed the risk of default in the model to be "very low' that led to the assumption of independence of individual households in suburbs of USA. Clearly, this assumption is a first attempt to model risk and faulty as the suburbs are usually inhabited by those who are alike in a number of ways; such as rather similar income levels, socio-economic levels or class of work etc. The assumption of independence between individuals could not be applied seriously to the valuing the real life tranches but it was done. The critical factor was not taken into account in the final modeling of risk as such the tranches were given a high credit rating. Now one would think a mathematician could identify this error but the process of modeling and subsequent calculations relating to risk appears to have been a "quick fix". The developments of such options are now done in house (in large Financial Institutions) and possibly "kept under wraps" rather than being under peer review or undergo some in depth scrutiny. Such an in depth critique work would be based on years of analysis and research work similar to those studying for PhD's. It seems the development of the option and risk pricing models have been a rather quick fix for a major company willing to pay large funds for the development of such exotic options or investment derivatives. Such a short term process of development does not and possible cannot lead to a "full proof" model. The time framework that is needed for a new application model to be rigorously developed and tested can be understood by understanding for example the Black-Scholes model and even then their work is limited in their applications. Clearly, the time taken for the development of tranches that seems to have started the GFC in US would not have passed this test of "time taken for analysis". Serious models in science have required years development to a model stage that would subsequently have undergone extreme testing over years so that proofs could be developed. In this example, serious consideration of correlations or taking in account incidents such as extreme cases or all other possible risk events as noted by Taleb (2010) in his book "Black Swan" could not have been conducted in this framework.

It is also important to note that there is much research done "in house" in recent times. In many instances research developments in large institutions are not published especially if they involve cutting-edge research that has possibly large monetary gains for a company. As such the research work may only be available for critical analysis after the money making opportunity has passed. This is a real problem for finance and business more generally in modern times. Another issue is to do with the provision of large funds of money usually fed by larger institutions particularly to attract high achievers in mathematics. Such a solution works most of the time but can fail as it did in the case above. In most cases it is due to the lack of appreciation of the critical aspects of the "real financial space of operation" on the modeler's part. A student in mathematics may not be totally aware of all on goings in the real financial world. It is during this critical financial space analysis when the need for the "expert or best" of the financiers is required to interrogate the model. The expert financiers therefore must take responsibility for they also had a role in the global financial crisis in that they should have appreciated serious flaws in the risk analysis and modelling process. Clearly, the experts in the finance and economics fields did not and one possible cause for this may be attributed to the lack of knowledge and appropriate training in the mathematics of finance. If this is the case, it can be traced back to the learning phase of the experts during which their instructors failed to show the critical importance and understanding the development of more complicated mathematical models. If the best of the financiers could not mathematically oversee, analyse or critically test the models or any assumptions underpinning them but instead assume them to be correct then there is certainly a serious need to rethink financial training.

Finance has also changed mathematics research and teaching

There is no doubt that financial mathematics has changed mathematics research and mathematics teaching (Roca and Tularam, 2012; Roca et al, 2010). It would difficult to find mathematics faculties not conducting any research in finance and economics around the globe (Roca and Tularam, 2012; Roca et al, 2010; Tularam et al, 2010). Almost all new algebra and calculus texts now written include financial application examples. Mathematics ingrained nature in finance and the use of mathematical concepts has changed finance research and teaching (as it should have) but at the same time finance has influenced mathematics research and teaching considerably. There are a number of courses in universities that integrate the teaching of finance theory and mathematics. Prichett and Feinstein (1999) argued that that this integrated nature produces a deeper understanding in both related fields. It is not only simply a one way "granger" causality between finance and mathematics but since the practice of finance has changed mathematical teaching and research a two way g-causality exits!!! (not proven). However, it is clear that modern finance simply cannot do without mathematics but reverse is debatable (you may wish to consider the reverse proposition more carefully). How mathematics has changed financial research in particular can be comprehended by a recent statement on research topics for finance theses:

It is possible to keep the thesis completely theoretical (i.e. no computers & numbers), but it is more natural to involve some numerical work. In fact, depending on how deep you go into the PDE-approach, you can get "as heavy numerics as you like"
Even then many continue to argue that mathematics is not a way in which financial work should be understood and that it should not be the focus or the use of models ought to be discouraged and so on. There have been other strong comments made in this regard in recent times in many cases blaming the mathematics and modeling using them. One the most famous mathematicians of all time Bertrand Russell lecture would have behaved differently it seems; his behaviour is worth quoting from Imre Salusinszky's article (1996) "Academic squabbles are feud for thought":
Bertrand Russell, after continual interjection during a Cambridge lecture by Ludwig Wittgenstein "dragged him across the floor to the exit, and slammed the door on his head several times. There were no further disputations..., noted Russell in his diary that night.

The author hopes that disputes and concerns are dealt with a much better fashion today than those days! but the recent heated discussion about mathematics and its importance in finance is a relevant point. The critical views expounded earlier appear to be mainly by those who have not understood mathematics or learning of it. There is nothing that can be further from the truth given the risk analysis of the tranches should have been more wisely reviewed or assessed by the world's best financiers even when the calculations were developed by mathematicians. The best financiers and managers are being paid large funds for their work, and thus should have been well versed in the modeling processes. The financial researchers should have critiqued the model with all their assumptions given that they are the so called world's "best financiers" but clearly this did not occur. It seems that this was due to their inability to understand the mathematical complexity of the "exotics' but more importantly it was probably due to their lack of appreciation of the underlying assumptions in the modeling process. The mathematicians working in the field are trained in applied mathematics and therefore were able to easily develop exotic money making options or develop valuation models for options - tranches. However, what appears to be occurring is that very high paid modeling positions are advertised by the world's major

financial institutions - so called “big players” who investing large amounts of funding for the development of exotica such as tranches in the first place using young “guns” who may be less experienced in the “finance space of operation”.
Evidence of the existence of such financial experts in both quantitative and quantitative are readily found. If one searches the internet there are many financial experts and holding rather prestigious positions after having a highly distinguished careers, for example:

Case 3: One of the many experts found in financial engineering and referred to as the master of static hedging:

Dr. X is a Managing Director at Morgan Stanley with 15 years of experience in the derivatives industry. … was also a finance professor for 8 years at Cornell University, after obtaining his PhD from UCLA.… is the Executive Director of the Math Finance program at NYU's Courant Institute, the Treasurer of the Bachelier Finance Society, and a trustee for the Museum of Mathematics in New York. … has over 70 publications in academic and industry-oriented journals and serves as an associate editor for 8 journals related to mathematical finance. … was selected as Quant of the Year by Risk Magazine in 2003 and shared in the ISA Medal for Science in 2008. Last December, the International Association of Financial Engineers (IAFE) and Sungard jointly announced that they have selected Dr. X as its 2010 Financial Engineer of the Year.
This case is just to show an example of experts in the field of quantitative analysis in finance. Clearly, the case is not presented to blame or point fingers at any particular person but to highlight the expertise, achievements and rather high public standings of the best financiers around the world. Since there are so many experts in the field there should be some who would have highlighted the difficulties the tranches posed but this was only done after the event and indeed by mostly blaming the mathematicians or so called quants in the field.

Simple ideas in finance may lead to complex mathematics – not easily understood by finance personnel

As a point of interest rather simplistic ideas in finance may be translated into mathematics easily but solution to the mathematics problems may be rather difficult to find without making certain simplifications or assumptions. Black has stated in one of his interviews that it took them approximately 6 months to develop the mathematical equivalent problem for asset pricing and its change over time; and then another 3 years to successful solve the same mathematically. An often quoted example is sufficient to bring to attention how a simple type of problem can lead to a complex mathematics one that requires higher order mathematics to solve. For example, there is a lot of data now available in the stock market and it is easily possible to collate all low, high and closing price for a number of years. If the best estimate of the price p was required from the data then order statistics and maximum likelihood methods would be sufficient to present a mathematical model:

$$\frac{\partial l\left(p^*,\sigma\right)}{\partial p^*}=\frac{1}{\sigma}\sum_{i=1}^{3}y_i\left(p^*,\sigma\right)-\frac{n-1}{\sqrt{2\pi\sigma}}\sum_{i=1}^{2}\frac{\left(-1\right)^i}{1-Q_i\left(p^*,\sigma\right)}e^{\frac{1}{2}y_i^2\left(p^*,\sigma\right)}$$

where $y_1 = \left(\mathrm{High}-\mathrm{p}^*\right)/\sigma$, $y_2 = \left(\mathrm{Low}-\mathrm{p}^*\right)/\sigma$, $y_3 = \left(Close-p^*\right)/\sigma$, and Q is the standard error function.

This equation appears to be rather difficult to solve when posed in this form but a simplified form leads to the best estimate being the average of the high and low prices that is not so surprising. Many such examples can be stated to show that simple financial ideas can lead to much more complex mathematical propositions to deal with. Once posed in a mathematical framework many of the tools of mathematics that has been developed over centuries can be applied to solving the problems but the answers found may or may not be useful in the real world. What is gained however is a proof of what may have been only a hunch or belief of financiers in the first place; in this way mathematics allows such beliefs or hunches to be proved based on sound logical statements and arguments. It may seem that the extreme values or outliers are not taken into account in this averaging process but whenever new information becomes available, a new high or low is formed that in itself now contains the most recent and important information content and the average captures all the relevant information.

It should be noted that a mathematical solution can be developed by simplifying the model that allows solutions that are based a somewhat simple but realistic model. The solution depends on sound statistical tools that can be applied with reasonable approximations. In most cases, mathematicians can provide solutions that may be simple but most importantly and in most cases allow significant insights into the real processes at work.

The next section highlights the nature of skills that are required both in finance and mathematics and it will be argued that the base skills and processes may be easily gained by mathematicians over years of learning of proofs, accuracy in problem solving and application of mathematical techniques to novel contexts. When seen from this perspective it seems that modern students in finance may lack many of these particularly in their preparation years. The skills and processes required are based on problem solving methods routinely or daily practiced in mathematics classrooms. Indeed, many of the skills overlap with the “generic type” of the so called higher order and critical thinking skills that are critically important and required for expertise in problem solving in any field of study. The next few sections address this issue in some detail and show in how pure and applied mathematics learning and training over years provide an edge over the others types of training in this area. That is, the training of mathematicians routinely involve the use of such critical higher order thinking skills whenever mathematicians develop and solve either simple or complex problems in classrooms, tutorials, workshops, and lectures.

Higher order thinking skills (HOTS) and mathematics learning and training

Essentially, mathematical finance is concerned with the discipline of finance and economics with its underlying theory. Financial mathematicians tend to derive, extend and model aspects of theories and applications involved in economics. Clearly, mathematical finance relies on other areas such as statistics and computer science. For example, a financial economist would study structural aspects to reach a particular price of a stock while a mathematician will use his or her knowledge of pure and applied mathematics to obtain the fair value of a derivative based on an underlying asset. Many of the mathematical processes involved require the use higher order thinking skills (HOTS) such as creativity, logical arguments, deduction, metacognitive analysis and so on, during a single applied problem solving exercise (Tularam, 1997, 1998; Tularam and Kelson, 1998; Tularam, 2011, 2013). Such an exercise may take some time to solve as aptly noted in Black’s comments earlier; Black stated that it took them 6 months to develop the Black Schole’s equations and then another three years to solve the model. This shows solving of mathematical models takes time sometimes years. However, it also shows that mathematics experts are highly trained in the use of such higher order thinking skills that includes the affective traits of persistence and motivation to pursue their goals. Although solutions may take some time mathematicians are able to search and develop logical approaches using their problem solving and lateral thinking to create solution opportunities in most cases.

Critical thinking skills include comparison, classification, sequencing, cause/effect, patterning, webbing, analogies, deductive and inductive reasoning, forecasting, planning, hypothesizing, and critiquing; while creative thinking involves flexibility, originality, fluency, elaboration, brainstorming, modification, imagery, associative thinking, attribute listing, metaphorical thinking, forced relationships (Tularam, 1997). More detailed definition of higher order thinking involved in mathematics can be stated as critical thinking and problem solving skills involving exercising sound reasoning in understanding, making complex choices, understanding the interconnections among systems, and framing, analysing and solving problems. Many of the skills that are termed higher order critical thinking skills are often practiced "routinely" during mathematics learning and training as stated earlier. Though they are generic type problem solving skills it seems that little attention may be paid to such learning in the routine sense in other courses when compared to mathematics daily learning.

Andrew Wiles proved Fermat's last theorem and his work took 7 years to complete. Evidence of the relationship between higher order critical thinking and mathematical is made rather obvious and clear in his documentary (see references). A deep analysis of this work shows that mathematical problem solving involves thinking that

- uses evidence skilfully and impartially;
- organizes thoughts and articulates them concisely and coherently;
- distinguishers between logically valid and invalid inferences;
- suspends judgment in the absence of sufficient evidence to support a decision;
- understands the difference between reasoning and rationalizing;
- attempts to anticipate the probable consequences of alternative actions;
- understands the idea of degrees of belief;
- sees similarities and analogies that are not superficially apparent;
- can learn independently and has an abiding interest in doing so;
- applies problem-solving techniques in domains other than those in which learned;
- can structure informally represented problems in such a way that formal techniques, such as mathematics, can be used to solve them;
- can strip a verbal argument of irrelevancies and phrase it in its essential terms;
- habitually questions one's own views and attempts to understand both the assumptions that are critical to those views and the implications of the views;
- is sensitive to the difference between the validity of a belief and the intensity with which it is held;
- is aware of the fact that one's understanding is always limited, often much more so than would be apparent to one with a non-inquiring attitude; and
- recognizes the fallibility of one's own opinions, the probability of bias in those opinions, and the danger of weighting evidence according to personal preferences (Schafersman, 1991; Tularam, 1997, 1998, 2002,2011, 2013);

that are often referred to as higher order - critical creative and problem solving thinking skills. Critical thinking appears to be more left-brain while creative thinking appears to be more right brain. The often quoted Bloom (1956) studied cognition and meta-cognition using cognitive, psychomotor, and affective domains. The cognitive domain included knowledge, comprehension, application as well as analysis, synthesis, and evaluation. Bloom's analysis showed:

1. Knowledge refers mainly to memory of facts;
2. Comprehension refers to understanding of facts, demonstrated by organizing or interpreting them;
3. Application refers to using understanding to solve problems;
4. Analysis refers to recognizing patterns suggested by facts);
5. Synthesis involves producing something new; and
6. Evaluation involves judging quality of a solution or theory.

The first three are lower order thinking skills and this include application, which many think is higher order. However, the higher-order thinking skills are the ability to analyse, synthesise, and evaluate that also includes a number of reflective thinking skills. For the benefit of financial education, the author has included some more specific thinking skills in each of the six categories above that can be easily identified in the examples in this regard later (challenge yourself to identify them). In addition, the often ignored affective domain is also included in this higher order skill base.

Knowledge

collect	describe	identify	list	show	tell	tabulate
define	examine	label	name	retell	state	quote
enumerate	match	read	record	reproduce	copy	select

Examples: dates, events, places, vocabulary, key ideas, parts of diagram;

Comprehension

associate	compare	distinguish	extend	interpret	predict	differentiate
contrast	describe	discuss	estimate	group	summarize	order
cite	convert	explain	paraphrase	restate	trace	

Examples: find meaning, transfer, interpret facts, infer cause and consequence, examples

Application

apply	classify	change	illustrate	solve	demonstrate
calculate	complete	solve	modify	show	experiment
relate	discover	act	administer	articulate	chart
collect	compute	construct	determine	develop	establish
prepare	produce	report	teach	transfer	use

Examples: use information in new situations, solve problems

Analysis

analyze	arrange	connect	divide	infer	separate
classify	compare	contrast	explain	select	order
breakdown	correlate	diagram	discriminate	focus	illustrate
infer	outline	prioritize	subdivide	points out	interogate

Examples: recognize and explain patterns and meaning, see parts and wholes

Synthesis

combine	compose	generalize	modify	invent	plan	substitute
create	formulate	integrate	rearrange	design	speculate	rewrite
adapt	anticipate	collaborate	compile	devise	express	facilitate
reinforce	structure	substitute	intervene	negotiate	reorganize	validate

Examples: discuss “what if” situations, create new ideas, predict and draw conclusions

Evaluation

assess	compare	decide	discriminate	measure	rank	test
convince	conclude	explain	grade	judge	summarize	support
appraise	criticize	defend	persuade	justify	reframe	

Examples: make recommendations, assess value and make choices, critique ideas

Affective domain

accepts	attempts	challenges	defends	disputes	joins	judges
contributes	praises	questions	shares	supports	volunteers	persists

Domain Attributes: self-motivation, interpersonal relations, emotions, attitudes, appreciations, and values

The importance of the critical and higher order thinking skills at the university level cannot be overstated as noted in the research results of fourth year physics students. The quantum physicists were used as a control group while the atomic physicists were specially taught HOTS – the experimental group. The results showed that experimental group demonstrated a high level of HOTS while the control group were average. Also, the pre and post testing showed significant differences in performance between the groups (Molina et al., 2008; Tularam and Amri, 201; Tularam, 2013)

In most cases, mathematics lecturers argue that more demands should be placed on student early mathematics learning and teaching yet it seems little attention has been paid generally when both the lower and higher order skills highlighted above are exposed or taught. Nonetheless, what is clearly evident in that mathematics classrooms, lectures, tutorials, workshops or problem solving sessions is that they typically include more than their fair share of practice and time spent on the above skills (Tularam and Amri, 2011); much of time spent in sessions are spent on activities that require the active use of the above thinking skills. It can be said with high probability that the above thinking skills are almost "routinely" practiced during mathematics learning in each and every mathematics classroom, university lecture or during tutorial and workshops around the world. Therefore, if simply the length of time spent on learning and practising as well as training in the use of higher order skills is considered then over the many years it is not surprising that mathematicians are accomplished in thinking or are considered to be rather "sharp" in thinking; and talented in the use of such skills when dealing with demanding problems and when solving questions posed in abstract prime number theory or real life problem solving and indeed in applied financial mathematics.

Higher order thinking skills in theory of financial mathematics

The use of Ito's lemma from discrete stochastic calculus has been an important breakthrough in mathematical financial studies generally and this is a point cannot be disputed and as such the author introduces and exposes Ito's thinking in the following. The mathematics used in Ito's work can be demanding in that the derivation involves many of the skills mentioned above as will be noted in the derivation of the univariate case below.

Consider a stochastic process defined by $B(0) = 0$ (more generally $B(0) = B_0$ a fixed starting point) where $B(t + 1) = B(t) + \varepsilon(t + 1)$ and $t \in \{0, 1, 2, \ldots\}$. The innovations in B are independent standard normal random variables: $\varepsilon(t + 1) \sim NIID(0, 1)$ for all t. That is, a special version of a random walk with normally distributed increments. If the process is more frequent in each fixed time interval then for example; if $\Delta t = 1/n$ for some arbitrary integer $n > 1$ then $B(t + \Delta t) = B(t) + \varepsilon(t + \Delta t)$ with $B(0) = 0$ and $\Delta B = B(t + \Delta t) - B(t) = \varepsilon(t + \Delta t)$; that is normally distributed as $N(0, \Delta t)$. Over n periods of length Δt [ie. $n\,\Delta t = 1$] the new process has the same expected change and the same variance as the original has over one fixed time "interval" or period. Making by n large allows $\Delta t \to 0$ [ie. $\Delta t \rightarrow dt$] and by letting dt be the smallest positive real number such that $dt^{\alpha} = 0$ whenever $\alpha > 1$ the value $dB(t) = B(t + dt) - B(t) = \varepsilon(t + dt)$ can be defined as the increments in the process B(t). dB(t) may be thought of as a normally distributed random variable with mean 0 and variance dt; dB is often referred to as white noise and B(t) is often referred to as a standard Wiener process. A number of rules can then be developed based on the statistical arguments of mean and variance such as:

1. $E[dB(t)^2] = dt$; that is, $E[dB(t)^2] = Var[dB(t)] = dt$ since $dB(t) = \varepsilon((t + dt)$ and $\varepsilon(t + dt) \sim N(0, dt)$ and since $E[\varepsilon(t + dt)] = 0$ implies that $E[\varepsilon(t + dt)^2] = Var[\varepsilon(t + dt)]$
2. $Var[dB(t)^2] = 0$; $Var[dB(t)^2] = E[dB(t)^4] - (E[dB(t)^2])^2 = E[\varepsilon(t + dt)^4] - dt^2 = 3dt^2 - dt^2 = 0$

It is noted the term $\varepsilon(t + dt)$ is $N(0, \sigma^2)$ the expectation: $E[\varepsilon(t + dt)^4] = 3\sigma^4$ (using $\mathrm{kurt}(X) = [E(X^4) - 4\mu E(X) + 6\mu^2 E(X^2) - 3\mu^4] / \sigma^4$. The above shows that $dB(t)^2 = dt$ is a constant. If the variance of a function of a random variable vanishes the expectation sign becomes redundant: for example, $E[f(dB(t))] = f(dB(t))$ when $Var[f(dB(t))] = 0$. A number of rules can be generated such as $dt^2 = 0$ since dt is small and $dB(t)dt$ is also 0, while $dB(t)^2 = dt$. It can be said that a standard Brownian motion denoted by $B(t)$ is a stochastic process defined by: $B(0) = B_0$ (= 0) (with probability 1) and has increments $B_s - B_t$ that is normally distributed with $N(0, s - t)$ for all s and t with $s > t$. The increments are independent normal random variables; $dB(t)$ is a differential representation of $B(t)$. However, the standard Brownian motion is not a good model for stock price movements in that it has no drift. A process that allows for drift in prices thus allowing the expected to change over any future interval (non-zero) would be more useful.

To introduce a drift let X be an Ito process defined by $\mathrm{dX_t} = \alpha_t \mathrm{dt} + \sigma_t \mathrm{dB_t}$ where the dependence of α (drift) and σ (diffusion) on (X_t, t) is normally not shown in equations (note: X_t and B_t are used for $X(t)$ and $B(t)$ respectively). If a function (f) is defined on f: $\Re \times [0, T] \rightarrow \Re$ then a function $Y_t = f(X_t, t)$ is an Ito process. The change in Y (dY) may now be defined using Equation 1 based on Taylor series as shown later:

$$dY_t = [f_x(X_t, t)\alpha_t + f_t(X_t, t) + \tfrac{1}{2} f_{xx}(X_t, t)\sigma_t^2]dt + f_x(X_t, t)\sigma_t dB_t; \quad (1)$$

f must be twice continuously differentiable in both X_t and t but we only need f_x, f_{xx}, and f_t to exist and to be continuous. However, how can Equation 1 be established? Using a 2nd order Taylor series for $f(X_{t+dt}, t + dt)$ that is close to the point (X_t, t), the change $dY_t = f(X_{t+dt}, t + dt) - f(X_t, t)$ can be determined.

$$\begin{aligned} f(X_{t+dt}, t+dt) = {} & f(X_t, t) + f_x(X_t, t)dX_t + f_t(X_t, t)dt \quad (2) \\ & + \tfrac{1}{2} \cdot f_{xx}(X_t, t)(dX_t)^2 + \tfrac{1}{2} \cdot f_{tt}(X_t, t)(dt)^2 \\ & + 2 \cdot \tfrac{1}{2} \cdot f_{xt}(X_t, t)dX_t dt + R(residual) \end{aligned}$$

The remaining terms (R) tend to 0 as higher order dt's and most of the higher order dX's tends to 0. In Equation 2, dX_t, dt are non zero while dt^2 is zero. The terms $(dX_t)^2$ and $dX_t dt$ is determined by squaring both sides of dX (Equation 3) giving $(dX_t)^2$ to be $\sigma_t^2 \mathrm{dt}$;

$$\begin{aligned} (\mathrm{dX_t})^2 &= (\alpha_t \mathrm{dt} + \sigma_t \mathrm{dB_t})^2 \quad (3) \\ &= \alpha_t^2 \mathrm{dt}^2 + \sigma_t^2 \mathrm{dB}_t^2 + 2\alpha_t \sigma_t \mathrm{dB_t dt} \\ &= 0 + \sigma_t^2 \mathrm{dt} + 0 = \sigma_t^2 \mathrm{dt}; \end{aligned}$$

the product $\mathrm{dX_t dt}$ is 0 since $\mathrm{dX_t dt} = (\alpha_t \mathrm{dt} + \sigma_t \mathrm{dB_t})\mathrm{dt} = 0 + 0 = 0$. Now Equation 1 may be derived using dY and Taylor's series expansion and $\mathrm{dX_t} = \alpha_t \mathrm{dt} + \sigma_t \mathrm{dB_t}$;

$$\begin{aligned} dY_t &= f(X_{t+dt}, t+dt) - f(X_t, t) = f(X_t + dX_t, t+dt) - f(X_t, t) \qquad (4) \\ &= [f_x(X_t, t)\alpha_t + f_t(X_t, t) + \tfrac{1}{2} \cdot f_{xx}(X_t, t)\sigma_t^2]dt + f_x(X_t, t)\sigma_t dB_t. \end{aligned}$$

In Equation 4, the higher order terms (R) vanish. In ordinary calculus dX is small enough and dX^2 vanishes as it is much smaller still. However in stochastic calculus, the dX term is a random variable and dX^2 cannot be assumed to vanish but rather converges to $\sigma_t^2 dt$. The higher order terms such as dX^3 or dXdt etc do in fact vanish. Therefore, Equation 1 (Equation 4) have been established as a way of representing the change in a function Y(t) derived using Taylors series and Ito's theoretical work. A brief examination of the above works would suggest many skills have been used such as analysing, questioning, creativity, constructing, collecting terms, justifying, persuading, synthesis, and logical presentation and finally judgement (for more specific skills see earlier under analysis, synthesis and evaluation).

Development of financial models using higher order mathematical thinking

In finance, the trading of options is an essential activity. An option is a contract that gives the right to the buyer of the option to buy or sell a primary asset (a stock or a bond) at a price with maturity date fixed at the time of purchase. This option is similar to an insurance contract in that the holder of the option is somewhat protected against changes in the underlying asset price. A number of finance agents considered the question of whether there exists a theoretical price of any derivative or option in a given economy (Duffie, 2001). The process of derivation of such pricing and model can developed based on Ito and stochastic calculus and the notion of no arbitrage (no free lunch).

In an ideal situation the model predicts the price to be the initial amount of money invested as a portfolio exactly replicates the payoff of the option at its maturity date. Clearly, a number of assumptions need to be taken into account in this pricing formula. Just as in chemistry the ideal gas equation (pv = nRT) has been used to explain behaviour of gases in general where as in fact none of the gases themselves ever behave ideally but vary with different degrees from the model prediction. Even when this is the case, a number of advances have been made in chemistry in terms of predicting structures and properties of gases after the development of the ideal gas equation. The model is able to provide the opportunity to develop higher order rules. This model was made possible with the use of mathematics and indeed mathematics models are at the forefront of development of cutting edge knowledge in many fields. The model constructed relied on the skills of the developers in that they were able to use their training in higher order thinking and critical and creative skills to reach a higher level of abstraction and understanding in another field. In many cases, the physicists and chemists were mathematicians first and applied mathematicians later in their respective fields. The same is noted in finance with the work of Ito and others. The initial work on Brownian motion and stochastic process is due to Bachelier (1900) who modelled stock price but it is for certain that Black and Scholes (1973) can be credited for the use of this knowledge and develop it to a higher level when they developed the European options pricing formula. Later, Harrison and Kreps (1979) and Harrison and Pliska (1981) provided a general conceptual framework to the option pricing problem based on the developed stochastic integration theory. The rest is history but the development of the model is a great advance in the field and was achieved using mathematical thinking that is highlighted in the following.

Black-Scholes model and modelling derivative pricing

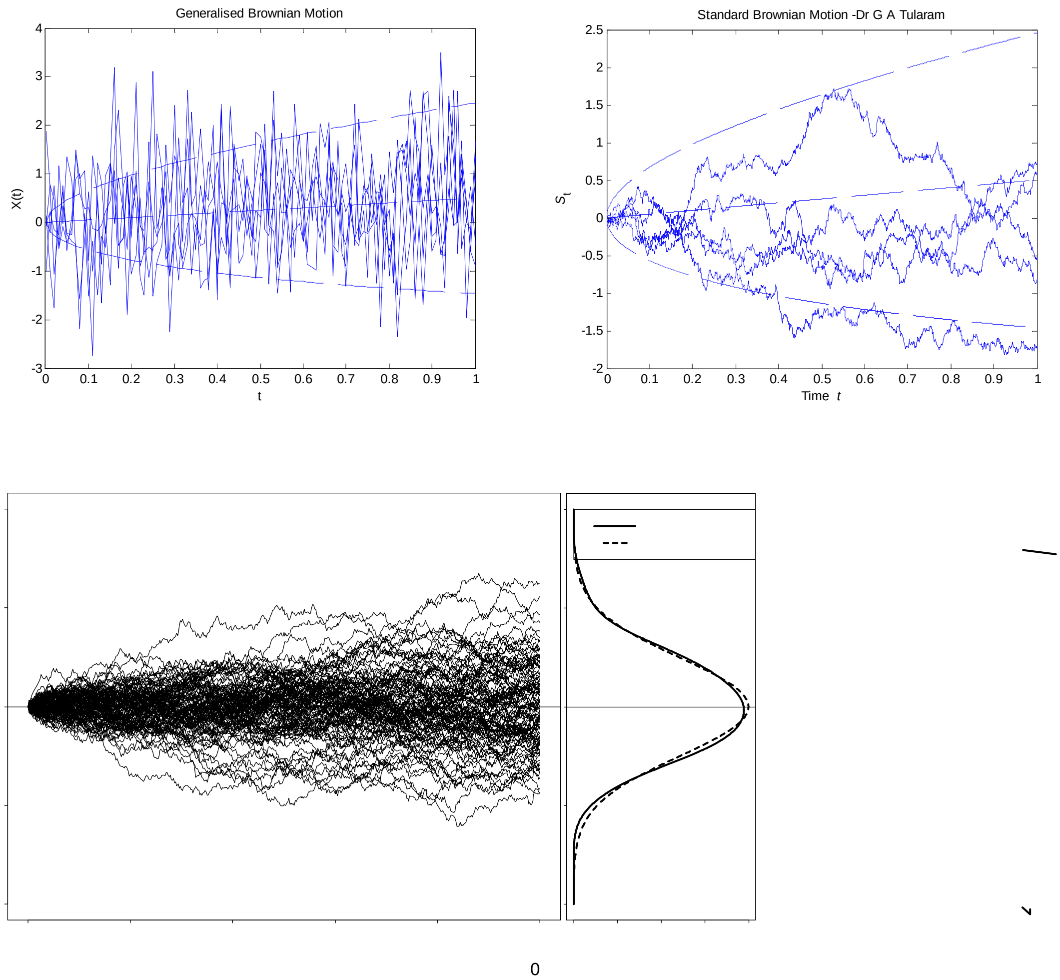
The partial differential equation that changed mathematical finance forever is analysed in some detail in a case study manner to show how higher order critical thinking skills routinely applies to mathematical thinking during development of creative ideas and solution processes. Most if not all of the models and techniques applied to pricing of assets or valuation of options are based on the model developed by Fischer Black and Myron Scholes in 1973 and Robert Merton (1973). The following notes are neatly adapted from Leeds and Strathmore University's lecture notes. One of the most important aspects of higher order thinking is to deal with the use of letter for variables and constants. Simply the use of letters does not change the affect mathematicians as it does when others view equations; this is an essential aspect of abstraction and critical to mathematical thinking – seeing through the problems. A stochastic process $X(t)$ is more formally a family of random variables $\{X(t) | t \in T\}$ and is defined on a given probability space. In mathematics a space is defined by a series of mathematical rules that the objects in the space obey to the letter. The process is indexed by a parameter t that varies over an index set T providing the time-line. It was noted that Brownian motion can be used to model asset returns having a zero mean yet it fails as the growth rate of the returns is nonzero. There was need then to extend the standard Brownian motion to have some non-zero drift term (mean) and a 1D generalised stochastic differential equation

(SDE): $\mathrm{dX(t)} = \mathrm{adt} + \mathrm{bdZ(t)},\ \mathrm{X(0)} = \mathrm{X_0}$ was initiated where a (drift rate) and b (standard deviation) are constants and $\mathrm{Z(t)} = \varepsilon\sqrt{\mathrm{t}}$ is a Wiener process with ε being a standard Wiener process with mean zero and variance one. Various notations are noted in texts so in the following dZ is defined in place of dB. The mean of $\mathrm{dX(t)}$ is given by adt and its variance is given by b^2dt (see earlier derivations). Therefore, for an interval $[0,t]$ the integrated form the SDE can be given as

$$\begin{aligned} &X(t) = X_0 + a\int_0^t ds + b\int_0^t dZ(s) \text{ where} \\ &X(0) = X_0 \end{aligned} \qquad (5)$$

The solution in Equation 5 is given as: $\mathrm{X(t)} = \mathrm{X_0} + \mathrm{at} + \mathrm{bZ(t)}$ where $Z(0) = \mathrm{Z_0} = 0$ and $X(0) = X_0$. Here $X(t)$ has a linear trend (or drift) given by $X_0 + at$ and the variance term: $b^2\,dt$. The parameters a and b are usually referred to as drift velocity (μ) and volatility (σ). The author has developed some trajectories using Matlab of a generalised Brownian motion (Figure 1) and it is noted that $X(t)$ takes both positive and negative values. This was a major problem with the Bachelier model as stock prices cannot take below zero values.

Figure1:
Typical trajectories of a generalised 1-D Brownian motion with $drift = 0.5$ *and standard deviation = 1 and 95% envelope (limiting curves) i.e.* $\pm 1.96\sigma\sqrt{t}$ *envelope about the trend line,* $x_0 + at$.



Changing parameters a and b to become functions of the underlying variable $X(t)$ and time t allowed researchers to generalize Brownian motion. For example, if $a = a(X(t), t)$ and $b = b(X(t), t)$, a 1D generalised Itô process can be written as $dX(t) = a(X(t), t)dt + b(X(t), t)dZ(t)$, where $a(X(t), t)$ is the instantaneous drift rate that measures the expected rate of change in $X(t)$ and $b(X(t), t)$ is the instantaneous variance that measures the amount of random diffusion. It is assumed that both drift and variance are functions of time. The expected drift is $a(X(t), t)dt$ and the expected variance is given to be $b^2(X(t), t)dt$. The change over the interval $[0, t]$ can then be expressed in an integrated form as a linear function of drift and volatility terms are considered (Equation 6):

$$X(t) = X_0(t) + \int_0^t a(X(s),s)ds + \int_0^t b(X(s),s)dZ(s) : X_0 > 0 \tag{6}$$

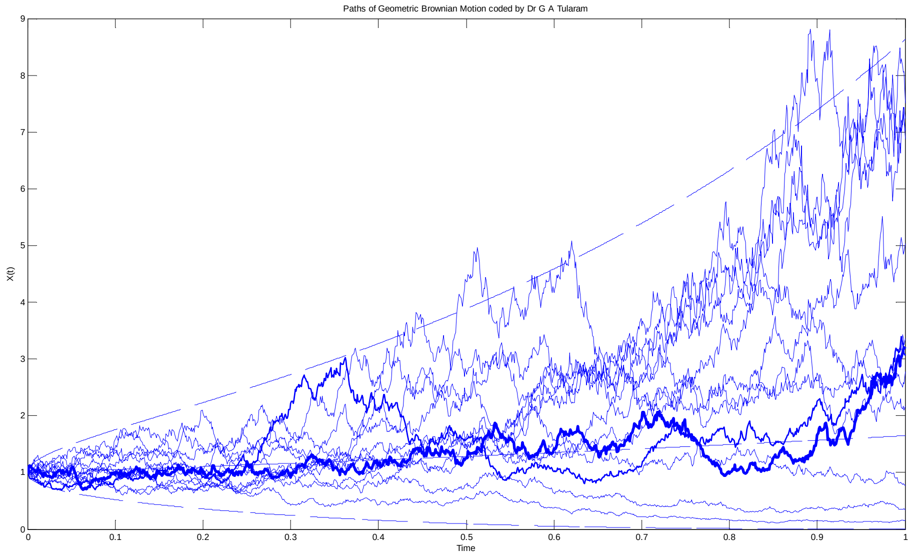
In the above, an important decision was made based on the observation that the option price is dependent on the stock price; the price movement over time must be related to the stock's movement over time. Moreover, a geometric Brownian motion process was also considered for stock price modelling and Itô's lemma allowed the change in the option price to be determined as the following SDE shows (Black and Scholes, 1973). In a simple case of a geometric Brownian motion it can be assumed that $a(X_t, t) = \alpha X_t$ and $b(X_t, t) = \sigma X_t$ thus forming $dX_t = \alpha X_t dt + \sigma X_t dB_t$ *or* $\frac{dX_t}{X_t} = \alpha\, dt + \sigma\, dB_t$; a process with a constant expected return over time and a constant variance of return. Although simplified this is a more natural model of stock prices (why?).

$$\frac{dX(t)}{X_t} = adt + bdZ(t);\; X(0) = X_0 \tag{7}$$

$$\mathrm{S}(\mathrm{t}) = log(\mathrm{X}(\mathrm{t})/\mathrm{X}_0) = \int_0^t \mathrm{ads} + \int_0^t \mathrm{bdZ}(\mathrm{s})$$

Here $X(t)$ is the price of a traded asset at time t while $dX(t)/X(t)$ is the rate of return on the asset. Equation 7 cannot be solved directly since it is not linear in $X(t)$ and contains parts that involve standard Wiener process $Z(t)$. The integral however, shows that the logarithmic stock return executes the relatively simple generalised Wiener process now called 'geometric Brownian motion'. First introduced by Samuelson (1965) it has an advantage over the Bachelier model in that $X(t) = X_0 \exp(S(t))$ never assumes negative values as shown in Figure 2. The creation of a model that allowed only non-negative values was a major improvement in the modelling of prices (Figure 2).

Figure 2:
Some trajectories of a 1-D geometric Brownian motion with X(0) = 1, drift = 0.5 and sigma or 1, and envelope (limiting curve), $\pm 1.96\sqrt{\left(e^{2t}\left(e^{t}-1\right)\right)}$ *about the expected mean,* $X_0 e^{0.5t}$ *(middle dotted curve).*



Suppose that the random variable $X(t)$ follows a diffusion process, with a predictable rate of return $a(X(t),t)dt$ and instantaneous rate of variance $b^2(X(t),t)dt$ and if one considers $b=0$ then a simple solution can be developed: $dX(t)/X(t)=a(X(t),t)dt$ but when $b \neq 0$, $X(t)$ has sample paths not differentiable anywhere as shown and ordinary calculus cannot be used to integrate for values of $X(t)$ for the integral include the Wiener processes, namely: $Z(t)$.

However, it is possible to reformulate a function based on a random variable $X(t)$ and a new function may be developed that is a smooth function *f* after which the Itô's lemma can be used to integrate say $\mathrm{f}(\mathrm{X}(\mathrm{t}),\mathrm{t})$. Consider $\mathrm{f}(\mathrm{X}(\mathrm{t}),\mathrm{t})$ be twice continuously differentiable function in X and once in t, then using Taylor series expansion and ignoring higher orders of expansion, the stochastic differential of $\mathrm{f}(\mathrm{X}(\mathrm{t}),\mathrm{t})$ is given by as shown earlier (that is, the change in f or df(X(t), t)):

$$\mathrm{df}(\mathrm{X}(\mathrm{t}),\mathrm{t}) = \frac{\partial \mathrm{f}}{\partial \mathrm{t}}\mathrm{dt} + \frac{\partial \mathrm{f}}{\partial \mathrm{X}}\mathrm{dX}(\mathrm{t}) + \frac{1}{2}\frac{\partial^2 \mathrm{f}}{\partial \mathrm{X}^2}\mathrm{dX}^2(\mathrm{t}) \tag{8}$$

$$\mathrm{df}(\mathrm{X}(\mathrm{t}),\mathrm{t}) = \left(\mathrm{a}\frac{\partial \mathrm{f}}{\partial \mathrm{X}} + \frac{\partial \mathrm{f}}{\partial \mathrm{t}} + \frac{1}{2}\mathrm{b}^2\frac{\partial^2 \mathrm{f}}{\partial \mathrm{X}^2}\right)\mathrm{dt} + \mathrm{b}\frac{\partial \mathrm{f}}{\partial \mathrm{X}}\mathrm{dZ}(\mathrm{t})$$

Note that in Equation 8, $\mathrm{dX}(\mathrm{t}) = \mathrm{adt} + \mathrm{bdZ}(\mathrm{t})$ was used to replace the term dX(t) and the expression simplified. Thus $\mathrm{f}(\mathrm{X}(\mathrm{t}),\mathrm{t})$ is an Itô process with diffusion coefficient

$b\frac{\partial f}{\partial X}$ and drift rate $\left(a\frac{\partial f}{\partial X}+\frac{\partial f}{\partial t}+\frac{1}{2}b^2\frac{\partial^2 f}{\partial X^2}\right)$. The integral form of Equation 8 can be given by:

$$f(X,t)=f(X_0,0)+\int_0^{t_i}\left(a\frac{\partial f}{\partial X}+\frac{\partial f}{\partial s}+\frac{b^2}{2}\frac{\partial^2 f}{\partial X^2}\right)ds+\int_0^{t_i}b\frac{\partial f}{\partial X}dZ(s)\,; \tag{9}$$

$$X(0)=X_0$$

The appropriate mathematical machinery now exists as shown above that can allow the Black-Scholes PDE to be developed based on the work presented above. The pricing of options model development has to be based in the context of finance and therefore various assumptions and boundary conditions need to be developed in some detail to help formulate the problem appropriately so that a solution may be derived. A most important aspect of a model is its assumptions and Black and Scholes derived the option pricing formula based on the following assumptions:

- the underlying asset (stock) price follows a geometric Brownian motion with stock price expected return and stock price volatility remaining constant over time;
- there are no transactions costs or taxes;
- all securities are perfectly divisible;
- there are no dividends on the stock during the life of the option;
- there are no riskless arbitrage opportunities;
- security trading is continuous;
- investors can borrow or lend at the same risk-free rate of interest; and
- the short-term risk-free rate of interest is constant over time of trade.

The Black-Scholes formula has been extended to options that have transaction costs as premium and to other assets such as currencies and dividend-paying options (Hull 2006; Nielsen 1999; Wilmot 1998). The model has also been applied to asset prices that follow nonlinear geometric Brownian motion (Onyango, 2004) among others. The above process that new applications were forged and complexity of the model increased with the knowledge gained from the development of much simplified models.

The step by step process described above shows how a mathematician works at each stage of the development of a model and then works on a path to the solution. The above mathematical machinery and processes were in some instances available but the PDE took more than 6 or more months to develop to an appropriate first model stage and another 3 years for the solution proper.

There are a number of different option types but in this paper focuses only the European version where the exercise can occur only at the end of the time period. Changing the value of underlying asset from X to S allows the following to be written for finance students; S is the stock price and the return is given by $\frac{\Delta S}{S}$, where ΔS is a small increment in price S. If the mean rate of return of S is μ over a given short period of time Δt then the return may be written as: $\frac{\Delta S}{S}=\mu\Delta t$ within that period. An

ODE can now be developed in continuous time as $\frac{dS}{S} = \mu dt$ and the solution is $S(t) = S(0)e^{\mu t}$. But as noted earlier a better model includes a noise term: $\frac{\mathrm{dS}}{\mathrm{S}} = \mu \mathrm{dt} + \mathrm{noise} = \mu\,\mathrm{dt} + \sigma \mathrm{dZ}$ where $dZ = \varepsilon\sqrt{dt}$ is N(0, dt) because ε is N(0,1). The changing stock price (*S*) may now be written as: $\mathrm{dS} = \mu \mathrm{S}\,\mathrm{dt} + \sigma \mathrm{S}\,\mathrm{dZ}$. An option value may now be determined based on the underlying asset price S; namely the value of the option in price terms is $\mathrm{P}(\mathrm{S},\mathrm{t})$. Now using the equations developed earlier using Ito's work and Taylor's expansion the value of dP may be derived as the following (note here a = $\mu\mathrm{S}$ and b = $\sigma\mathrm{S}$):

$$\mathrm{dP}(\mathrm{S},\mathrm{t}) = \left(\mu \mathrm{S}\frac{\partial \mathrm{P}(\mathrm{S},\mathrm{t})}{\partial \mathrm{S}} + \frac{\partial \mathrm{P}(\mathrm{S},\mathrm{t})}{\partial \mathrm{t}} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 \mathrm{P}(\mathrm{S},\mathrm{t})}{\partial \mathrm{S}^2}\right)\mathrm{dt} + \sigma \frac{\partial \mathrm{P}(\mathrm{S},\mathrm{t})}{\partial \mathrm{S}}\mathrm{dZ}(\mathrm{t}); \tag{10}$$

where $\frac{\partial \mathrm{P}}{\partial \mathrm{t}}$ = instantaneous change in option's price value per unit time

$\frac{\partial \mathrm{P}}{\partial \mathrm{S}}$ = sensitivity of option's price relative to the asset price

$\frac{\partial^2 \mathrm{P}}{\partial \mathrm{S}^2}$ = change of sensitivity relative to change in asset price

"*A good critical thinker (a) raises vital questions and problems, formulating them clearly and precisely; (b) gathers and assesses relevant information, using abstract ideas to interpret it effectively comes to well-reasoned conclusions and solutions, testing them against relevant criteria and standards; (c) thinks openmindedly within alternative systems of thought, recognizing and assessing, as need be, their assumptions, implications, and practical consequences; and (d) communicates effectively with others in figuring out solutions to complex problems*" (Har, 2011)

At this point the important skill of considering a new situation that replicates the earnings of an option is considered. The idea of continuously adjusting the hedge portfolio *P* for it to become a risk free option is an advance in mathematical finance. A hedged portfolio will earn the risk-free rate in equilibrium situations in efficient markets. Assuming continuous trading, a portfolio may be formed using the underlying asset and a call whose change can be determined.

Consider the construction of a portfolio consisting of one option and say *n* of the underlying asset. The value of the portfolio may be given as $\pi = \mathrm{P}(\mathrm{S},\mathrm{t}) - \mathrm{nS}$ and the change in portfolio as $\mathrm{d}\pi = \mathrm{dP}(\mathrm{S},\mathrm{t}) - \mathrm{ndS} = \mathrm{dP}(\mathrm{S},\mathrm{t}) - \mathrm{n}(\mu \mathrm{Sdt} + \sigma \mathrm{SdZ}(\mathrm{t}))$ holding *n* is fixed during the time-step. A number of critical thinking skills are applied above and again here by assuming $\mathrm{n} = \frac{\partial \mathrm{P}(\mathrm{S},\mathrm{t})}{\partial \mathrm{S}}$ (to make the expression on the right in Equation 10 risk free) simplifies the change in value of the portfolio to be:

$$d\pi = \left(\frac{\partial P(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(S,t)}{\partial S^2} \right) dt \tag{11}$$

The no-arbitrage argument implies that the percentage return of the portfolio over the time interval dt should equal the risk-free interest rate (r): $d\pi(t) = r\pi(t)dt$. Another equation may now be written for this equality as:

$$\left(\frac{\partial P(S,t)}{\partial t} + \frac{\sigma^2}{2} S \frac{\partial^2 P(S,t)}{\partial S^2} \right) dt = r\left(P(S,t) - \frac{\partial P(S,t)}{\partial S} S \right) dt \,; \tag{12}$$

Only a little more algebraic manipulation work leads to the following partial differential equation:

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0 \tag{13}$$

Equation 13 is the PDE known as Black-Scholes equation but often Merton is also added in the naming of the PDE as he had a lot to do with its development as well. A striking feature of this PDE is that it is not dependent on μ. Black said it took them a further 3 years for them to solve this PDE. The PDE was solved by a series of variable transformations and may in fact be manipulated to form the well known heat diffusion equation and solved used the Green's method. Although it may be said that this PDE may not be the flavour of the month today due to many inappropriate applications of it, the PDE is nevertheless accepted and respected worldwide for its role in advancing finance theory and derivative pricing analysis and indeed the formula with its solution used by financial analysts daily. A number of more complex models have been developed based on this PDE such as the stochastic volatility model. Other researchers have relaxed some of the assumptions given above. Some variants of the Black-Scholes model can be used when risk-free rate of interest and stock volatility varies with time, meaning that they are time-dependent and thus not constant. The model has also been adjusted to take dividends into account (Merton 1986).

A more sophisticated model can be developed by considering volatility itself by making it stochastic in nature. Using the above theoretical knowledge, we can model stochastic volatility by considering again a single asset *S* that follows a log-normal Brownian process: $dS = \mu S dt + \sqrt{\upsilon} S dZ_1$ and the variance ν evolves according to a mean-reversion stochastic process: $d\upsilon = \kappa(\theta - \upsilon)dt + \sigma\sqrt{\upsilon} dZ_2$ where, μ is the drift of single asset *S*, κ is the rate of mean-reversion for variance ν, and θ is the long term mean level of the variance, σ is the volatility of the variance ν. And *t* is the time, but there may be a correlation between the stochastic process; Z_1 and Z_2 are two Wiener processes that may be correlated with a coefficient: ρ.

As a result, most of the traders in trading rooms are now using stochastic processes to model the primary assets and deduce theoretical optimal hedging strategies which help to take management decisions. The related questions are various and complex, for example: is it possible to identify stochastic models precisely, can one efficiently approximate the option prices (usually given as solutions of PDES or as expectations of functionals of processes) and the hedging strategies, or whether one can evaluate risks of severe losses corresponding to given financial positions or the risks induced by the numerous mispecifications of the models. Such questions are subjects of intensive research by both in academic and financial institutions. They require competences in statistics, stochastic processes, PDE solving, numerical analysis, software engineering among other fields of social sciences to study human behaviour.

The above analysis has in detail shown without doubt the critical importance of mathematical skills and processes in finance studies. Stephen Burke from the University of Kent at Canterbury confirms that mathematics is critical to the finance sector. He argues that sophisticated mathematical tools such as time series forecasting are frequently used to study trends in the world stocks and futures markets. Actuarial science graduates evaluate and manage financial risks relating to insurance and pension funds using mathematical models. Many of the graduates also work for consultancy practices, government departments, stock exchanges, industry and commerce and universities. Others also employed in high-level management positions to advice policy and strategies. It is then not surprising the level of mathematical competency is paramount in the teaching of finance and economics. Indeed, Stephen said that many accountancy companies prefer graduates to have competency in mathematics rather than just a straight accountancy qualifications. Not surprisingly, many UK and other university institutions offer mathematics and accounting majors in their undergraduate programs. Options to do degrees and higher studies in financial mathematics, econometrics or business mathematics are included in world's top ranked university's mathematics and business degree programs. Finally, a quick glance through any of the finance journals of economics or accounting demonstrates the ingrained nature of pure and applied mathematics and thus not many can argue against the fact that mathematics forms an integral as well as an important part of finance, economics and accounting.

Future of mathematics in finance

The complexity of financial transactions in the world of real stock markets and industry at large cannot be understated. The world is full of complexities but humans have forever tried to understand the nature of our world by studying the relationships amongst the various systems no matter how complex. Many of the physical and chemical aspects have been more clearly understood over time due to the time spent researching the fields yet many are still out of reach. A similar learning process is presently occurring in finance but the historical analysis shows that the involvement of mathematics in finance is fairly recent and this rather early nature of applications is probably the key for the lack of an understanding of the more complex aspects and as such the advancement when compared to areas such as physics and chemistry that has undergone centuries of mathematical involvement is small. It is almost certain that growth of new knowledge or major revolutions in thinking based on new ideas other than topics of Ito calculus, Black-Scholes, including the more recent advances, are yet to occur. If the present argument of blaming quants and modelers and using their failures as reasons to significantly lower the place of mathematics in finance or mathematical competency and content in finance and economics degree programs or courses were to become a reality, then it is clear that opportunities for significant insights or growth in the area of mathematical finance or indeed any possible major revolution in thinking will most certainly be missed; and finance related fields will suffer for it in that they will not be able to advance to stages that fields such as physics, astronomy, chemistry, and biology have all achieved thus far.

Many mathematical and statistical applications are yet to be improved to take into account of the intrinsic complexities in finance and related fields. Many of the statistical tests it seems do not sufficiently discriminate. For example, statistical tests usually fail to contradict the random-walk hypothesis for prices. It is certain that more work is needed to cope with the large effects of noise in financial time series analysis. A number other aspects it seems needs work such as the assumption that participants act rationally and aim to maximise returns. The work on neural-psychology and behavioural finance may help provide significant insights and advances in thinking. It is certain that if all of the above are incorporated into the modeling process a higher level of mathematics will be required to deal with aspects such as "real" market participants, ideas of random walks, market interdependencies, correlations, and so on.

Actions required in finance degree studies

The recognition of the importance of stochastics and modelling is becoming apparent in finance. An experienced financier adding to his comments said: "*Another thing, mathematical finance is not all that mathematical. At worst, it involves stochastic differential equations. Probably the most valuable skills are probability/statistics, basic linear algebra, and numerical computer programming*". Clearly finance is not all mathematical but importantly, SDE, linear algebra and numerics are the core of both pure and applied mathematical learning and such the comment significantly values mathematical work that is needed in the lecturing and teaching in finance. It seems that it is critically important to provide a sound mathematical basis for finance students in the first years so that the above mentioned topics can later be appreciated and easily acquired. Therefore, an introduction to financial mathematics, mathematics of risk are critically important for the first year requirements for finance graduates while financial linear algebra and matrices, time series and third year stochastics should also be compulsory second and third year courses in finance. It has also been noted that in recent years, the pricing of options and other contingent claims are areas of much activity for mathematicians. However, it has been said that "*financial formulas are only helpful to an extent - they can't truly predict the true risk of any financial product. As we have witnessed by the credit-market meltdown, no mathematical formula can predict the human element in a financial transaction*." This is a fair assessment and it is a challenge for both the financiers and mathematicians working in the finance fields. Advancements can only be achieved through serious involvement in higher studies in mathematical finance by younger graduates. For the above statement to permeate the financial degrees, a major attempt must be made to advocate more sophisticated understanding of the mathematical concepts procedures, processes, mathematical machinery and tools rather than be an incompetent user of all.

Summary and Conclusion

In conclusion, the extensive qualitative, historical and quantitative analyses show the ingrained nature of mathematics in finance. The nature of thinking and reasoning used in the advancement of finance studies include the use of many higher order thinking and creativity skills often taught routinely in mathematics classes. Indeed, higher order thinking is at the heart of mathematical thinking and reasoning although this is true in all disciplines but seemingly routinely practiced in mathematics. The in depth presentation was needed to show how various skills needed in finance are related to mathematical way of thinking and reasoning and as such they should be taught by those in with a high level of mathematics training such as mathematicians who have undergone years of routine and complex problem solving studies. Clearly, given the nature of the 2007 global crisis, a lot of areas need attention including mathematics teaching in finance (as math has been blamed in many instances by finance personnel); that is, student learning and teaching needs to be reshaped to include well designed financial mathematics courses that are compulsory yearly in finance and business degree programs. The author believes this can be taught within departments but it is proposed the teaching possibly be conducted by mathematicians as the mathematical reasoning patterns, thinking, explanations, simplifications required in transfer of knowledge is more clearly passed on to others if experts in the field are teaching, showing or presenting proofs; that is, to teach the essence of the conceptual development and solutions to models is often beyond the training of non-mathematics personnel mostly due to their lack of time spent on such mathematically related work. It is hoped that over time this may not be the case more generally if the methodology is adhered to strictly so that our future finance students are well exposed to such thinking and reasoning in their courses in degree programs.

Finally, the author cites some quotes by famous finance academics around the world in that they do make rather important points regarding the applications of mathematical and/or neural and social science tools (also mathematically based) in finance .

"A colleague and I recently wrote a research paper entitled "Artificial Neural Networks: Cerebrally smart but lamentably dumb", in which we discussed the power of neural nets for nonlinear and robust modelling, but also pointed out some of their dramatic failures."

"If applied intelligently and with great care, genetic algorithms show great promise for non-linear optimisation in finance. Their greatest virtue is their ability to find global maxima, provided sufficient genetic diversity is present in the initial population of solutions"

"Nobel Prize-winning "Modern Portfolio Theory" is now more than 50 years old and suffers from an inappropriate measure of risk. Reworking MPT, using a more realistic risk measure and including re-balancing costs, would be a worthy challenge for any aspiring financial mathematician" (Moore, 1997).

"In summary, it had to be admitted that the nature of crisis still eluded understanding and that governments could sink into bankruptcy for any number of reasons. It was not possible to identify culprits or to allocate blame, and it might be unwise to try. It was necessary to move on from asking 'what happened?' to 'what to do?' as the crisis was still happening. Perhaps the 'best' next step would be the construction of a new generation of 'first-generation' models" (CEPR, 2011).

"Financial mathematics needs to tell not only what people ought to do, but also what people actually do. This gives rise to a whole new horizon for mathematical finance research: can we model and analyse ... the consistency and predictability in human flaws so that such flaws can be explained, avoided or even exploited for profit?" (Johnson, 2011).

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