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Delaware Longitudinal Study of Fraction Learning: Implications for Helping Children With Mathematics Difficulties

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Abstract

The goal of the present article is to synthesize findings to date from the Delaware Longitudinal Study of Fraction Learning. The study followed a large cohort of children (*N* = 536) between Grades 3 and 6. The findings showed that many students, especially those with diagnosed learning disabilities, made minimal growth in fraction knowledge and that some showed only a basic grasp of the meaning of a fraction even after several years of instruction. Children with low growth in fraction knowledge during the intermediate grades were much more likely to fail to meet state standards on a broad mathematics measure at the end of Grade 6. Although a range of general and mathematics-specific competencies predicted fraction outcomes, the ability to estimate numerical magnitudes on a number line was a uniquely important marker of fraction success. Many children with mathematics difficulties have deep-seated problems related to whole number magnitude representations that are complicated by the introduction of fractions into the curriculum. Implications for helping students with mathematics difficulties are discussed.

Keywords

mathematics difficulties, mathematics disabilities, fraction knowledge, elementary and middle school

The Delaware Longitudinal Study of Fraction Learning examined fraction development between Grades 3 and 6—the primary instructional period for teaching fractions to students in U.S. schools. In particular, the study documents children's development of fraction concepts and procedures over eight points in time. The longitudinal design allows us to evaluate development before, during, and immediately after formal fraction instruction. We were especially interested in identifying component processes and skills that predict learning difficulties in fractions, as well as the relationship of growth in fraction knowledge to later mathematics difficulties. The goal of the present article is to summarize our key findings to date and draw implications for helping students who have mathematics learning difficulties and disabilities.

We began the project with >500 children in the winter of Grade 3. Students were drawn from schools in two adjacent school districts that followed a curriculum based on the benchmarks of the Common Core State Standards–Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), beginning when students were in Grade 4. All third graders whose parents gave informed consent and who provided assent were included in the project. Students represented a diverse range of ethnicities, socioeconomic status, and ability levels. Because we were especially interested in students with or at risk for mathematics difficulties, however, we oversampled in schools serving low-income communities. Demographics for study participants are summarized in Table 1.

Based on the integrated theory of numerical development discussed by Tian and Siegler (2017, this series), we hypothesized that knowledge of relative magnitudes of whole numbers—especially, accurate representations of these relative numerical magnitudes—would be a uniquely important predictor of learning fractions and, probably, most mathematics skills. To test this hypothesis, we examined the extent to which general cognitive and mathematics-related skills predict

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Table 1. Demographic Information for Study Participants (N = 536).

Characteristic	%
Gender	
Male	47.0
Female	53.0
Race	
White	51.8
Black	40.0
Asian/Pacific Island	5.7
American Indian/Alaskan Native	2.5
Hispanic	17.7
Low income	60.9
English language learner	10.6
Special education[a]	10.6
Learning disability	5.8
Mean age, months	105.9

Note. In third grade, 517 students were recruited for participation in this study. Thirty-six opted out before the study commenced. We replenished the sample in fourth grade (n = 27 new children) and fifth grade (n = 28 new children), resulting in a final sample of 536 students.
[a]Includes students with a diagnosed learning disability.

fraction outcomes. General predictors included measures of attentive behavior, working memory, and verbal and nonverbal ability. Mathematics-related predictors included measures of whole number line estimation acuity, nonsymbolic proportional reasoning, and whole number calculation. Our set of fraction outcomes consisted of a fraction number line estimation task, a measure of fraction concepts, and a measure of fraction arithmetic procedures. We also considered students' special education status, including whether they had been diagnosed with a learning disability, as well as income status, gender, and age.

Rationale for Processes and Skills Examined

General Cognitive Foundations

The central executive system, the language system, and the visual spatial system have all been identified as general cognitive processes that influence the acquisition of mathematical knowledge (e.g., Geary, 2004; LeFevre et al., 2010).

The *central executive* is responsible for attentional processes and working memory. Attentive behavior—specifically, students' ability to attend to instruction—is strongly associated with math achievement (Fuchs et al., 2005) and co-occurs with mathematics difficulties (Zentall, Smith, Lee, & Wieczorec, 1994). Indeed, observed differences between students with mathematics learning difficulties and those with typical achievement in fraction knowledge—which requires simultaneous attention to the numerator and denominator and multiple procedures—are mediated by students' attentive behavior (Hecht & Vagi, 2010). Working memory—or the capacity to store and manipulate information in short-term memory (Baddeley, 1986)—predicts accuracy in problem solving (Swanson, 2011), and there is a strong association between working memory and mathematics disabilities (Peng, Namkung, Barnes, & Sun, 2015).

The *language system* is crucial for comprehending mathematical terms relevant to fraction learning (e.g., "numerator," "denominator," "equivalent"). It is important during instruction and within applied word problems (Seethaler, Fuchs, Star, & Bryant, 2011).

Finally, the *visual spatial system* helps students to reason about numerical information in a spatial context, which relies in part on nonverbal or spatial reasoning (Gunderson, Ramirez, Beilock, & Levine, 2012). Nonsymbolic proportional reasoning, for example, involves seeing multiplicative relations among equivalent fractional quantities (e.g., a solution of 2 parts juice to 6 parts water will taste the same as a solution of 3 parts juice to 9 parts water; Boyer & Levine, 2012; Möhring, Newcombe, Levine, & Frick, 2015)—an ability that is related to success with fractions (Boyer & Levine, 2012).

Numerical Magnitudes

We also considered cognitive processes related to numerical magnitude representations that underlie mathematical learning and learning difficulties (e.g., Jordan, Kaplan, Ramineni, & Locuniak, 2009). The ability to represent magnitudes on a number line follows a gradual process of learning such that all real numbers have magnitudes that can be ordered along a number line for an increasingly wider range and type of numbers, and it has been identified as a unifying feature of numerical development (Case & Okamato, 1996; Siegler & Lortie-Forgues, 2014). The acquisition of fraction magnitude knowledge in particular is an important part of this process because some properties of fractions do not extend to all numbers (Siegler & Lortie-Forgues, 2014; Siegler, Thompson, & Schneider, 2011).

Children with mathematics difficulties—especially those with a specific kind of learning deficit in number and arithmetic operations called *dyscalculia* (e.g., Butterworth, 1999, 2005; Butterworth & Reigosa-Crespo, 2007; Landerl, Bevan, & Butterworth, 2004)—are characterized by a weak understanding of magnitude. For example, children with dyscalculia perform more poorly than their typically achieving peers on tasks requiring them to identify which of two numerals is larger (Butterworth & Reigosa-Crespo, 2007; Landerl et al., 2004). It has been suggested that magnitude understanding develops somewhat independently from other cognitive skills, such as language and spatial skill (Gelman & Butterworth, 2005). Although estimation of numerical magnitudes improves with age in typically achieving students, 10-year-old children with dyscalculia exhibit levels of accuracy on numerical estimation tasks similar to those of 5-year-old typically achieving children (Piazza et al., 2010).

Neurological evidence supports the idea that magnitude comprehension plays an important role in dyscalculia. In individuals without dyscalculia, the intraparietal sulcus—the region of the brain implicated in magnitude processing—is active when one processes numerical magnitude tasks. By comparison, the brains of individuals with dyscalculia show a weakened level of activation (Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007). This finding suggests a lack of parietal modulation in magnitude processing for individuals with dyscalculia. Magnitude-based tasks have been found to be sensitive diagnostic tools for identifying children with math learning difficulties and disabilities, such as those with dyscalculia (Reigosa-Crespo et al., 2012).

Whole Number Calculation Skills

In addition to numerical magnitude understanding, other mathematics-related skills are likely associated with fraction learning. We examined calculation fluency and long division skill in particular. Students with fluent knowledge of arithmetic are better able to apply this knowledge to more complex mathematical procedures (Locuniak & Jordan, 2008), such as those needed for fractions. Division of whole numbers is closely related to fractions, not only mathematically (fractions are essentially a form of division; 5/6 is the same as 5 divided by 6), but also in terms of student performance (Siegler et al., 2012; Siegler & Pyke, 2013). When using the traditional long division algorithm, students must remember to use other arithmetic operations and remember a series of steps, similar to how students must solve problems requiring advanced fraction operations. Interpreting the remainder in a long division problem also may support fraction concepts. Students must be sure that the remainder is less that the divisor (numerator < denominator): When the remainder equals the divisor, another whole group is made, and the dividend increases by 1.

Fraction Concepts and Procedures

Conceptual and procedural knowledge is relevant for learning in any mathematical domain (Geary, 2004), including fractions. *Procedural knowledge* refers to the ability to solve fraction-based problems accurately: addition, subtraction, multiplication, and division (Hecht & Vagi, 2012; Siegler, Fazio, Bailey, & Zhou, 2013). *Conceptual knowledge* involves understanding the properties of fractions, such as the meaning of the relationship between the numerator and the denominator, that an infinite number of fractions exist between any two fractions, and the difference between relative and absolute magnitudes (Hansen, Jordan, & Rodrigues, 2015; Siegler et al., 2013; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015).

Fraction concepts and procedures are intertwined; acquiring knowledge of fraction concepts supports learning of procedures and vice versa. For example, Hecht, Close, and Santisi (2003) found that fifth graders' fraction conceptual knowledge uniquely contributed to performance in fraction computation. A student with strong conceptual understanding may be more likely to detect errors when completing procedural algorithms than a student who is "following orders" in executing an algorithm, with little idea about the underlying logic or rationale (Hecht, 1998; Tian & Siegler, 2017). Conversely, students with relatively stronger procedural knowledge can use this knowledge to compensate, to a degree, for their weaker conceptual understanding (Hecht & Vagi, 2012). Not surprising, there is evidence that the development of conceptual and procedural knowledge in mathematics is bidirectional (e.g., Baroody & Ginsburg, 1986; Rittle-Johnson & Alibali, 1999; Siegler & Stern, 1998; Sophian, 1997). We hypothesized that students may demonstrate different strengths and weaknesses in fraction knowledge; therefore, we assessed fraction concepts and procedures separately.

Predictor and Outcome Measures

General Cognitive Predictors

Language (Grade 3) was assessed with the *Peabody Picture Vocabulary Test* (Dunn & Dunn, 2007), which measures receptive vocabulary. To assess attentive behavior (Grades 3 and 5), students' mathematics teachers completed the Inattention subscale of the *SWAN Rating Scale* (Swanson et al., 2006). The SWAN consists of items based on the inattention criteria for attention-deficit/hyperactivity disorder from the fourth edition of the *Diagnostic and Statistical Manual of Mental Disorders* (American Psychiatric Association, 1994). To assess nonverbal reasoning ability (Grade 3), we used the Matrix Reasoning subtest of the *Wechsler Abbreviated Scale of Intelligence* (Wechsler, 1999). Working memory (Grades 3 and 5) was assessed via the Counting Recall subtest of the *Working Memory Test Battery for Children* (Pickering & Gathercole, 2001). The students were asked to count collections of increasing length and then to recall the number of items that were counted.

Mathematics-Related Predictors

Whole number line estimation (Grades 3 and 5) was assessed with a task adapted from Siegler and Opfer (2003). Students estimated where whole numbers (one, two, and three digits) should be placed on a number line labeled "0" at the left end and "1000" at the right end. Nonsymbolic proportional reasoning (Grade 5), adapted from Boyer and Levine (2012), assessed a student's ability to judge the proportional equivalence of two proportions that are visually depicted. Items required students to either scale down from a larger target to a smaller match or scale up from a smaller target to a larger match. Addition fluency (Grade 3) and multiplication fluency (Grade 5) were assessed with the *Wechsler Individual Achievement Test* (Psychological Corp., 1992). Students had 1 min to solve one-digit addition or multiplication combinations. Long division (Grade 5) was assessed through a series of six written problems involving division with whole numbers, adapted from Siegler and Pyke (2013).

Fraction Outcomes: Grades 3–6 and 4–6

Fraction magnitude understanding was assessed with an established fraction number line estimation task (Siegler et al., 2011). For the 0–1 number line, students estimated the locations of fractions <1. For the 0–2 number line, students estimated the location of fractions <1, 1, and >1, as well as mixed numbers. Fraction concepts were assessed primarily with released fraction items from recent reports of the National Assessment of Educational Progress (2007, 2009). The items covered a range of fraction concepts, such as identifying parts of wholes, parts of a set, and equivalent fractions. Students were also asked to reason about fractions in word problems. Fraction procedures, adapted from Hecht (1998), included written fraction computation problems involving addition and subtraction (like and unlike denominators), multiplication, and division.

Mathematics Achievement

General achievement in mathematics was assessed on a subtest of the *Delaware Comprehensive Assessment System* (American Institutes for Research, 2012), which is a statewide standardized test.

A summary of the measures given at the different time points is presented in Table 2, along with overall findings from each study, further described in the next section.

Key Findings

Although we are continuing to analyze our data, the following is a summary of our key findings to date.

Finding 1

Fraction learning involves a range of cognitive processes and math-specific skills. Controlling for background variables of age, gender, and income status, results of multiple regression analyses suggest that fraction learning involves a set of numerical, behavioral, and general cognitive processes. Highlights of each are discussed here.

Grade 3 predictors of Grade 4 fraction outcomes. Overall, the ability to estimate placement of whole numbers on a number line in Grade 3 was the most important predictor of general fraction concepts and fraction procedures at the end of Grade 4, holding the other number-specific and general competencies constant (Jordan et al., 2013). As anticipated, this finding suggests that insights with whole number magnitudes give children a clear advantage in the early stage of learning fractions. The integrated theory of numerical development suggests that whole number magnitude understanding provides a foundation from which children continually expand the size and type (e.g., fractions) of magnitudes that they can accurately represent (Tian & Siegler, 2017). The whole number line estimation task and fraction knowledge both require strong proportional reasoning skills.

On general fraction concepts, number line estimation ($\beta = .27$), attentive behavior ($\beta = .23$), calculation fluency ($\beta = .17$), and verbal ($\beta = .20$) and nonverbal ($\beta = .11$) ability all made significant and unique contributions. Using many of the same or similar measures, Vukovic et al. (2014) found that general abilities in Grade 1 indirectly supported the acquisition of fraction concepts in Grade 4 via skill in whole number line estimation in Grade 2, providing further support for the findings from our study. On fraction procedures, number line estimation ($\beta = .24$), working memory ($\beta = .14$), attentive behavior ($\beta = .14$), and calculation fluency ($\beta = .13$) contributed reliably and independently to the model. Overall, the complete set of number-related and general cognitive processes predicted fourth graders' fraction concepts better than fraction procedures (56% vs. 30% of explained variance). Unobserved variables explaining the remaining variance may include instructional influences, which may account for more variance in learning procedures than concepts. For example, early in fraction learning, children may learn to carry out fraction arithmetic operations in a relatively rote fashion and not intuit any real understanding of the concepts and ideas.

Grade 5 predictors of grade 6 fraction outcomes. Whole number line estimation ($\beta = .36$), nonsymbolic proportional reasoning ($\beta = .19$), attentive behavior ($\beta = .17$), long division ($\beta = .11$), and working memory ($\beta = .11$) all contributed uniquely and significantly to fraction concepts in Grade 6 (Hansen, Jordan, Siegler, et al., 2015). On a Grade 6 measure of fraction procedures, attentive behavior ($\beta = .25$) whole number line estimation ($\beta = .18$), calculation fluency ($\beta = .17$), and long division ($\beta = .17$) all made unique and significant contributions. Overall, the entire set of Grade 5 predictors accounted for 58% of the explained variance for general fraction concepts and 40% for fraction

Table 2. General and Number-Specific Predictors of Fraction Knowledge: Findings From the Longitudinal Study.

	Predictors of fraction knowledge, spring				Predictors of low-growth trajectory membership		
	Grade 4		Grade 6		Grades 3–6		Grades 4–6
Fraction knowledge measure	Concepts	Procedures	Concepts	Procedures	Concepts	Procedures	FNLE
Predictors administered Grade 3							
Language	✓	—			✓	—	
Attention	✓	✓			✓	✓	
Nonverbal ability	✓	—			✓	—	
Whole number line estimation	✓	✓			✓	—	✓
Calculation fluency (addition)	✓	✓			✓	✓	
Working memory	—	✓			—	—	
Predictors administered Grade 5							
Attention			✓	✓			✓[a]
Whole number line estimation			✓	✓			
Working memory			✓	—			
Long division			✓	✓			
Calculation fluency (multiplication)			—	✓			✓[a]
Nonsymbolic proportional reasoning			✓	—			

Note. A checkmark (✓) indicates a significant unique predictor of fraction knowledge; a dash (—) indicates nonsignificance; and blank cells indicate that the predictor was not included in the analysis. FNLE = fraction number line estimation.
[a]These predictors were administered in Grade 4.

procedures in Grade 6. The finding that whole number estimation (which involves symbols) and nonsymbolic proportional reasoning both make independent and important contributions to fraction concepts is especially important. Proportional reasoning may be reflective of a “spatial sense” that helps students determine plausibility when comparing two fractions (Möhring et al., 2015). For example, 8 cookies for 4 people are proportionally the same as 16 cookies for 8 people (2:1). The ability to see or visualize these multiplicative relationships between equivalent fractional quantities underpins success with fractions (Boyer & Levine, 2012).

The unique association of long division to fraction knowledge, when controlling for other variables, contradicts the common assumption that skill with long division primarily reflects procedural learning. Long division problems involve place value of numbers, integration of operations, and understanding the relationship among the remainder, divisor, and dividend, which seem to be important to fraction knowledge (Siegler & Pyke, 2013). Multiplication fluency, in contrast, predicted fraction procedures but not concepts, which reflects the importance of fast fact knowledge for finding common denominators when adding and subtracting fractions (e.g., 3/5 + 2/9) and simplifying to lowest terms.

Finding 2

A significant number of children have persistent difficulties with fractions and show low growth in fraction knowledge, which predicts later mathematics failure. Our longitudinal design with multiple time points allowed us to examine growth on the fraction outcome measures. We examined growth in knowledge of fraction magnitudes as well as general fraction concepts and fraction procedures over multiple time points between Grades 4 and 6.

Growth in fraction magnitude estimation acuity. Over the course of the study, students generally increased their accuracy in estimating the location of fractions on a number line, which revealed that the fraction number line estimation acuity task is a developmentally sensitive measure (Resnick et al., in press). However, despite this overall trend toward increased accuracy across the whole sample, latent class growth analyses revealed three empirically distinct growth trajectory classes: Class 1—students (32%) who were highly accurate from the start of the study and became even more accurate; Class 2—students (26%) who were inaccurate at first but showed steep growth during the course of instruction; and, strikingly, Class 3—students (42%) who started inaccurate and showed minimal growth. Students with poor calculation fluency, weak classroom attention, and inaccurate whole number line estimation skill at the start of the study were much more likely to fall into the low-accuracy, minimal-growth group (Class 3). Importantly, growth trajectory membership predicted performance on the statewide mathematics test at the end of Grade 6—that is, 67% of students in Class 3 (starts inaccurate, ends inaccurate) did *not* meet state mathematics standards versus only 5% of Class 1 (starts accurate, ends more accurate) and 17% of Class 2 (starts inaccurate, ends accurate).

Growth in general fraction concepts versus fraction procedures. We also looked at growth on our National Assessment of Educational Progress fraction concepts measure in relation to growth on fraction procedures and uncovered underlying classes of students who showed similar growth trajectories over time (Hansen, Jordan, & Rodrigues, 2015). For fraction concepts, 23% of students fell into a low-growth trajectory. For fraction procedures, 13% of students fell into a low-growth trajectory. Growth on both measures, considered separately, predicted mathematics achievement in Grade 6: 86% of students showing low growth in fraction concepts and 79% showing low growth in fraction procedures did not meet state mathematics standards at the end of Grade 6. There was also a statistically independent relationship between fraction concepts growth trajectory membership and fraction procedures growth trajectory membership. Interestingly, about half the students who showed persistently low growth in fraction concepts showed average growth in fraction procedures. Of the students who showed persistently low growth in fraction procedures, only 24% showed average growth in fraction concepts. These findings indicate that there may be some children who learn procedures for solving fractions problems despite a weak grasp of fraction concepts. On the surface, students who calculate reasonably well with fractions in elementary school may seem prepared for more advanced mathematics. However, if their conceptual understanding is not adequately addressed, they may be at risk for experiencing later difficulties.

Implications for Students With Mathematics Difficulties

Children who were reported to be receiving special education services in school, many of whom had diagnosed learning disabilities, were disproportionately represented in all of our low-growth fraction groups, making them especially at risk for broader mathematics failure. Relative to their peers who were not reported to be receiving special education, these students were 1.6 times more likely to show low growth in fraction magnitudes, 2.5 times more likely to experience low growth in general fraction concepts, and 11.5 times more likely to experience low growth in fraction procedures.

Examination of performance on individual items further illuminates the results and the common misunderstandings among these students who demonstrate low growth in fraction understanding. Overall, we found that, relative to average-growth students, students with persistent learning difficulties with fraction concepts struggle in several areas: placing fractions on a number line, determining equivalent fractions, comparing and ordering fractions, and estimating sums of two fractions (Hansen, Jordan, & Rodrigues, 2015). Students who exhibited minimal growth in fraction magnitude understanding typically estimated fractions that are <1 and >1 as both being <1 on a number line, failing to base estimates on the relation between numerator and denominator (Resnick et al., 2016). The emphasis on fractions <1 in early fraction instruction may lead some students to view all fractions as numbers between 0 and 1 (Vosniadou, Vamvakoussi, & Skopeliti, 2008). Moreover, children with persistent problems with fraction procedures tended to operate on the numerators and denominators of fractions as though they were four separate whole numbers (e.g., 1/5 and 1/5 = 2/10), even in sixth grade.

Helping Students With Mathematics Difficulties Learn Fractions

The longitudinal research identifies cognitive factors and mathematics-related skills that affect the learning of fractions; these findings provide insight regarding a theory of change that can be used for developing fraction interventions for struggling learners. Weaknesses in whole number knowledge—especially related to numerical magnitude judgments and calculation fluency—are hallmarks of mathematics difficulties more generally (Locuniak & Jordan, 2008).

Our longitudinal research supports a number line approach to teaching fractions. Students who develop an understanding that all real numbers, including fractions, are assigned to their own location on a number line have an advantage in learning not only fractions but also algebraic concepts (Booth, Newton, & Twiss-Garrity, 2014). In this way, the student can see that fractions have unique magnitudes and, as such, that the numerator and denominator together represent one magnitude rather than two separate numbers.

Number line activities require reasoning about proportions and multiplicative relations (Barth & Paladino, 2011; Vukovic et al., 2014); thus, practice on number line tasks may improve such reasoning. Until recently, a part–whole interpretation of fractions (e.g., pie models) has been dominant in the U.S. mathematics curriculum, especially for students with learning disabilities (Siegler, Fuchs, Jordan, Gersten, & Ochsendorf, 2015). However, instruction that incorporates visual number line activities (i.e., linear representations of fractions) successfully supports at-risk students' learning of whole numbers (Ramani & Siegler, 2008) and fractions (e.g., Fuchs, Malone, Schumacher, Namkung, & Wang, 2017, this issue; Fuchs et al., 2013; Fuchs et al., 2014).

Whole number fluency also is important and should continue to be reinforced in late elementary and even middle school (Gersten et al., 2009). Fast and accurate multiplication skills facilitate reasoning about fractions (Hecht, Close, & Santisi, 2003; Seethaler et al., 2011) and can help students reason through fraction number line activities. For example, fluency with multiplication facts helps students to see, with minimal cognitive effort, that equivalent fractions (1/4, 2/8, 4/16) have the same location on the number line, as well as the mathematical relationship between improper fractions and mixed numbers (6/4 is the same as 6 × 1/4, or 1 1/2).

Long division skill emerged as a unique predictor of fraction outcomes (Hansen, Jordan, Siegler, et al., 2015). There is a natural relationship between fractions and division (3/7 is the same as 3 divided by 7). Moreover, long division emphasizes the integration of multiple operations, a skill that may be useful when solving fraction procedural problems (Siegler & Pyke, 2013). It also supports the concept that when the remainder equals the divisor, another whole must be added to the quotient.

Nonmathematical factors also may influence the effectiveness of a student's response to fraction instruction. For example, given that attentive behavior is a unique predictor of fraction knowledge and overall mathematics achievement (Fuchs et al., 2005; Hansen, Jordan, Siegler, et al., 2015; Hecht & Vagi, 2010; Jordan et al., 2013), an intervention approach that encourages students to exercise attentive behavior is likely to be beneficial. Attentive behavior allows students to stay on task and to acquire relevant knowledge and skills in their mathematics classrooms. Attentive behavior also facilitates effective strategy application on number line tasks; to understand the magnitude of a given fraction, students need to attend to both the numerator and the denominator and inhibit ineffective whole number strategies (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Meert, Grégoire, & Noel, 2010). Students must also attend to the varying end points of the number line (e.g., 0 to 1, 0 to 2, 0 to 5) when estimating fraction magnitudes; for example, low-performing students tend to always place 1/2 in the midpoint of a number line, irrespective of the end points (Rodrigues, Hansen, Jordan, & Dyson, 2015).

Preliminary Field Trial

Special education services may not be adequately addressing children's mathematics difficulties, given the prevalence of students with diagnosed learning disabilities in the low-growth fraction groups. To address this concern, our team carried out a small exploratory field trial during the final year of the Delaware study, based on our findings as well as those from the other center sites (e.g., Fuchs et al., 2013; Siegler & Pyke, 2013). Our intervention aimed to provide in-depth instruction related to the meaning of a fraction and fraction relations, two key components of "fraction sense." The meaning of a fraction refers to understanding that the relation between the fraction's numerator and denominator determines its magnitude rather than either number alone; that fraction magnitudes increase with numerator size and decrease with denominator size; that the closer the numerator is to the denominator, the closer the fraction is to 1; that a fraction with a numerator larger than the denominator is always >1 and vice versa; and that all fractions can be represented on the number line. Fraction relations refer to comparing the magnitude of two or more fractions, understanding that fractions have magnitudes that can be ordered along a number line, and finding equivalent fractions for a single magnitude on the number line.

Our intervention targeted change in numerical magnitude understanding, a critical underlying cognitive numerical structure for learning fractions, as described previously (Case & Okamoto, 1996; Siegler & Lortie-Forgues, 2014; Siegler et al., 2011). Our intervention approach emphasized several key features in terms of the presentation of mathematics content, motivation, and principles from cognitive science. In terms of mathematics content, intervention components were taught concurrently through a set of relatively few denominators to deepen understanding; specifically, students learned the denominator 2, then 4, then 8, and finally 3 and then 6. Additionally, instruction explicitly taught and connected the intervention components. To support student attention and motivation, instruction was anchored in a meaningful story line for thinking about fractions (i.e., a color run race for charity where runners have colored powder thrown on them at various points during the race). We integrated key principles from cognitive science by including activities centered on visual number lines by using gestures and other materials that minimize cognitive load and by providing systematic, cumulative practice activities to consolidate learning, develop fluency, and monitor progress. We used formative assessment in every lesson to monitor student progress.

Grade 7 math teachers in participating schools recommended students from their RTI (response to intervention) classes who specifically needed help with fractions. Although we aimed to retain low-performing students from the longitudinal portion of the study, a majority of the final sample for the intervention had not been followed longitudinally. These students were randomly assigned to our intervention group or to a comparison group. Trained researcher instructors met with small groups of seventh graders (two to four students per group) for a total of 14 lessons. Each lesson was 30 min in duration. Overall, 21 students participated in the intervention lessons offered by our research team, and 23 students were in the comparison group and received their school's mathematics intervention. Controlling for pretest scores, students in the intervention group showed significantly and meaningfully higher scores at posttest than the business-as-usual control group on measures of fraction concepts (effect size [ES] = 0.97), fraction comparisons (ES = 0.86), and fraction number line estimation (ES = 0.49). Importantly, the intervention group continued to outperform the control group 8 weeks later, with ESs exceeding 0.50 for each of the three measures.

However, students did not appear to make meaningful gains in fraction procedures. Although it has been suggested that improvements in fraction magnitude understanding can directly increase fraction arithmetic accuracy, this was not the case. Students with mathematics difficulties likely benefit most from instruction that explicitly connects procedures to concepts (Bottge et al., 2014; Fuchs et al., 2013). Our future research will address how best to help students connect fraction concepts with procedures within an RTI program, the extent to which observed gains hold over time, whether they transfer to other areas of mathematics learning (e.g., algebra), and how the findings can be replicated in authentic settings where school personnel carry out the intervention rather than trained researcher instructors. We will also target students who show the lowest performance levels on our reliable fraction screening measures (Hansen, Jordan, & Rodrigues, 2015).

Concluding Remarks

Fractions represent a crucial component of middle school mathematics. Proficiency with fractions facilitates algebra learning as well as daily life functioning. Our findings show that many students, especially those with diagnosed learning disabilities, make minimal growth in fraction knowledge between Grades 4 and 6 and that some show only a basic grasp of the meaning of a fraction after several years of instruction. Although a range of general cognitive and number-specific competencies predict fraction outcomes, the ability to estimate numerical magnitudes on a number line is a key marker of fraction success.

Children with mathematics difficulties may have deep-seated problems related to whole number magnitude representations and calculation fluency that are further complicated by the introduction of fractions into the curriculum. Although fractions differ from whole numbers in key ways (e.g., multiplication of fractions can yield an answer smaller than either multiplicand, whereas multiplication of whole numbers always leads to a larger product), both types of numbers can be assigned specific locations on a number line.

Multiplicative reasoning helps students with fraction-to-fraction equivalency; for example, when trying to determine if 4/8 is equivalent to 1/2, students must know that $2 \times 4 = 8$ (or 4 is half of 8). Arithmetic fluency reduces the cognitive load of arithmetic processes on working memory, allowing children to focus their attention on the problems being solved. Fortunately, our children's fraction sense appears to be malleable and can be supported with activities revolving around visual number lines.

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