Addition chains: A reSolve lesson

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This article draws on some ideas explored during and after a writing workshop to develop classroom resources for the reSolve: Mathematics by Inquiry (www.resolve.edu.au) project. The project is well into its development phase, is funded by the Australian Government Department of Education and Training and conducted by the Australian Academy of Science in collaboration with the Australian Association of Mathematics Teachers. The project develops classroom and professional learning resources that will promote a spirit of inquiry in school mathematics from Foundation to year ten.

The centrepiece of reSolve is the reSolve Protocol, a framework that both underpins the development of the resources and provides a vision for excellence in teaching and learning mathematics generally. The Protocol has three key elements:

- reSolve mathematics is purposeful;
- reSolve tasks are challenging yet accessible; and
- reSolve classrooms have a knowledge-building culture.

The activity described in this article, and particularly the way in which it was developed, exemplifies these three elements. It starts with a curious number activity that students are asked to explain. The clear purpose is to show the power of algebra as generalised arithmetic, as it is only by expressing what is happening algebraically that students can explain why the ‘trick’ works. This leads to a task in which students explore other possible relations made with a similar process. The use of spreadsheets and the numerical approach make the activity accessible, while the results themselves, as will be clear later in the article, provide a level of challenge that extends well beyond school-level mathematics.

The article attempts to convey the sense of discovery and excitement that was experienced by some of the people involved in developing the lesson. It was only by talking through the problem and testing some ideas that we were able to discover that there was much more to the problem than first met the eye. This led to an exploration of some sophisticated mathematical relationships that were completely new to us that we were able to research and work on further. This is the essence of a knowledge-building culture. It is to be hoped that the same enthusiasm will infect the students for whom the lesson was designed and will inspire further investigation and generalisation both for students and for teachers.

The activity: Addition chains

The teacher invites students to think of two numbers, write them down one below the other on a whiteboard and to add them, putting the sum on a third line. Then, the second number is added to the third and the sum is appended to the list. The process continues until the list contains ten numbers. Some students are asked to write their ten numbers on the board, during which time the teacher is mysteriously and unerringly able to announce the sum of all ten numbers as soon as the seventh number in the addition chain is calculated. Table 1 shows an example of some possible numbers in the addition chain, which in this case sum to 1364.

After seeing several examples using different seed numbers, some students may realise that the sum is eleven times the seventh number. (The arrangement of numbers and sum can be implemented on a spreadsheet so that many different pairs of seed numbers can be tried quickly to test this observation.)

Table 1. The Addition chain.

n	Addition chain
1	4
2	13
3	17
4	30
5	47
6	77
7	124
8	201
9	325
10	526

This is by no means an original activity, and is shown on a number of websites and YouTube clips. However, we have never seen any further explorations of whether there might be other multiplicative relationships of this kind. Here began a gradual departure from the immediate requirements of the lesson.

In thinking about how we might extend the lesson we decided to develop a spreadsheet and look not only at the sum of ten numbers, but at the cumulative sums as they progress. We created a table such as that shown in Table 2 to look for relationships.

Table 2. Addition chain and cumulative sums.

n	Addition chain	Cumulative sums
1	4	4
2	13	17
3	17	34
4	30	64
5	47	111
6	77	188
7	124	312
8	201	513
9	325	838
10	526	1364

Looking at the numbers in the table, some other relationships may appear to be true. For example, the fourth cumulative sum appears to be 16 times the first number in the addition chain, or the seventh cumulative sum appears to be

26 times the second number in the addition chain, but it is clear that these are mere coincidence as the first two numbers in the addition chain are arbitrary. However, the sixth cumulative sum appears to be four times the fifth number in the addition chain, a result which may be coincidence but is potentially a general result. This search for possible relationships forms the body of the lesson.

Algebra enters the exercise as the tool for confirming the result for the seventh number and the sum of all ten numbers, and for confirming or rejecting other conjectures. This is where the mathematical purpose of the lesson is captured, in that algebra is essential to developing and testing generalisations. Table 3 shows the algebraic representation of the addition chain and cumulative sums.

The algebraic notation makes it clear that the tenth sum is always eleven times the seventh number no matter what the initial choices of seed numbers, since $55a + 88b = 11(5a + 8b)$.

If the sequence of numbers in the first list is given the notation t_n and those in the cumulative sum list are called s_n, we might write the following after looking at the lists as far as they go:

$s_1 = t_1$
$s_2 = t_3$
$s_3 = 2t_3$
$s_6 = 4t_5$
$s_{10} = 11t_7$

While the first result is obvious, it is curious that, if we ignore the third result, the subscripts on the t_i in the relations found above are consecutive odd numbers while there is a hint that the subscripts on the s_i might increase in jumps of 4. Curiosity led us to extend the lists to look for a pattern beyond ten numbers (Table 4).

We checked and found that indeed, this pattern continues as far as this extended list goes.

$s_{14} = 29t_9$
$s_{18} = 76t_{11}$
$s_{22} = 199t_{13}$

Table 3. Algebraic representation of the addition chain and cumulative sums.

n	Addition chain	Cumulative sums
1	a	a
2	b	$a + b$
3	$a + b$	$2a + 2b$
4	$a + 2b$	$3a + 4b$
5	$2a + 3b$	$5a + 7b$
6	$3a + 5b$	$8a + 12b$
7	$5a + 8b$	$13a + 20b$
8	$8a + 13b$	$21a + 33b$
9	$13a + 21b$	$34a + 54b$
10	$21a + 34b$	$55a + 88b$

Table 4. Extension of the addition chain and cumulative sums beyond 10 entries.

n	Addition chain	Cumulative sums
11	$34a + 55b$	$89a + 143b$
12	$55a + 89b$	$144a + 232b$
13	$89a + 144b$	$233a + 376b$
14	$144a + 233b$	$377a + 609b$
15	$233a + 377b$	$610a + 986b$
16	$377a + 610b$	$987a + 158b$
17	$610a + 987b$	$1597a + 2583b$
18	$987a + 1597b$	$2584a + 4180b$
19	$1597a + 2584b$	$4181a + 6764b$
20	$2584a + 4181b$	$6765a + 10945b$
21	$4181a + 6765b$	$10946a + 17710b$
22	$6765a + 10946b$	$17711a + 28656b$

Neither of us was aware of this result, and of course, it was begging for an explanation. We decided to first see if the coefficients of the t_i that had been revealed so far, excluding those corresponding to s_1 and s_3, {1, 4, 11, 29, 76, 199…} formed a known sequence. So we typed it into the Online Encyclopaedia of Integer Sequences (OEIS) at https://oeis.org and found that the sequence of numbers was contributed by Lekraj Beedassy on 31 December 2002. There is also a list of the first 200 numbers in the sequence at https://oeis.org/A002878/b002878.txt .

Given the appearance of the Fibonacci numbers in the lists we might suspect a Fibonacci-like process in the coefficients. According to the OEIS this sequence is the bifurcation of a particular Lucas sequence. A Lucas sequence is a generalisation of the Fibonacci sequence. In this case

$L_1 = 2$
$L_2 = 1$
$L_n = L_{n-1} + L_{n-2},\ n > 2$

Specifically, by taking every second term of the sequence of Lucas numbers {2, 1, 3, 4, 7, 11, 18, 29…} we get the sequence of coefficients above.

Alternatively, and more usefully, we observe that the sequence c_n of coefficients of the t_i is given by

$c_1 = 1$
$c_2 = 4$
$c_n = 3c_{n-1} - c_{n-2},\ n > 2$

This is a second-order linear homogeneous recurrence relation with characteristic equation $t^2 - 3t + 1 = 0$. Solving this gives $t = \frac{3\pm\sqrt{5}}{2}$.

If we now call the two distinct roots r and s, and let $c_n = Ar^n + Bs^n$, by substitution into the values for c_1 and c_2 we obtain the explicit formula for c_n:

$$c_n = \frac{\sqrt{5}-1}{2}\left(\frac{3+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}+1}{2}\left(\frac{3-\sqrt{5}}{2}\right)^n$$

This seems appealing. However, although a sequence has been named and has been characterised in various ways, the challenge remains of showing that it is correct.

It will be necessary to prove that for positive integers n, $c_n t_{2n+1} = s_{4n-2}$.

To do this, it will be helpful if each term in this proposed equation can be expressed in terms of Fibonacci numbers, F_i.

We have, for $i > 2$,

$$t_i = F_{i-2}a + F_{i-1}b \text{ and}$$
$$s_i = F_i a + (F_{i+1} - 1)\,b.$$

By inspection, it appears that $c_i = 2F_{2i} - F_{2i-1}$. This is proved by induction. It is true when $i = 3$ and we suppose it is true that $c_k = 2F_{2k} - F_{2k-1}$ for a positive integer $k > 3$.

We have $c_{k+1} = 3c_k - c_{k-1} = 3(2F_{2\mathrm{k}} - F_{2k-1}) - (2F_{2(k-1)} - F_{2(\mathrm{k}-1)-1})$.

And after several applications of the fact that $F_{n-1} = F_{n+1} - F_n$, we find that $c_{k+1} = 2F_{2k+2} - F_{2k+1}$ as required.

Thus, to prove that $c_n t_{2n+1} = s_{4n-2}$ we have to prove that

$$(2F_{2n} - F_{2n-1})(F_{2n-1}a + F_{2n}b) = F_{4n-2}a + (F_{4n-1} - 1)b$$

for all values of a and b.

This can only be the case if both the following equations hold:

$$(2F_{2n} - F_{2n-1})F_{2n-1} = F_{4n-2}$$

and

$$(2F_{2n} - F_{2n-1})F_{2n} = \mathrm{F}_{4n-1} - 1$$

We will need three lemmas about Fibonacci numbers:

(1) $F_{2k} = F_{k-1}F_k + F_kF_{k+1}$

(2) $F_{2k+1} = F_{k+2}F_{k+1} - F_kF_{k-1}$

(3) $F_{k-1}F_{k+1} = F_k^2 + (-1)^k$

Using (1), we have

$$\begin{aligned} F_{4n-2} &= F_{2(2n-1)} \\ &= F_{2n-2}F_{2n-1} + F_{2n-1}F_{2n} \\ &= (F_{2n} - F_{2n-1})F_{2n-1} + F_{2n-1}F_{2n} \\ &= F_{2n-1}(2F_{2n} - F_{2n-1}) \end{aligned}$$

as required.

Using (2), we have

$$\begin{aligned} F_{4n-1} - 1 &= F_{2(2n-1)+1} - 1 \\ &= F_{2n+1}F_{2n} - F_{2n-1}F_{2n-2} - 1 \\ &= (F_{2n} + F_{2n-1})F_{2n} - F_{2n-1}(F_{2n} - F_{2n-1}) - 1 \\ &= F_{2n}^2 + F_{2n-1}^2 + (-1)^{2n-1} \end{aligned}$$

Then, using (3),

$$\begin{aligned} F_{4n-1} - 1 &= F_{2n}^2 + F_{2n-2}F_{2n} \\ &= F_{2n}(F_{2n} + F_{2n-2}) \\ &= F_{2n}(2F_{2n} + F_{2n-1}) \end{aligned}$$

as required.

For completeness, we should prove the three lemmas concerning Fibonacci numbers that were used in the proof that $c_n t_{2n+1} = s_{4n-2}$. This can be done using induction arguments that we leave to the reader.

The addition chains lesson activity led to a chain of enquiries about a pattern of common factors found in the coefficients of a and b in the sequence of cumulative sums compared to the coefficients in the addition chain. While the explanation was not immediately obvious, persistence led to success. The activity exemplifies the three central elements of the reSolve Protocol:

purposeful mathematics, challenging yet accessible tasks, and a knowledge-building culture. In particular it promotes persistence on a challenging task, the focus of one of the reSolve professional learning modules.

The potential of this activity has not been exhausted. The algebraic lists of numbers and cumulative sums still hold mysteries. For the students, it could be noted that the coefficients of a and b in the sequence t_n are successive Fibonacci numbers and they seem never to have common factors.

This could be explained by arguing that if two adjacent Fibonacci numbers had a common factor other than 1, then because of the way the sequence of Fibonacci numbers is constructed, all Fibonacci numbers would have the same common factor. But, since successive Fibonacci numbers without common factors are easy to find, it must be that no successive pairs of Fibonacci numbers have common factors.

On the other hand, the coefficients of a and b in the sequence s_n of cumulative sums are regularly both even and do sometimes have other common factors, namely the numbers c_n. Questions remain: When, other than in the observed instances, do the coefficients of s_n have common factors and what can these factors be?

Notes

- We used Anderson (1989), *A first course in combinatorial mathematics*, as a reference to assist in the mathematical derivations.
- The addition chains activity can be found at http://www.1728.org/fibonacci.htm or http://www.pleacher.com/mp/puzzles/tricks/fibo.html.
- A YouTube clip is at https://www.youtube.com/watch?v=CWhcUea5GNc.

Reference

Anderson, I. (1989). *A first course in combinatorial mathematics*. Oxford, UK: Clarendon Press.