Maths games

A universal design approach to mathematical reasoning

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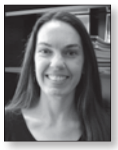

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By incorporating mathematical games in the classroom, through the principles of Universal Design for Learning, teachers are able to address mathematical content, reasoning and problem solving as well as tailoring games to address individual student's needs.

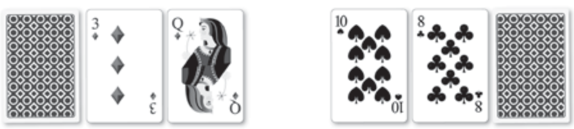
It's a tale of two tasks. Look at the activities in Figure 1. How are these tasks similar and different? Each of these activities promotes computational fluency and provides opportunities to compare sums or evaluate expressions or equations (e.g., 3 + 10 = 15, 10 + 8 = 18). In addition, both activities have the potential to promote mathematical reasoning; yet, the presentation of the tasks is very different. Task A is a sample problem on a typical worksheet, whereas Task B is structured as a game.

Task A

Are the equations true or false?
3 + 10 = 15
10 + 8 = 18

Task B

How to play: Players split a deck of cards and simultaneously flip over their top two cards.



Player 1: Sum is 13 **Player 2: Sum is 18**

The highest sum wins all 4 cards.

Figure 1. Two types of addition tasks.

In the pencil and paper version (i.e., Task A), students might find themselves working independently with few opportunities to discuss or challenge one another's reasoning. However, in Task B, pairs or small groups of students naturally engage in mathematical talk as they play the game—evaluating responses, negotiating problem-solving strategies, and collectively exploring mathematical relationships—which can extend students' reasoning and sense-making.

Bragg, Loong, Widjaja, Vale, and Herbert (2015) argued that mathematical reasoning is foundational to students' conceptual understanding. Viewing mathematical reasoning and problem solving as a critical element of mathematics instruction is evident in curricular documents such as the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2017) and the *Common Core State Standard for Mathematics* (2010). In addition, these mathematical processes are explicitly addressed as essential components of effective mathematics teaching in seminal reports such as the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* (2000), and NCTM's *Principles to Actions* (2014). For example, in *Principles to Actions*, the authors describe a strong mathematics program as one that exemplifies reasoning and sense-making, which is achieved through authentic discussions, experiences, and tasks.

Providing students with an opportunity to explore mathematical content through games allows teachers to include tasks that:

- present alternative representations of the content;

- welcome various expressions of mathematical reasoning; and
- incorporate variations that empower all students to engage in the problem solving process.

Games not only motivate students to learn mathematical concepts, but games also embody the principles of Universal Design for Learning (UDL)—a framework designed to optimise instructional decisions to meet students' needs. In this article, we describe how a teacher of six- and seven-year-old children used the game, Double Compare, to apply the principles of UDL and engage her students in mathematical reasoning and problem solving.

Games in the mathematics classroom

At first glance, incorporating games into mathematics instruction may seem arbitrary—much more like a reward for students or a haphazard, unplanned decision to fill time. However, games can be used as an effective instructional tool when the tasks involved in the game directly align to planned mathematical goals and provide all students opportunities to engage in high levels of mathematical thinking and reasoning. When teachers use mathematical games that are:

- grounded in mathematics (Swan, 2004);
- self-directed and engaging; and
- appropriately challenging to all;

they provide students with opportunities to extend their mathematical reasoning and understanding (Jackson, Taylor, & Buchheister, 2013).

Moreover, using games to explore complex mathematical ideas empowers students with a wide range of mathematical experience. As students participate in games, they exhibit critical thinking related to specific mathematical concepts, and the rich discussions that may occur during the game help deepen students' mathematical understanding as they justify their solutions and strategies (Jackson, Taylor, & Buchheister, 2013).

Games include opportunities for variation and modification, which provide multiple entry points so all students—regardless of their mathematical proficiency—can participate in strategy development and the problem solving process. Even with rules, games are flexible and embody multiple variations to accommodate students' individual needs and interests, especially when integrated with discussion questions that encourage reflection and representation (Buchheister, Jackson, & Taylor, 2015; Dockett & Perry, 2010; Jackson, Taylor, & Buchheister, 2013). As a result, presenting mathematical concepts through games becomes a valuable context in which a diverse population of students can interpret different strategies and representations of mathematical ideas through an engaging and motivating problem-solving environment that reflects the mathematical proficiencies (e.g., communication, modelling, quantitative reasoning) as described in the *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014).

Universal design for learning

Classrooms reflect widespread diversity including students with disabilities, students exceeding grade-level expectations, students from various cultural backgrounds, and students whose home language is not English (Subban, 2006). Therefore, a challenge classroom teachers' face, is how to incorporate mathematical tasks that appropriately and effectively engage all students in high quality mathematical experiences. Universal Design for Learning provides an instructional planning process that encourages classroom teachers to consciously embed appropriate accommodations and incorporate multiple entry points into cognitively demanding tasks, in order to promote reasoning and problem-solving while meeting the needs of students with a broad range of interests and skills.

Researchers (Israel, Ribuffo, & Smith, 2014; Subban, 2006) described how UDL embeds appropriate accommodations through multiple modes of:

- presentation (e.g., various representational forms such as pictorial representations or physical manipulatives);
- expression (e.g., sharing thinking through various student-selected modalities); and
- engagement (e.g., incorporating students' strengths and interests; see Figure 2).

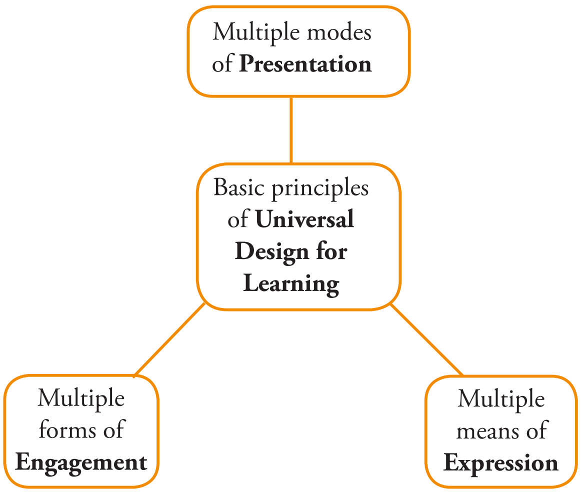


Figure 2. Basic principles of Universal Design for Learning Framework (Buchheister, Jackson, & Taylor, 2014).

By explicitly attending to these basic principles in planning and instruction, teachers can better meet the diverse needs of their students and promote strategic reasoning and problem solving through high quality tasks that effectively, and appropriately, challenge individual learners.

Incorporating multiple means of presentation supports students by representing mathematical content in various modes including discussions: stories, songs, or poems; virtual manipulatives; concrete objects; or real world contexts. When mathematical content is presented through a variety of instructional materials, such as in the context of a game and the mathematical tools used to solve problems embedded in the game, students have the opportunity to apply and conceptualise content in a way that corresponds to their learning preference and previous experiences. Students also have opportunities to translate content across multiple representational forms, thus developing a deeper conceptual understanding of the underlying mathematics (Buchheister, Jackson, & Taylor, 2014; 2017). Universal Design for Learning not only includes various ways to present content to students, but it also encourages the learner to express his or her understanding through means other than traditional pencil and paper formats, which may include manipulatives or technological tools. As students communicate their strategies and evaluate the responses of others, they can begin to articulate their mathematical knowledge more precisely and effectively. Finally, supporting the mathematical development of all learners requires that teachers engage students by attending to their interests while also providing flexible tasks, which include variations that appropriately challenge a diverse population of learners.

Double Compare – game directions

In Double Compare (also known as Addition Top-It in some curricula), students equally distribute a deck of 40 cards (digits 0–9) among 2–4 players. In each round, the players turn over the first two cards in their deck and place them in the centre. The child with the greatest sum keeps all of the cards displayed. If there is a tie for the greatest sum, the "owners" of those cards place two additional cards in the center face up, and compare the sums of the new cards. The person with the most cards at the end of the round (e.g., 15 minutes, or until the deck is "captured" by one player) wins. The game addresses the Year 1 ACARA standard in Number and Algebra in that as students play the game they are representing and solving addition problems and can be encouraged to explore a variety of strategies to manipulate and compare addends and sums (ACMNA015). Furthermore, the game provides opportunities for students to engage in reasoning and problem solving through modelling problems, explaining patterns and relationships, and discussing the reasonableness of various solutions and strategies, which are emphasised in ACARA's standards.

As part of the UDL framework, teachers anticipate obstacles students may experience during game play prior to implementing the game in the classroom. By anticipating potential barriers, teachers may embed instructional strategies and scaffolds that support students' needs thus providing opportunities that allow all students, including those who struggle with mathematics, access to rigorous mathematics. In Table 1, we provide an overview of some potential barriers for Double Compare, possible solution strategies, and how each of these correspond to the tenets of UDL.

Table 1. Potential barriers for Double Compare

Potential barrier	Solution strategy	UDL tenet
Students have difficulty adding numbers without concrete representations.	Include pictorial ten-frame cards or blocks to help students connect mathematical equations (number sentences). Use numeral cards (0–9) to support more efficient strategies like counting-on.	Multiple modes of presentation.
Students have difficulty identifying the relationship among different solution strategies.	Encourage students to use two different strategies for finding and comparing the totals for each round.	Multiple modes of expression.
Activity is not appropriately challenging to students.	Modify the game by comparing single digit or two digit numbers (without adding). Extend the activity by increasing the number of players (e.g., from two to three or four) or type (three digit vs two digit) of addends.	Multiple modes of engagement.

Double Compare–game episodes

In this section, we use the three tenets of UDL to highlight how a first grade teacher promoted reasoning and problem solving with her students as she attended to:

- presentation by scaffolding students as they translate among multiple solutions/strategies,
- expression by accepting various student-selected strategies and methods of explanation while encouraging more efficient strategies, and
- engagement, which included multiple entry points that stimulated rich mathematical discussions, while generating student interest and participation.

Presentation

Presenting and connecting multiple modes of representation during the context of game play allows learners to use their particular learning preference (e.g., kinesthetic learners prefer manipulating concrete materials such as counters or blocks; visual learners prefer to draw their interpretation of the problems). However, teachers also need to ensure they provide opportunities that move students to use more sophisticated strategies such as counting-on, making tens, or doubles. For example when Ms Aya noticed two students consistently counting the icons on the playing cards, she provided them with single-digit numeral cards (0–9) to encourage counting-on rather than counting all each time. Various modes of representation support what content is learned and how learners assign meaning to what is seen and recognised (Israel, Ribuffo, & Smith, 2014). In the following discussion, two students, Kennedy and Lennox, work with the classroom teacher using blocks (i.e., snap cubes) and ten frames to make sense of basic facts.

Figure 3. Lennox's justification attending to single addends.

Kennedy: I win again. I have 8 and you have 7.

Lennox: No. 5 + 3 = 9. I have the most. This 5 makes it one more.

Ms. Aya: You have a different answer than each other for 5 + 3. How do we know which is correct?

Lennox: 4 + 4 = 8. So we know she has 8 because that's a double. I have 5 + 3 and 5 is one more than 4 so I have one more, which makes 9. (See Figure 3).

Kennedy: But I have 4 on this side and you have 3 here. So I have one more. You have 7 and then I have 8.

Ms. Aya: Let's use our blocks and the ten frames to look at this one. It's tricky.

Kennedy: Okay. We know 4 + 4 = 8 because it's a double—easy peasy. 4 + 4 = 8 [places cubes on the ten frame with 4 in the top row and 4 below it in the bottom row]. (See Figure 4.)

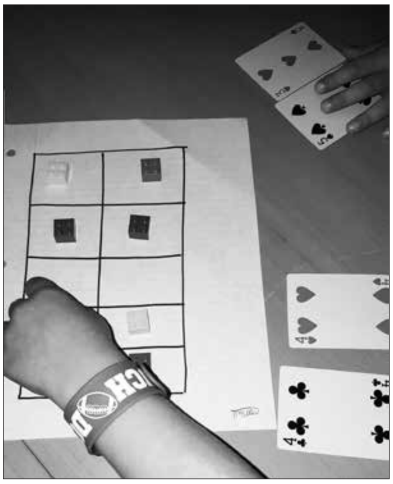


Figure 4. Ten frame showing 4 + 4.

Lennox: And I have [removes cubes from ten frame] 5 + 3 so it's 9 [places 5 on top row and 3 on bottom row]. Wait, it's 1, 2, 3, 4, 5, 6, 7, 8. It's 8, too.

Kennedy Yeah, so we tie.

Ms. Aya: Okay, let's think about this. You had 4 + 4 [puts two rows of 4 on ten frame and writes equation 4 + 4 = 8 on whiteboard]. Now Lennox, you took off all the blocks from the ten frame, but look at what we can do. We can take this one ...

Lennox: And move it here so 5 on top and 3 on bottom [moves one cube from the bottom row to the top row to fill the top row in the ten frame]. That's like mine!

Kennedy: So 5 + 3 is the same because we just moved the blocks around. 4 + 4 is the same as 5 + 3, it just looks different?

Ms. Aya: That's right. [Writes equation 4 + 4 = 5 + 3 as she talks] Four plus four is equal to five plus three. The total is the same—remember when we built it with the blocks and you both had eight? When we moved the blocks around on the ten frame we could see that the total was the same—it was still eight—but the parts that made up the total were just moved around.

Kennedy and Lennox, under the guidance of their teacher, used blocks and ten-frames to represent their mathematical reasoning. Guided by prompts and questions from Ms. Aya, the students connected two representations of the addition situation and used the structure of the ten-frame to begin making sense of a critical component of part-part-whole relations, compensation. Here, the students used the different representations and the connection to the doubles fact (4 + 4) to note the relation between the addends and the sum—when one addend is increased while the other is decreased by the same amount, the sum remains the same.

Expression

During game play, the students were encouraged to express their thinking through strategies that made sense to them such as counting-all, counting-on, drawing pictures, or using manipulatives. However, as the game progressed, they began to seek shortcuts after continuous calculations. As such, the students worked to demonstrate their thinking and reasoning through more complex number relationships.

Shawn: [finishing his tally marks for 3 + 6] See. I told you they were the same [points to Grace's cards, 5 and 4.]

Grace: But mine has the biggest number first. Mine should be more big- more than yours.

Shawn: No see. Here's my dots [points to his tallies] for 3 and 6. I have these cards and when I count them they make 9 dots. And when you do your dots [draws 5 tallies and an additional 4 tallies] and count them you get 1, 2, 3, 4, 5, 6, 7, 8, 9 too. (See Figure 5).

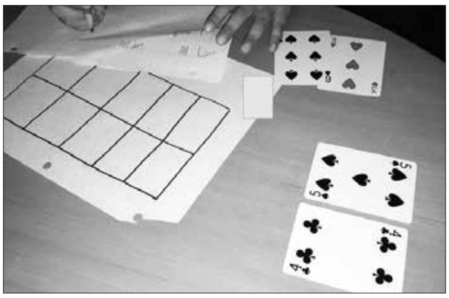


Figure 5. Shawn uses tally marks to explain.

Grace: That's weird. Our numbers are different, but we get the same answer.

Ms. Aya: That is interesting. Let's look at this idea with our blocks. Why do you think these two problems turned out to be the same? [The students each build their own equation. Grace takes her 5 counters and puts them over Shawn's 6 counters and also places her 4 counters over Shawn's 3 counters.]

Grace: I matched my biggest and his biggest and my littlest and his littlest.

Ms. Aya asked Grace what she noticed when she matched her counters to Shawn's counters. After lining the counters up with one-to-one correspondence, Grace stated that the answers were the same number. Ms. Aya prompted the students to see if they noticed anything else about the counters. Grace reasoned, "You can move this one [takes one from the group of 4 counters] over here [and moves it to the group of 5 counters]. Now I have the same number problem as Shawn!" Like Kennedy and Lennox, Grace was beginning to note more complex mathematical relations embedded in the comparison task—even when demonstrating the less sophisticated strategy of counting-all.

As the game progressed, several students sought refuge from the more tedious calculating or counting methods. Their longing for a shortcut encouraged them to express their mental maths through more sophisticated reasoning that applied part-part-whole relations. Similar observations of number patterns helped the first graders in the next example begin to move beyond counting-on or counting-all and explore alternative, more advanced solution strategies.

Jeremiah: Me! Mine two are the most. I have 4 and 6 makes [counts on fingers] 7, 8, 9, 10. What you got?

David: I don't know. I only have a 5 and a 3. Those two are smaller than both your numbers.

Kim: I have 6 plus 2 makes 8. I need two more to tie you, Jeremiah. Too bad I don't have 8 or 4.

In this exchange, Kim and David began to explore some big ideas within part-part-whole relationships (Irwin, 1996). David's comment implies he recognises he does not have to compute his addition problem because his two parts are less than Jeremiah's parts. This observation demonstrates that he understands a substantial relationship between the whole and its corresponding parts. When both of the parts are less than the two parts in another addition situation the sum will also be less. Furthermore, Kim notes that her total is two less than Jeremiah's and states that if she had drawn an eight instead of a six, or a four instead of a two, then she would have tied her competitor. In each of these examples, the students did not have to calculate to determine the winner; they were able to apply more advanced reasoning and number sense to determine and justify the greater sum.

Engagement

In the context of competing in the game, students are motivated to reinforce their knowledge of basic facts, while building more sophisticated reasoning strategies such as making tens or using doubles. As they participate, students note successes in solving equations and identifying the greater sum and when they justify their thinking strategies—or challenge the reasoning of others—the students become more engaged in the activity.

In addition, the game may be adapted with minimal modifications to appropriately challenge students with diverse mathematical experiences. Several variations of the game can include varying the process for creating numbers, altering the number of addends, or modifying the type of operation involved. For example, students can use three number-cards and compute with multiple addends. To further explore number sense and algebraic reasoning, students can play the game where they must strategise to find the greater number without calculating. Specifically, when students reveal the cards 5 and 3 in one hand and 3 and 7 in the other hand, they can reason that the sum of 3 and 7 is greater than the sum of 5 and 3 because one of the addends is the same, while the second addend is two greater. Therefore, the sum of 3 and 7 must be two greater than the sum of 5 and 3. Additionally, students may find the smallest or greatest difference, or create double-digit numbers that may be compared with base-10 blocks. These variations provide an opportunity for students to have a "new game" each time; thus continuing to engage learners in the underlying mathematics of Double Compare.

Conclusion

Effective mathematics teachers implement tasks and activities that allow all students opportunities to engage in high levels of mathematical thinking and reasoning (NCTM, 2014); however, there is often a disconnect between the needs of individual students and the type of instruction teachers implement in the classroom (Buchheister, Jackson, & Taylor, 2014; Kroesbergen & Van Luit, 2003). By incorporating games in the classroom through the principles of UDL, teachers can bridge the type of instruction alongside the needs of the individual student to positively impact students' mathematical development (Buchheister, Jackson, & Taylor, 2014; Kroesbergen & Van Luit, 2003).

The game, Double Compare, provides multiple means of presentation to support the ways in which meaning is assigned to what we see and recognise (i.e., what we learn), multiple means of expression to support strategic ways of learning (i.e., how we learn), and multiple means of engagement to support affective learning (i.e., why we learn). As teachers, it is imperative that we anticipate our students' diverse needs and subsequently provide tasks and activities that include multiple entry points so that all students may engage in mathematical tasks that promote reasoning and problem solving.

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