Integers for algebra

Playing your cards right: Integers for algebra

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The number and algebra strand of the *Australian Curriculum: Mathematics* (2015) advocates for holding together the study of number and algebra across years K–8—a position that mathematics educators have endorsed in many countries (e.g., Kaput, Carraher, Blanton, 2008; Li & Lappan, 2014). This recommendation along with the report *Shape of the Australian Curriculum: Mathematics* (2009), which states that during years 7–10 students' understandings of mathematics, "include(s) a greater focus on the development of more abstract ideas (p. 8)", led us to ask the following questions about our instruction for year 7 students on integers (ACMNA, 280): What models do we commonly use with students to teach them about integers and integer addition? Do these models support students for success in algebra? How can instruction about integers and integer operations help students prepare for more abstract understandings of integers that are useful for algebra? The key issue in extending from addition with whole numbers (the positive integers) to addition with integers, generally, is the establishment of negative integers.

In our classroom, the two ways we most typically worked on integers and integer addition involved representing integers as concrete quantities and integer addition as actions on these quantities. The first way involved representing an integer on a number line as a position in relation to 0. For integer addition, the first integer in an equation is represented as a position on the number line and the second integer is represented as a movement to the left or right on the number line, depending on the sign of the second integer (e.g., Nurnberger-Haag, 2007). For example, in the expression, +7+(-3), a student would start at +7 and then move three units to the left to reach +4 (Figure 1).

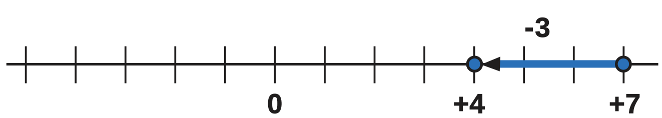


Figure 1. A number line representation of +7 + (-3) = +4

The second way we worked with integers was using an amount of chips where the sign of the number is determined by the color of the chip. For example, a red chip is considered positive and a yellow chip is considered negative. To add integers using this way, red and yellow chips are used to cancel each other out; a student puts two collections of chips together, and sees what chips are left over after cancellation takes place. While these concrete models can be used with younger students to help them start to make generalisations about our number system (Bofferding, 2014), we have found that a more abstract model for integers and their operations are useful for Year 7 students who are preparing for algebra (Ulrich, 2012; Moses & Cobb, 2001). We use this article to outline a card game that helps us develop a more abstract model for integers and integer addition with students in a fun and engaging context.

How are integers and integer addition defined in the card game?

In contrast to treating integers as direct measures of observable quantities (number of chips or distance from zero), in the card mame integers represent the difference between two positive card values. For example, one student might draw a card whose value is 8 and another student might draw a card whose value is 5. Students learn to represent this situation with either +3, if they drew the 8, or -3, if they drew the 5. This is different from the number line or chip model because it means a student has to compare the value between two positive numbers to determine what integer will represent a given situation. We consider this an important feature of how integers are produced in the card game because it provides students an opportunity to create integers (a new type of number) from whole numbers (a familiar type of number).

We then work with students to generalise this understanding by helping them see that -3 can represent the difference between an infinite number of card pairs ($-3 = 1-4 = 2-5 = 3-6 = 4-7 = \ldots$) similarly to how $\frac{1}{3}$ represents the ratio between an infinite number of number pairs ($\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \ldots$). We describe this conception of -3 as a generalised difference because -3 comes to represent any situation in which one student has a value of a card that is three less than his or her opponent, as opposed to representing a particular pair of cards that have a difference of -3. This is a more abstract, and general, understanding of the meaning of -3 than when it is represented as a specific number of chips or as a specific location on the number line. Students then learn to interpret integer addition as the sum of two generalised differences. In what follows, we discuss how the card game works, what key classroom discussions take place as the game unfolds, and how developing integers and integer addition in this way prepares students for algebra.

How the card game works

To play the card game, students pair up, each student places a deck of playing cards face down, and each student flips over one card from his or her deck. This is considered one round of the game. The points they receive for the round is the value of the card they flip where the Jack is worth 11 points, the Queen 12 points, and the King 13 points. The students repeat this process for each round of the game.

On the card game handout (Figure 2), students initially track four different values: the score of each round, the total game score, how much they won or lost the round by (round differential), and how much they are winning or losing the game by (game differential).

Round score		Total game score		How much you won or lost the round by	How much you are winning or loosing the game by
M	T	M	T		
8	5	8	5	+3	+3
7	9	15	14	-2	+1

Figure 2. Example card game handout.

To use the card game handout, we have found it helpful to have students list their initials and scores at the top of the first two columns. Figure 2 represents correct scoring if Megan and Tim are playing and draw an 8 and 5, respectively. Megan records an 8 and 5 to represent their round scores. Because this is the first round, their total game scores will be the same. In the third and fourth columns, she writes a +3 to indicate that she won the round and is winning the game by three, respectively. The game then continues to a second round. In the second round, Megan draws a 7 and Tim a 9. Megan records these round scores and then adds them to the previous round scores to get each player's total game score. She next determines the difference, 7−9, in her score and Tim's score to determine she lost the round by two, and so records a −2 in this column. To find the amount Megan is winning the game by, we ask students to use the total game scores, 15−14, to determine she is winning the game by +1. Computing the round and game differential familiarizes students with the meaning for integers in the card game: integers are created from finding the difference between two whole numbers.

Key discussion 1: Why can the same situation be represented with two different integers?

On day 1 students begin the game by playing a few rounds to become familiar with recording their total game score, round score, round differential, and game differential. Once they are familiar with these features of the game, it is important to stop the class for a discussion about how students are recording the amounts they are winning or losing by. It is important to have this discussion because we have noticed some students do not associate the positive sign with winning and the negative sign with losing. Instead, these students simply record the difference in round and game score with a positive number regardless of whether they have won or lost the round or game (Whitacre, Bishop, Philipp, Lamb & Schappelle, 2014).

This provides a good opportunity to have an explicit discussion with students about how positive and negative signs are used in the game; the positive sign means one person is ahead relative to the other person, and the negative sign means one person is behind relative to the other person. For example, when one student draws a 9 and his partner draws a 7, the winning student can record "+2" and the losing student can record "−2" because the student who drew a 9 is ahead two points relative to his partner and the student who drew a 7 is behind two points relative to her partner. Because integers represent a relative relationship, the same situation (drawing a 9 and a 7) can be represented by two different integers (+2 and −2). In our experience students have come to express this relative relationship using expressions like "your loss is my win" or "your win is my loss."

Key discussion 2: Why can two different situations be represented by the same integer?

Another key discussion for day 1 involves pushing students to think about round or game differentials as generalised differences. To accomplish this, we usually wait until at least two pairs of students have represented different round situations (e.g., 6 and 9, and 7 and 10) with the same integer (i.e., both can be represented by −3). We then stop the game for a whole-class discussion to explore these cases. We ask students who have recorded a round differential of −3, "What cards did you draw to arrive at this difference in scores?"

Then we ask the class questions like, “Why can the same number, –3, represent both situations even though they had different cards?” These questions help students understand that –3 can mean any loss of 3, without having to refer to particular card values.

Once students have developed this idea we have found it helpful to pose a hypothetical round differential that is different from the one that we already explored. For example, we ask students to generate pairs of cards where there is a round differential of +2. This activity underscores the idea that multiple pairs of cards can create the same round differential and helps students come to see the integers they are producing as a generalised difference (Figure 3).

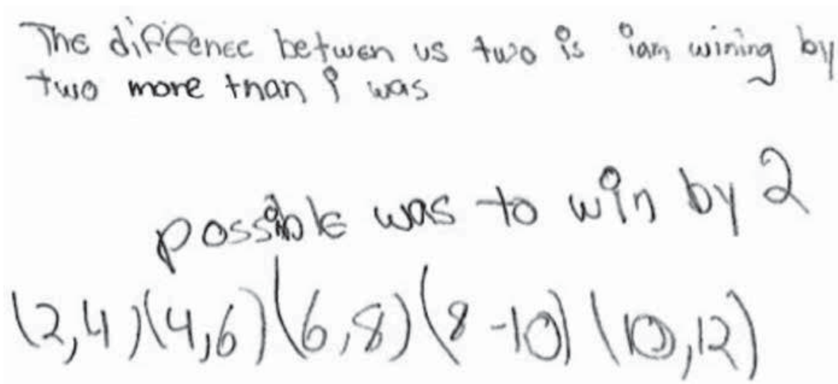
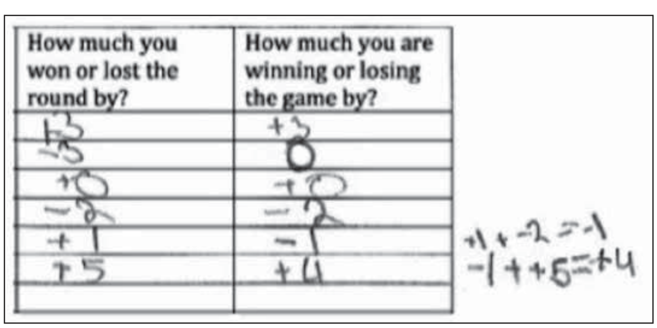


Figure 3. Possible card combinations that would allow the student to win by +2

Key discussion 3: Moving from establishing integers to working on integer addition

Up to this point, students have not used the card game to think about integer addition; they have only created integers by finding round differentials and game differentials. To use the card game to develop integer addition, we aim for students to see that they could determine the current game differential by adding the current round differential to the previous game differential. This is a shift in how students have been calculating the current game differential—they have been calculating the current game differential by finding the difference between their total score and their partner’s total score. To make the transition to integer addition, we give the whole class a problem to discuss where they can no longer find the current game differential using the method they had been using.

> **Addition in the card game**: Suppose you are losing the game by one point. You draw an 8 from the deck and your partner draws a 3. How much are you winning or losing by now?



How much you won or lost the round by?	How much you are winning or losing the game by?
+3	+3
–3	0
+0	+0
–2	–2
+1	–1
+5	+4

Figure 4. An example of using round score differentials to determine who is winning.

This problem does not allow students to use the method they had originally been using because the total scores are not given—students are only given the information that they were losing the game by one point along with the values for the cards in the current round. Figure 4 shows student work on this problem where the student came up with the equation $(-1)+(+5)=+4$ because they added the difference in round score to the prior amount they were losing the game by. This transition in thinking is often difficult for students, and so once we ask students to discuss a situation like the one above, we ask students to discuss a situation where both students draw cards whose values are the same:

> **Addition problem 2**: Suppose you are winning the game by 12 points. You draw a 6 and your partner draws a 6. How much are you winning or losing the game by now?

We have found this kind of problem to be very helpful for students who are struggling to understand the meaning of integer addition in the card game because it helps them have the insight that the game differential does not change since both people got the same score in the round. This situation can be symbolized as $+12+(0)=+12$.

Once we have had a classroom discussion about this kind of thinking we have students return to playing the card game with their partner and we circulate around the class to determine which students are now using this thinking and which students may still be struggling with this thinking. This check is how we close the first day of instruction because it helps us know who will need additional support during day 2.

Key discussion 4: Representing integer addition on the number line

Starting day 2, the card game handout is modified so that the "total game score" column is eliminated (Figure 5), which means that students now need to use the round differentials to determine who is winning or losing the game and by how much.

Round score		How much you won or lost the round by	Show your work	How much you are winning or loosing the game by
E	Mr. G			
12	3	+9		+9
13	2	+11	+9 + (+11)	+20
8	5	+3	+20 + (+3)	+23
9	11	-2	+23 + (-2)	+21

Figure 5. An example of the modified card game handout for day 2.

Because this thinking may be new for students, we have found it is helpful for them to learn to represent their answer to these questions using a number line. Therefore, we start the day by having students answer hypothetical questions like those that we used at the end of day 1. Figure 6 illustrates how a student developed a number line representation for the fourth round in Figure 5. In Figure 6 the student first marked the current game differential, +23, using an arrow between the lower Mr. G and the lower *E* on the number line; the student then used arrows to show that Mr. G scored 11 points and that *E* scored 9 points in the round; finally the student showed the new game differential with a bracket between the upper Mr. G and the upper *E* on the number line, showing that the game differential decreased by the amount of the round differ-

ential. He then symbolised the integer addition problem as (+23)+(−2)=+21.
We have found that the number line representation helps support students to treat the previous game differential and the current game differential as a generalised difference because the students represent each as the difference between their score and their partner's score without actually representing the total score of either player. This is shown in Figure 6 where the previous game differential, +23, is represented as a difference between Mr. G and *E* without numerically identifying the total score that either person had.

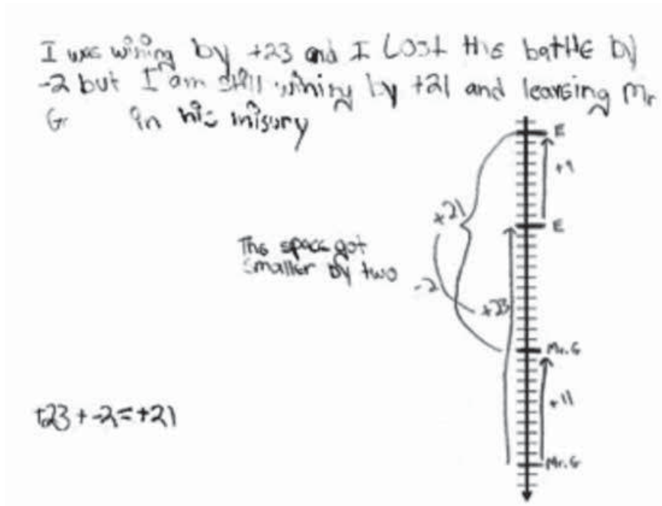


Figure 6. A student uses a number line to represent integer addition.

Once we have helped students develop a number line representation at the beginning of class, we have them continue to play the game, and circulate the classroom asking them to justify their integer addition calculations by making number line representations.

Key discussion 5: Treating all three addends as generalised differences

A final goal on day 2 is to help students develop the understanding that integer addition can be seen as a sum of two generalised differences that creates a third generalised difference. Students have begun to make this transition by treating the previous game differential and current game differential as generalised differences. However, up to this point, students have been given particular values of cards in order to calculate the round differential. To help them treat the round differential as a generalised difference we end the second day by playing a variation of the original game, which we call the "Secret Card Game". We usually model this variation of the game by having the whole class play against the teacher. The teacher and a student representative each select a card, but do not turn the cards over. The teacher looks at the cards and writes down the round differential. This is followed by a discussion similar to day 1, in which students have to come up with possibilities for the two cards' values. The teacher then allows the student to flip over his or her card. Then, the class guesses the teacher's card and determines what the current game differential would be given that they have determined the round differential. This allows students to begin treating the round differential as a generalised difference because they are determining multiple card values that could produce a particular round differential (e.g., –2 could be produced from drawing a 1 and 3, a 2 and 4, a 3 and 5, etc.). At this point, we then have students play this variation with their partner by having them alternate between the role of teacher and guesser.

Conclusion: How does this prepare students for algebra?

Two major goals of the card game are to have students understand integers as generalised differences and to have students operate with these generalised differences by adding them. The reasoning students use in this card game has the potential to support them to understand aspects of important algebra topics like slope. For example, Key discussion 1—"my win is your loss"— links to the idea in algebra that if you are finding the gradient of a line using two points, say (1,4) and (3,−2), you can consider either point as your starting or reference point (ACMNA, 214, 294). In other words, the sign of the difference in y-coordinates and the difference in x-coordinates depends on which point is the starting point. This means to determine the gradient of the line that goes through the points (1,4) and (3,−2) we want students to be able to interpret the difference between the y-coordinates, 4 and −2, as either the integer, +6 or −6, and the difference between 3 and 1, as either the integer, −2 or +2 (Figure 7). Furthermore, Key discussion 2—different situations can be represented with the same integer—leads to the idea of generalised difference and connects to understanding gradient as a measure of the ratio of differences between any two points on a line (ACMNA, 294). The reason understanding integers as generalised differences is important to the idea of gradient is that we want students to see that, for example, for any change of -6 in the y-coordinates there would be a corresponding change of +2 in the x-coordinates rather than thinking of −6 as a particular difference between y-coordinates or −2 as a particular difference between x-coordinates. While this understanding requires more than treating integers as generalised differences, understanding integers as generalised differences is an essential part of this understanding.

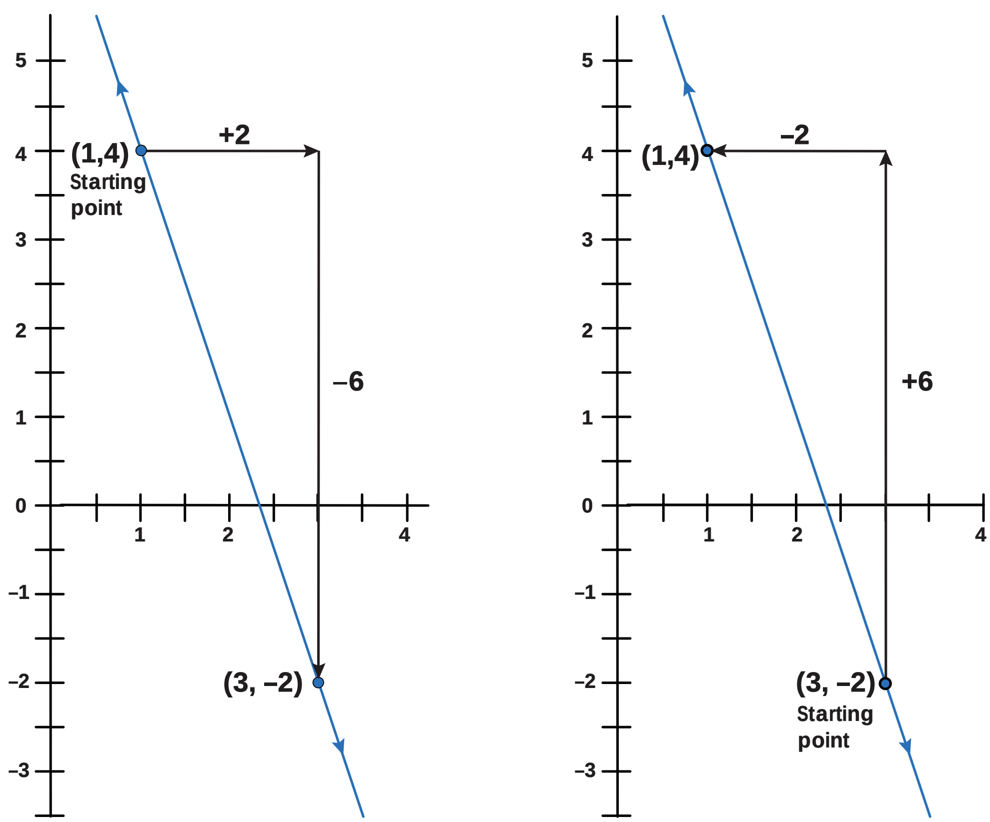


Figure 7. Finding the slope of the line in two different ways.

Key discussions 3,4, and 5 further encourage students to treat integers as generalised differences by helping them establish relationships between two numbers whose values are unknown. For example, Figure 6 in Key discussion 4 shows how +23 is simply represented as a relationship between two people's scores neither of whose values is known (ACMNA, 175). This type of thinking we consider to be of an algebraic character because it is both abstract and general, and because it pushes students to use relational thinking (Empson, & Levi, 2011). These qualities are all important to algebra because algebraic thinking relies on establishing fundamental mathematical relationships among quantities where the relationships among the quantities rather than the specific values are most important. It is in these ways that we consider this approach to integers to prepare students for the more abstract ideas they will encounter in algebra.

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