

Reasoning for all: An instructional model



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An instructional model based on a learning progression that uses student reasoning as well as concept development is described using the comparison of fractions. How this model allows all students to access mathematical reasoning is explained.

All students should have opportunities to develop their ability to reason mathematically.

Gersten and Chard (1999) support this stance with respect to students with learning disabilities: “Even if students are not automatic with basic facts, they still should be engaged in activities that promote the development of number sense and mathematical reasoning” (p. 25). However, the typical instructional model for struggling learners does not incorporate these opportunities (Gersten & Chard, 1999). Further, although the *IES Practice Guide, Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*, advocates for “explicit and systematic” instruction that involves “verbalization of thought processes,” little emphasis is placed on the role of peers (Gersten et al., 2009, p. 6). In this article, we share an instructional model that incorporates discussion between all students and the teacher. We implemented the model to help low-performing students develop their fraction sense in the context of comparing fractions.

Comparing fractions

Number and operation sense about fractions involves the ability to compare fractions, building upon the ideas of a unit fraction (e.g., $\frac{1}{3}$) and benchmark fractions (0 , $\frac{1}{2}$, and 1), rather than relying on algorithms such as finding a common denominator (Clarke, Roche, & Mitchell, 2008; Van de Walle & Lovin, 2006). Cramer, Wyberg, and Leavitt (2009) describe four strategies that students use to reason about comparing fractions tasks. The same denominator approach (e.g., $\frac{3}{8}$ and $\frac{4}{8}$) involves reasoning about same-size parts of a whole

(e.g., eighths; four eighths are greater than three eighths). With the residual approach, fractions are considered in relation to the whole. For instance, $\frac{3}{4}$ is $\frac{1}{4}$ away from the whole and $\frac{4}{5}$ is $\frac{1}{5}$ away from the whole; therefore, $\frac{3}{4}$ is farther away from the whole and smaller than $\frac{4}{5}$ (see also Clarke et al., 2008). The other two strategies were incorporated into the four-stage learning progression designed for our work. Table 1 includes descriptions of the strategies as well as examples of student reasoning for each. The first approach, same numerator, extends the idea that unit fractions with larger denominators are smaller (Stage 1) to any set of fractions with the same numerator (Stage 2). A second approach, transitive, compares fractions to a benchmark fraction to determine the relationship between them (Stages 3 and 4) (see also Clarke et al., 2008). Within each stage, a sequence of different problem types scaffolds students’ reasoning and provides additional opportunities to apply that reasoning. To assess students’ movement through the learning progression, students completed a check-out problem at the end of each stage, and sometimes in the middle of a stage (Stages 3 and 4), as indicated by an asterisk in the third column of Table 1. (Refer to the next section for more details about the check-out problems.)

The instructional model

The *Australian Curriculum: Mathematics* Reasoning proficiency standard states: “Students are reasoning mathematically when they explain their thinking” (Australian Curriculum and Assessment Reporting Authority, 2016, *Mathematics Proficiencies*). In particular, opportunities for students to explain their thinking are important to the development and assessment of

Table 1. Description, sequence of problem types, and example of reasoning for each stage of the learning progression.

| Stage | Description | Problem types | Example | Reasoning |
|-------|--|---|---------------------------------|---|
| 1. | Same numerator limited to unit fractions. | (a) Ordering 2 fractions (b) Ordering 2 fractions (c) Ordering 3 fractions (d) Ordering 3 fractions * | $\frac{1}{5}$ and $\frac{1}{8}$ | A whole divided into five equal-size pieces will result in bigger pieces than the same whole divided into eight equal-size pieces. Therefore, $\frac{1}{5}$ is greater than $\frac{1}{8}$. |
| 2. | Same numerator not limited to unit fractions. | (a) Ordering 2 fractions (b) Ordering 3 fractions (c) Ordering 3 fractions * | $\frac{3}{5}$ and $\frac{3}{8}$ | As explained for Stage 1, $\frac{1}{5}$ is greater than $\frac{1}{8}$. Three of the bigger pieces will be greater than three of the smaller pieces. So, $\frac{3}{5}$ is greater than $\frac{3}{8}$. |
| 3. | Transitive with one fraction always equal to $\frac{1}{2}$. | (a) Ordering 2 fractions, even denominator, $\frac{1}{2}$ is smaller (b) Ordering 2 fractions, odd denominator, $\frac{1}{2}$ is smaller* (c) Ordering 2 fractions, even denominator, $\frac{1}{2}$ is greater (d) Ordering 2 fractions, odd denominator, $\frac{1}{2}$ is greater* (e) Ordering 3 fractions (f) Ordering 3 fractions* | $\frac{1}{2}$ and $\frac{3}{8}$ | $\frac{4}{8}$ is equivalent to $\frac{1}{2}$ and $\frac{3}{8}$ is less than $\frac{4}{8}$. Therefore, $\frac{3}{8}$ is less than $\frac{1}{2}$. |
| 4. | Transitive with neither fraction equal to $\frac{1}{2}$. | (a) Ordering 2 fractions, even denominators (b) Ordering 2 fractions, even denominators* (c) Ordering 2 fractions, odd denominators (d) Ordering 2 fractions, odd denominators* | $\frac{5}{6}$ and $\frac{3}{8}$ | As explained for Stage 3, $\frac{3}{8}$ is less than $\frac{1}{2}$. Further, $\frac{5}{6}$ is greater than $\frac{1}{2}$ because $\frac{3}{6}$ is equivalent to $\frac{1}{2}$ and $\frac{5}{6}$ is greater than $\frac{3}{6}$. Therefore, $\frac{5}{6}$ is greater than $\frac{3}{8}$. |

Note: An example of the problem type(s) in bold is provided in the fourth column. The * indicates that a check-out problem was implemented after all students demonstrated partial or complete reasoning for this problem type.

their fraction sense (Fennell, Kobett, & Wray, 2014). Therefore, our instructional model (Figure 1) is guided by the Mathematics Teaching Practices presented in NCTM's (2014) *Principles to Actions*, with a particular focus on elicit[ing] and us[ing] evidence of student thinking. In particular, a teacher works with a small group of three to four students, poses a problem, and asks the students to determine a solution and explain how they figured it out.

After the students document their reasoning, they share it with their peers. The teacher assesses their reasoning and moderates a discussion to remediate any misconceptions. If some or all of the students are incorrect and demonstrate faulty reasoning (no reasoning, guess attempt, or incomplete reasoning), the process is repeated with another similar problem. This cycle is broken once all students have the correct answer and partial or complete reasoning. (Refer to Table 2 for definitions

of each type of reasoning as well as an example of each in the context of comparing fractions.) At this point, all students either move on to the next problem type in the sequence and repeat the process or complete a similar check-out problem.

Check-out problems serve as benchmarks throughout the learning progression. They are completed independently, and the teacher assesses each student's documented reasoning according to the categories in Table 2. If some of the students do not get the correct answer and provide faulty reasoning, the students complete another check-out problem. If all of the students do not get the correct answer and provide faulty reasoning, the students re-enter the instructional cycle. If all of the students have the correct answer and partial or complete reasoning, they proceed to the next problem type or stage in the learning progression.

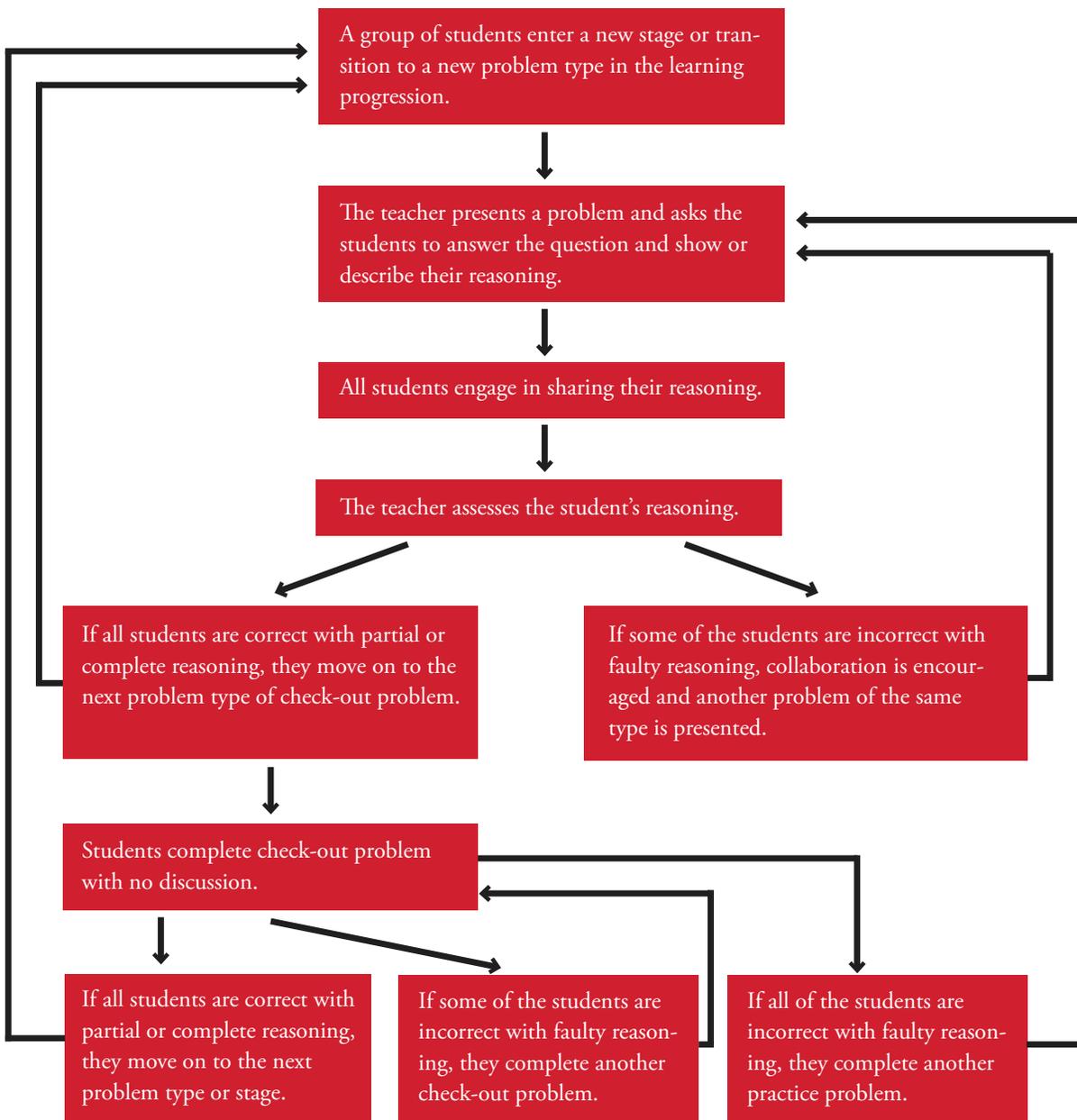


Figure 1. Flow chart representing the instructional model.

The instructional model in action

To demonstrate the instructional model in action, Figure 2 includes a discussion which occurred during field testing of the model, between a teacher and three low-performing fifth grade-students: Will, Dana, and Nora (pseudonyms). The teacher asked the students to determine whether $\frac{3}{8}$ or $\frac{1}{2}$ is bigger and explain their reasoning. The students documented their responses with a multi-modal iPad application that allowed for written (by hand or typed), audio-recorded verbal, and/or pictorial explanations. The teacher observed the students' recorded reasoning while they individually worked on the problem. When all students had the opportunity

to develop their thinking about the problem, the teacher asked students to share their ideas with the group.

Over the course of the discussion, all three students articulated their reasoning. Will (lines 4–5) displayed incomplete reasoning. Although his diagram (Figure 3) correctly depicted the fractions, he incorrectly concluded that $\frac{3}{8}$ was the larger fraction. In addition, as verbalised, Will's thinking was difficult to follow. Because the teacher had worked with this group on other fraction comparison problems, she realized Will was inappropriately generalising the idea that, even though $\frac{1}{2}$ is greater than $\frac{1}{8}$, $\frac{3}{8}$ is greater than $\frac{1}{2}$. In comparison, both Dana (lines 9–11, 13–15) and Nora (lines 17–18) exemplified complete reasoning. They both used the transitive property by equating $\frac{1}{2}$

1. **Teacher** This time we are going to figure out what is bigger: $\frac{3}{8}$ or $\frac{1}{2}$?
[Students work individually on their iPads.]
- Teacher** Will, can you share with us and show everyone your screen? Explain what you decided.
- Will** I finally get what I was supposed to do ... I put $\frac{3}{8}$ because 3 is closer to 8 and $\frac{1}{2}$ is smaller than more shaded pieces than $\frac{3}{8}$. So, yes, $\frac{3}{8}$ is greater.
5. **Teacher** And, do you guys agree or disagree?
- Dana** Disagree.
- Teacher** Why do you disagree?
- Dana** Because I feel like $\frac{1}{2}$ is greater than $\frac{3}{8}$ the same thing. Let's just say $\frac{1}{2}$ is equal to $\frac{4}{8}$.
10. **Will** So, you could see it better that way and...
 $\frac{4}{8}$ is reduced to $\frac{1}{2}$.
- Dana** $\frac{4}{8}$ is reduced to $\frac{1}{2}$. But, $\frac{4}{8}$, if you looked at it, it would be... I guess maybe it's easier for you to see that 4 is greater than 3 which makes 4 closer to one whole and it's also equal to $\frac{1}{2}$. So, it would be like this [points to picture] if you could see that.
15. **Teacher** So, what do you think Nora? We have two different things, we have...
- Nora** I have $\frac{1}{2}$ is greater because...I said $\frac{1}{2}$ is bigger than $\frac{3}{8}$ because $\frac{1}{2}$ of 8 is 4 and 3 is less than 4 so it would be less than.
- Teacher** So, if we think about that argument— and can you turn your [iPad] around again Will and look at Will's picture. Does that tie in with Will's picture— what you just said?
[Nora looks at picture.]
20. **Nora** Yes, yes, well, there, yes, because there is only 3 shaded, and, if it was half, then half of 8 would be 4, so...
- Teacher** So, do you understand what she just said? [Will nods] What did she just say?
25. **Will** That $\frac{1}{2}$ is greater.
- Teacher** Because?
- Will** Because $\frac{4}{8}$ is the same as $\frac{1}{2}$.
- Teacher** If we looked at yours— can you turn that around so we can all see it—if you were to shade in $\frac{4}{8}$...
30. **Nora** They would be equal.
- Teacher** They would be equal.
- Will** But, it's not $\frac{4}{8}$.
- Teacher** But, it's not $\frac{4}{8}$. So, what does that tell you?
- Will** That $\frac{1}{2}$ is greater. I am not doing so good today.
35. **Teacher** But, you know what? What does his picture tell us?
- Dana** The same, what we're trying to say.
- Teacher** So, your picture is saying it. You got it in your picture.
- Dana** But, maybe it would help if you would write down 8 equal pieces in your $\frac{1}{2}$ and then shade it in.
- Teacher** And we will get more opportunities to try.

Figure 2. Example of the instructional model in action. (Refer to Figure 3 for Will and Nora's diagrams.)

Table 2: Definitions and examples of each type of reasoning.

| Type of reasoning | Definition | Fraction comparison problem | Example(s) of student reasoning |
|-------------------|--|---|---|
| No reasoning | A student only provides the answer to the question OR explains a procedure without providing the conceptual underpinnings of the procedure. | $\frac{5}{11}$ and $\frac{7}{12}$ | $\frac{7}{12}$ is bigger OR $\frac{7}{12}$ is bigger because I multiply $7 \times 11 = 77$ and $5 \times 11 = 55$. |
| Guess attempt | A student guesses at the solution. | $\frac{4}{10}$ and $\frac{3}{4}$ | $\frac{3}{4}$ because I guessed. |
| Incomplete | A student uses reasoning that is incorrect (i.e., a logical fallacy). | $\frac{1}{2}$ and $\frac{7}{10}$ | $\frac{7}{10}$ are greater than $\frac{1}{2}$ because $\frac{1}{2}$ is smaller and $\frac{7}{10}$ is bigger than the squares of $\frac{1}{2}$. |
| Partial | The student uses reasoning that is logical and heading in the right direction but is not fully developed and/or articulated. | $\frac{1}{11}$, $\frac{1}{8}$, and $\frac{1}{10}$ | All of the numerators are the same and the denominator is not $\frac{1}{10}$ and $\frac{1}{11}$ is less than $\frac{1}{8}$ has greater pieces than $\frac{1}{11}$ and $\frac{1}{10}$. |
| Complete | The student uses reasoning that is founded on the intrinsic mathematical properties of the components of the problem with or without describing a procedure. | $\frac{5}{11}$ and $\frac{7}{12}$ | $\frac{5}{11} < \frac{1}{2}$ $5 \frac{1}{2} = \frac{1}{2} [\text{of } 11]$ $\frac{5}{11}$ is less than $\frac{1}{2}$ $\frac{6}{12} = \frac{1}{2}$ $\frac{7}{12}$ is more than $\frac{1}{2}$ |

to $\frac{4}{8}$ and then concluding that $\frac{3}{8}$ is less than $\frac{1}{2}$ because $\frac{3}{8}$ is less than $\frac{4}{8}$. Because Will had faulty reasoning, all three students would complete another problem of the same type. However, before moving on, the teacher attempted to remediate some of Will's misconceptions, an important component of the instructional model. Instead of directly addressing his errors, the teacher prompted a discussion between the three students. First, the teacher asked Dana and Nora if they agreed or disagreed with Will's reasoning (lines 6–8). This prompted Dana to disagree and share her thinking. Will's response (line 12) to Dana also revealed another potential misunderstanding. Will possibly did not understand the

concept of equivalent fractions; that is, he did not see their bidirectional nature reflecting the fact that the two fractions have the same value. The teacher encouraged Nora to participate in two ways. First, the teacher asked Nora to give input on the disparate views of Will and Dana (line 16). The teacher then encouraged Nora to make connections between her verbal representation of her reasoning and Will's pictorial representation of the two fractions (lines 19–21). To assess whether Will understood Nora's explanation, the teacher asked him to restate her argument (line 24). An interaction ensued (lines 25–34) that scaffolded Will in understanding that $\frac{1}{2}$ is equal to $\frac{4}{8}$ and $\frac{1}{2}$ is greater than $\frac{3}{8}$. When Will

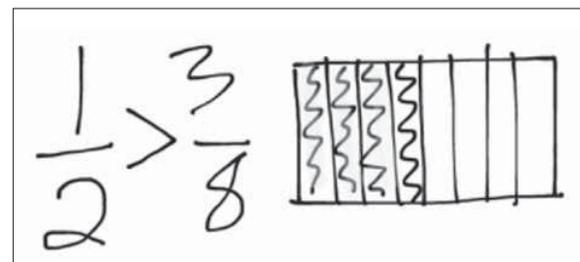
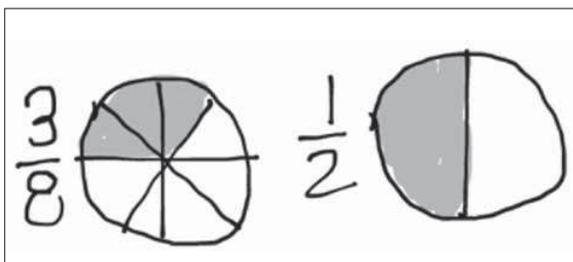


Figure 3. The diagrams from Will (left) and Dana's (right) recorded explanations.

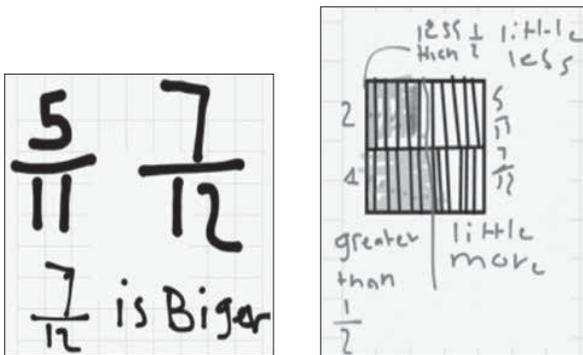


Figure 4. One student's responses for one of the questions on the pre- (left) and post-assessment (right).

acknowledged that he was struggling, the teacher provided positive reinforcement with respect to his pictorial representation of the fractions (lines 35–37) and Dana offered a suggestion for modifying his pictures to reflect the reasoning presented in the discussion (line 38). At this point, the students would re-enter the instructional cycle and complete another problem of the same type providing them the opportunity to apply what they learned (line 39).

Outcomes of the instructional model

We implemented this instructional model with 30 low-performing (below the 35th percentile) fifth- and sixth-grade students as measured on the *Wide Range Achievement Test-IV: Math Computation Subtest* (Wilkinson & Robertson, 2006). Over a five-week period (10 total hours), the students, in small groups, moved through the four stages of the comparing fractions learning progression. The instructional model guided all teacher decisions. Therefore, the way in which each group advanced through the learning progression varied depending on the reasoning and understanding of the particular students in a group.

To determine whether students' ability to reason about comparing fractions improved, we compared students' reasoning on all of the check-out problems and an assessment completed prior to and at the conclusion of the five weeks. As students proceeded through the check-out problems, the percentage of students with partial or complete reasoning increased. This result is notable as the type of problem increased in difficulty as well; that is, students demonstrated enhanced reasoning on more challenging problems. This finding was supported by the students' performance on the pre- and post-assessment. Overall, there was significant improvement in student reasoning. For instance, Figure 4 shows one student's responses on the pre- and post-test for the problem: Compare $\frac{5}{11}$ and $\frac{7}{12}$. The student shows no reasoning on the pre-test; he simply states which fraction is larger. In contrast, the student displays complete reasoning on the post-test.

He aligns two rectangles indicating his understanding that when comparing fractions the referent whole must be the same size. Although the divisions within each rectangle are inaccurate, he marks the one-half point for each fraction and uses the transitive property to determine the larger fraction.

Implications of the instructional model

We believe that the focus of the instructional model on communication, between all students and the teacher, based on student thinking in conjunction with purposeful teacher decisions, helped all students improve their reasoning about comparing fractions. We want to stress that these results occurred with low-performing students, a population for whom instruction often concentrates on repetitive practice with facts and procedures (Gersten & Chard, 1999). The instructional model can be paired with any learning progression grounded in students' reasoning around a particular concept. In addition, the model can be used with students at all ability levels in small group settings such as in-class rotations and supplemental instruction. All students, including those who struggle, have the ability to reason mathematically provided that they are given structured opportunities to share and refine their reasoning and listen to and critique the reasoning of others.

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