A comparison of two types of bank investments

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Introduction

Let us suppose a bank customer has $150 000 to invest. The bank says it can offer an account where the bank pays interest compounded daily at 3% per annum. As an alternative, the bank offers also another account, where the interest rate is 2.5% on the first $50 000 and 3.5% on any amount in excess of $50 000, where again the interest is compounded daily. The customer wishes to invest the $150 000 for, say, five years. Which account should the customer choose? If the customer were prepared to wait for 10 years instead of five, would this make a difference to the account the customer should choose? Is there much difference between the two choices? Does a small change in an interest rate lead to a possibly large change in the outcome? More generally, in what ways do the interest rates and the other variables affect the answers to such questions?

In this paper, the above type of investment problem is investigated in terms of elementary algebra, recurrence relations, functions, and calculus at high school level. The problem comes down to understanding the behaviour of a function associated with the problem and, in particular, to finding the zero of the function. A wider purpose is not only to formulate the problem mathematically and to make necessary calculations, but to think on a wider front, and to seek insight and mathematical modes of thinking in relation to general investment strategy.

An understanding of factors affecting financial and investment decisions is important both for individuals and business. In the *Australian Curriculum: Mathematics*, money and financial mathematics is a sub-strand in the *number and algebra* strand for years F–10 (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2015) and is a topic that could appear naturally within Specialist Mathematics at the senior level (ACARA, 2016). In NSW, compound interest and finance are in the Mathematics 2/3 unit HSC syllabus under applications of series (NSW Education Standards Authority, 2016). In Victoria for the VCE, recursion and financial modelling is a core part of one of the areas of study for Further Mathematics Unit 3, and an envisaged outcome is knowledge of "first-order linear recurrence relations to model compound interest investments and loans" (Victorian Curriculum and Assessment Authority, 2016).

The two investment types

The first type of investment is where a sum of money is invested with the bank and compound interest is calculated at the end of each period of a given duration. We call this a *type I* investment. The interest rate may vary over time, but we assume that it remains constant. A *fixed term account*, known also as a *term deposit* or a *term account*, offers a fixed interest rate over a given term or time and can be regarded as a type I investment, but one having a time limit. The time over which the investment is to occur may be significant, depending on the circumstances.

The second type of account is where the bank pays a certain interest rate on a given amount and then pays a generally higher interest rate on the amount in excess of the given amount, and it does this each time the interest is compounded. In this type of account, the interest rates may vary over time, but we will assume that they remain constant. This type of account we call a *type II* investment.

In fact, at least two Australian banks offer a more complicated investment than type II in that (to illustrate one case) there is one interest rate on the first \$2000 invested, a second interest rate on the next \$46 600, and a third interest rate on what is in excess of \$48 600. However, in fact it suffices to restrict analysis to the type II investment. We consider that we have a given amount to invest, and we are going to invest it all in either a type I or a type II investment.

Now if the interest rate for the type I investment is greater than the higher interest rate for the type II investment, the customer clearly is better off choosing a type I investment. On the other hand, if the interest rate for the type I investment is less than the lower of the two interest rates in the type II investment, the customer is better off investing in the type II investment. Typically, however, what one would expect is that the interest rate on the type I investment is greater than the lower rate for type II, but lower than the higher rate for the type II investment. In these circumstances one faces a decision as to which type of investment is preferable. The main question that arises is: if money were to be put into the type II investment, how long does it have to stay there before the return is greater than it would be in the type I investment?

The arguments here use the sum of geometric progressions, senior high school level calculus, the logarithm function, inequalities and compound interest (see, for example: Geha, 2000a, Topic 1 J(ii), Topic 12, Topic 14; Geha, 2000b, Topics 1(B), 8(B); Grove, 2000a, Chapter 2, Chapter 4, Chapter 7; Grove, 2010b, Chapter 3, Chapter 8). The arguments call upon a synthesis of ideas from these topics. As well, as an aim of the paper is to obtain mathematical insight, as distinct from simply carrying out calculations, there are descriptions of the thinking behind the technical results of the paper, and a qualitative pondering of the formulas arising in the comparison of the two types of investment.

Analysis of the two investment types

The type I investment is where the interest rate is fixed at, say, $a\%$ per annum, whatever the amount invested. As an ordinary fraction, $a\%$ is $\frac{a}{100}$ per annum. We assume that $a > 0$, and that the year is divided into r periods of equal duration where the interest is compounded at the end of each of these periods. Typically, the interest is compounded daily, in which case we would have $r = 365$. If we initially invest an amount $\$s_0$, after n periods there will be an amount s_n in the account where, for $n = 1, 2, 3\ldots$

$$s_n = \left(1 + \frac{a}{100r}\right)^n s_0 \tag{1}$$

This is the usual formula for compound interest. Putting

$$\gamma = 1 + \frac{a}{100r} \tag{2}$$

we can write (1) as

$$s_n = \gamma^n s_0 \tag{3}$$

for $n = 1, 2, 3\ldots$ Note that as $a > 0$, $\gamma > 1$.

The type II investment is where the bank specifies an amount u_0 and two interest rates $b\%$ and $c\%$ per annum, with $0 < b < c$. As ordinary fractions, these interest rates respectively are $\frac{b}{100}$ and $\frac{c}{100}$ per annum. The bank says that on the first u_0 dollars invested the interest is $b\%$ and on any amount in excess of u_0 the interest rate is $c\%$. The year is divided into r periods of equal duration, and the interest is compounded at the end of each of these periods. So, if we invest initially $\$t_0$, where $t_0 > u_0$, after one time period we will have an amount $\$t_1$ in the account, where

$$\begin{aligned} t_1 &= t_0 + \left(\frac{b}{100r}\right)u_0 + \frac{c}{100r}\left(t_0 - u_0\right) \\ &= \left(1 + \frac{c}{100r}\right)t_0 - \left(\frac{c-b}{100r}\right)u_0 \end{aligned} \tag{4}$$

Let us denote by t_n the amount in the type II investment after n periods. In going from the amount t_{n-1} after $n-1$ periods to the amount t_n after n periods we have, for $n \geq 1$,

$$\begin{aligned} t_n &= t_{n-1} + \left(\frac{b}{100r}\right)u_0 + \frac{c}{100r}\left(t_{n-1} - u_0\right) \\ &= \left(1 + \frac{c}{100r}\right)t_{n-1} - \left(\frac{c-b}{100r}\right)u_0 \end{aligned} \tag{5}$$

Note that the formulas (4) and (5) are intuitively ‘obvious’. The term

$$\left(\frac{c}{100r}\right)t_{n-1}$$

in (5) is the amount of interest that would be paid over period n if the interest rate of c% applied to *all* of the amount t_{n-1} available after $n-1$ periods. So, we have to subtract from this the interest we gained from applying the interest of c%, instead of b%, to the amount $\$u_0$. That is, we have to subtract

$$\left(\frac{c-b}{100r}\right)u_0$$

which is precisely the second term in (5). A similar comment applies in the case of (4), which is a special case of (5) anyway, obtained by putting $n = 1$. We see from (4) and (5) that $t_1 > t_0$ and that $t_n > t_{n-1}$ for $n = 1, 2, 3\ldots$

We would like to know how much will be in the account after n periods. So, we seek a formula for t_n. Let us put

$$\alpha = 1 + \frac{c}{100r} \text{ and } \beta = \frac{c-b}{100r} \qquad (6)$$

Note that $\alpha > 1$ and $\beta > 0$. Then from equation (5) we have for $n = 1, 2, 3\ldots$ that

$$t_n = \alpha t_{n-1} - \beta u_0 \qquad (7)$$

We see that if $n \geq 2$, using (7) with $n-1$ in place of n gives

$$t_n = \alpha t_{n-1} - \beta u_0 = \alpha(\alpha t_{n-2} - \beta u_0) - \beta u_0 = \alpha^2 t_{n-2} - \beta u_0(1+\alpha) \qquad (8)$$

Similarly, if $n \geq 3$, using (7) again with $n-2$ in place of n, and using (8), gives

$$t_n = \alpha^2 t_{n-2} - \beta u_0(1+\alpha) = \alpha^2(\alpha t_{n-3} - \beta u_0) - \beta u_0(1+\alpha) = \alpha^3 t_{n-3} - \beta u_0(1+\alpha+\alpha^2)$$

Continuing in this way, we see that

$$t_n = \alpha^n t_0 - \beta u_0\left(1+\alpha+\alpha^2+\ldots+\alpha^{n-1}\right) \qquad (9)$$

Note that, strictly speaking, the conclusion (9) requires an argument by mathematical induction (see, for example, Geha, 2000b, Topic 8).
Now, as $\alpha > 1$, we have

$$1+\alpha+\alpha^2+\ldots+\alpha^{n-1} = \frac{\alpha^n - 1}{\alpha - 1}$$

So, from (9) we deduce that

$$t_n = \alpha^n t_0 - \beta u_0\left(\frac{\alpha^n - 1}{\alpha - 1}\right)$$

which gives

$$t_n = \left(t_0 - \frac{\beta u_0}{\alpha - 1}\right)\alpha^n + \frac{\beta u_0}{\alpha - 1} \qquad (10)$$

As well, observe from (6) that

$$\frac{\beta}{\alpha - 1} = 1 - \frac{b}{c} \qquad (11)$$

Now, let us put $\theta = \dfrac{t_0}{u_0}$. As $t_0 > u_0$, $\theta > 1$. So, as $t_0 = \theta u_0$, using (6), (10) and (11) gives

$$t_n = u_0\left[\left(\theta - 1 + \frac{b}{c}\right)\alpha^n + 1 - \frac{b}{c}\right] \tag{12}$$

for all $n = 1, 2, 3\ldots$ Note that as $0 < b < c$ and $\theta > 1$, $\theta - 1 + \frac{b}{c} > 0$ and $1 - \frac{b}{c} > 0$.

In view of (3) and (12), we now know how much is in the type I account and the type II account after n time periods, when we invest $\$s_0$ in the type I account, and $\$t_0$ in the type II account. We will assume that we consider investing the same initial amount in both types of account, and then compare the outcome. So, in (5) we will take $t_0 = s_0$. Then, as $t_0 = \theta u_0$, by looking at (1) and (5) we see that the difference between the amounts in the two accounts after n periods is

$$t_n - s_n = u_0\left[\left(\theta - 1 + \frac{b}{c}\right)\alpha^n - \theta\gamma^n + 1 - \frac{b}{c}\right] \tag{13}$$

Calculus terminology and facts

The following are standard facts and terminology concerning functions at high school level (Geha, 2000a, Topic 12) and (Grove, 2010, Chapter 2). Given a function f, its derivative is denoted by f' and its second derivative is denoted by f''. Let h, k be given numbers with $h < k$. We say that a function f is *increasing* between h and k if whenever $h \le x$, $y \le k$ with $x < y$, we have $f(x) < f(y)$. Also a function f is decreasing between h and k if whenever $h \le x$, $y \le k$ with $x < y$, we have $f(x) > f(y)$. It is known from school calculus that a function is increasing when its derivative is positive, and decreasing when its derivative is negative. So, if $f'(x) > 0$ for all $h \le x \le k$ then f is *increasing* between h and k, and if $f'(x) < 0$ for all $h \le x \le k$ then f is *decreasing* between h and k. If f is increasing between h and k for all $0 < h < k$, we say that f is *increasing between* 0 *and* ∞, and if f is decreasing between h and k for all $0 < h < k$, we say that f is *decreasing between* 0 *and* ∞.

The natural logarithm of x is usually written as $\log_e x$ but, to avoid some equations appearing to be cluttered, we shall denote it by $\log x$. Note that if $x, y > 0$ having $x > y$ is equivalent to having $\log x > \log y$. Also, if $x > 0$, $x^y = e^{y \log x}$. If $g(x) = d^x$, we have $g(x) = e^{(\log d)x}$ and $g'(x) = (\log d)\, d^x$.

Analysing the difference between the two investments

We will assume that we consider investing the same initial amount in both types of account. That is, we take $t_0 = s_0$. Later we will compare the outcome. The difference between the two investments after n periods is $t_n - s_n$ and this is given explicitly by (13).

The main questions are the following: Will the type II investment eventually become more profitable than the type I investment? If so, how many periods of time must pass before the type II investment becomes more profitable than the type I investment?

The former question in mathematical form is to decide whether there is a value n such that $t_n - s_n > 0$. The latter question in mathematical form is to find the least such value of n. We will now use some calculus to examine the behaviour of $t_n - s_n$. Put

$$c_1 = u_0\left(\theta - 1 + \frac{b}{c}\right) \text{ and } c_2 = \theta u_0 \qquad (14)$$

and note that $0 < c_1 < c_2$. Now, define a function by

$$f(x) = c_1\alpha^x - c_2\gamma^x - c_1 + c_2 \text{ for all real } x \qquad (15)$$

Our interest in f derives from the fact that in view of (13), (14) and (15),

$$f(n) = t_n - s_n = u_0\left[\left(\theta - 1 + \frac{b}{c}\right)\alpha^n - \theta\gamma^n + 1 - \frac{b}{c}\right] \qquad (16)$$

for all $n = 0, 1, 2, 3\ldots$ That is, $f(n)$ is the profitability of the type II account above that of the type I account after n time periods. We call f the *profitability function.* So, if we understand something of the behaviour of the function f, we may be able to compare the two types of account.

Now, as $\alpha^0 = \gamma^0 = 1$, using (15) gives $f(0) = c_1 - c_2 - c_1 + c_2 = 0$. Also, differentiating, we see that

$$f'(x) = c_1(\log \alpha)\alpha^x - c_2(\log \gamma)\gamma^x, \text{ and}$$
$$f''(x) = c_1(\log \alpha)^2\alpha^x - c_2(\log \gamma)^2\gamma^x \qquad (17)$$

Observe from (17) that

$$f'(x) = c_1(\log\alpha)\gamma^x\left(\left(\frac{\alpha}{\gamma}\right)^x - \frac{c_2}{c_1}\left(\frac{\log\gamma}{\log\alpha}\right)\right) \qquad (18)$$

So, solving the equation $f'(x) = 0$, we see that there is a unique number x_1 such that $f'(x_1) = 0$. Similarly, we see that there is a unique number x_1' such that $f''(x_1') = 0$. Also, from (18) and with a similar argument for x_1', we see that

$$x_1 = -\frac{\log\left(\frac{c_2}{c_1}\left(\frac{\log\gamma}{\log\alpha}\right)\right)}{\log\left(\frac{\gamma}{\alpha}\right)} \text{ and } x_1' = -\frac{\log\left(\frac{c_2}{c_1}\left(\frac{\log\gamma}{\log\alpha}\right)^2\right)}{\log\left(\frac{\gamma}{\alpha}\right)} \qquad (19)$$

We see also from (18) that $f'(x) < 0$ for all $x < x_1$, so that f is decreasing between $-\infty$ and x_1. As well, it follows from (18) that $f'(x) > 0$ for all $x > x_1$, so that f is increasing between x_1 and ∞. Since $f'(x_1) = 0$ these facts imply that f has a unique minimum $f(x_1)$ at x_1, and that this minimum is 0 precisely when $x_1 = 0$. Note that if $c_2 \log \gamma < c_1 \log \alpha$, $x_1 < 0$; if $c_2 \log \gamma > c_1 \log \alpha$, $x_1 > 0$; and if $c_2 \log \gamma = c_1 \log \alpha$, $x_1 = 0$.

Using the formula for $f''(x)$ in (17), a corresponding argument yields the statement $f''(x_1') = 0$ and the facts that f' is decreasing between $-\infty$ and x_1' and increasing between x_1' and ∞. As $1 < \gamma < \alpha$,

$$0 < \frac{\log \gamma}{\log \alpha} < 1$$

and we see from (19) that the numerator in the expression for x_1 is greater than the numerator in the expression for x_1' and it follows that $x_1' < x_1$. Finally, we have noted that the minimum value $f(x_1)$ of f is 0 when $x_1 = 0$ but also, when $x_1 \neq 0$, $f(x_1) < f(0) = 0$.

As $f(n) = t_n - s_n$ from (16), we are also interested in the behaviour of $f(x)$ for positive and large values of x. We have

$$f(x) = c_1\alpha^x - c_2\gamma^x - c_1 + c_2 = c_1\alpha^x\left(1 - \frac{c_2}{c_1}\left(\frac{\gamma}{\alpha}\right)^x - \frac{c_1 - c_2}{c_1\alpha^x}\right) \tag{20}$$

As $1 < \gamma < \alpha$, the expression in the outside bracket in (20) may be made as close to 1 as we wish, for all sufficiently large x. Consequently, if $0 < \delta < 1$ it follows from (20) that for all sufficiently large x,

$$\delta c_1\alpha^x < f(x) < \delta^{-1}c_1\alpha^x \tag{21}$$

Thus, as x increases indefinitely, $f(x)$ becomes as large as we wish because α^x does so. Also, as x increases, $f(x)$ increases at the same rate as α^x.

Now, we are interested in the values of $t_n - s_n$ for $n = 0, 1, 2, 3\ldots$ So, in view of (16) we are interested only in the values of $f(x)$ for $x \geq 0$. We see that as f is increasing between x_1 and ∞, if $x_1 \leq 0$ the minimum of $f(x)$ for $0 \leq x < \infty$ is $f(0) = 0$.

On the other hand, if $x_1 > 0$, the minimum of $f(x)$ for $0 \leq x < \infty$ is $f(x_1) < 0$. In this case, we see from (21) that as $f(x)$ ultimately becomes as large as we wish, for some $x_3 > x_1$ we must have $f(x_3) > 0$. As $f(x_1) < 0$ and $f(x_3) > 0$, and as f is increasing between x_1 and ∞, there is a unique positive number x_2 such that $f(x_2) = 0$, and in this case $0 < x_1 < x_2 < x_3$. Note that, conversely, if there is $x_2 > 0$ such that $f(x_2) = 0$, then as $f(0) = 0$, f must have a maximum or a minimum at some point ξ with $0 < \xi < x_2$ at which we have $f'(\xi) = 0$. By the uniqueness of x_1 we must have $x_1 = \xi > 0$.

The following result summarises most of the above observations about the profitability function f.

Theorem 1

Let $0 < b < a < c$. Let α, β, γ be as given in (2) and (6). Let $u_0 > 0$ and $\theta > 1$. Let $c_1 > 0$, $c_2 > 0$ be as given in (14). Put $f(x) = c_1\alpha^x - c_2\gamma^x - c_1 + c_2$, for all real x. Then the following hold.

(i) $f(0) = 0$ and $f'(x) = c_1(\log \alpha)\alpha^x - c_2(\log \gamma)\gamma^x$ for all x.

(ii) If δ is a number with $0 < \delta < 1$, $\delta c_1\alpha^x < f(x) < \delta^{-1}c_1\alpha^x$ for all sufficiently large x.

(iii) Assume that $\frac{c_1}{c_2} < \frac{\log\gamma}{\log\alpha}$, and put

$$x_1 = -\frac{\log\left(\frac{c_2}{c_1}\left(\frac{\log\gamma}{\log\alpha}\right)\right)}{\log\left(\frac{\gamma}{\alpha}\right)} \tag{22}$$

Then $x_1 > 0$, the function f has a unique minimum at the point x_1 given by (22), and $f(x_1) < 0$. Also, there is a unique number x_2 such that $f(x_2) = 0$, and we then have $0 < x_1 < x_2$.

(iv) Conversely to (ii), if there is a number x_2 with $x_2 > 0$ and $f(x_2) = 0$, then $x_1 > 0$ and $\frac{c_1}{c_2} < \frac{\log\gamma}{\log\alpha}$.

(v) If $\frac{c_1}{c_2} \geq \frac{\log\gamma}{\log\alpha}$, f is increasing between 0 and ∞,

the minimum value of $f(x)$ for $0 \leq x \leq \infty$ is 0, and it occurs when $x = 0$.

Figure 1 below illustrates aspects of Theorem 1. It shows the general shape of the graph of a function f in the theorem. There is a horizontal asymptote at the positive number $-c_1 + c_2$ (recall that $c_1 < c_2$). The vertical dashed line indicates the point where the minimum of f occurs. Note the two zeros of f, one of which is zero. The two zeros may coincide at 0, but if there is a zero unequal to 0, it may occur on the right of the y-axis (as in the figure), or on the left, but the overall shape of the graph does not change. There is a point of inflection for f, as given in (19), and it always lies to the left of the point where the minimum occurs.

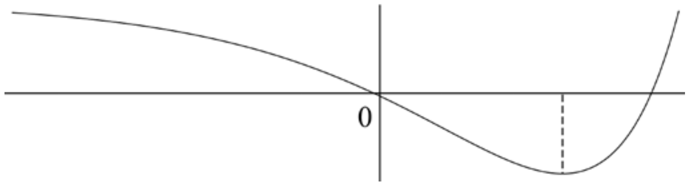

Figure 1. The general shape of the graph of a profitability function f as discussed in Theorem 1.

Comparing the two types of investment

As in the preceding section, we assume that we invest the same initial amount in both types of account, and then compare the outcome. So, we take $t_0 = s_0$. Then the difference between the amounts of the two investments after n periods is $t_n - s_n$ and this is given explicitly by (13). As mentioned, we will investigate the number of periods of time that pass before the type II investment becomes more profitable than the type I investment. This leads us to make the following definition.

Definition

The *profitability period* (in the comparison the between the type I and type II investments) is the first time period at the end of which the investment in the type II account becomes more profitable than the investment in the type I account. That is, it is the least value n such that $t_n - s_n > 0$.

There is a least value of n such that $t_n - s_n > 0$ and, as $t_0 - s_0 = 0$, the profitability period cannot be 0 and so must be a positive integer, which could be 1. We now apply the earlier analysis to comparing the two types of investments.

Let f be the function as given by (15) in the preceding section, and we use the earlier notations. We use the fact from (16) that $f(n) = t_n - s_n$ for $n = 0, 1, 2\ldots$

If $\frac{c_1}{c_2} < \frac{\log\gamma}{\log\alpha}$ by (iii) of Theorem 1, $x_1 > 0$,

there is a unique number x_2 such that $f(x_2) = 0$, and then necessarily $0 < x_1 < x_2$. We see from the definition that the profitability period is the least integer greater than x_2 (maybe check separately the cases where $x_2 \neq n$ for all $n = 1, 2, 3\ldots$ and $x_2 = n$ for some n).

If $\frac{c_1}{c_2} \geq \frac{\log\gamma}{\log\alpha}$, $x_1 \leq 0$ and by (v) of Theorem 1 f is increasing between 0 and ∞.

So, $t_0 - s_0 = f(0) = 0$, $t_1 - s_1 = f(1) > 0$, and the profitability period is 1. In this case, $t_n - s_n = f(n) > 0$ for all $n = 1, 2, 3\ldots$ and the type II investment is more profitable than the type I investment right from the beginning.

Note that in either of the preceding cases, as we see from (ii) of Theorem 1, $s_n - t_n$ increases at the same rate as

$$\left(1+\frac{c}{100r}\right)^n$$

in the sense that if $0 < \delta < 1$,

$$\delta\left(1+\frac{c}{100r}\right)^n < t_n - s_n < \delta^{-1}\left(1+\frac{c}{100r}\right)^n$$

for all sufficiently large n. Essentially, this means that in the long run the difference in profitability of the type II investment compared with the type I investment increases at the same rate as the value of the type II investment by itself.

In general it is not possible to calculate exactly the zero x_2 of the profitability function f, which means that we cannot necessarily calculate the profitability period in terms of elementary expressions. However, the profitability function may be plotted using a computer package such as Maple or Mathematica, and then the profitability period may be estimated.

In Figure 2, pictured are three profitability functions divided by the constant u_0. This division does not alter where the minimum or the zeros of the functions occur. In each case, $a = 3.0$, $b = 2.5$ and $c = 3.5$. Also, we assume interest is compounded daily, so $r = 365$. The functions are shown for $\theta = 2$, $\theta = 1.995$ and $\theta = 1.99$. The zeros of the functions respectively are exactly 1.00 and approximately 61.95 and 123.12. The profitability periods are 2, 62 and 124, measured in days. We see that the profitability period, that is the time at which the type II investment becomes superior to type I, is sensitive to θ, the amount θu_0 of the original investment when compared as a ratio with u_0.

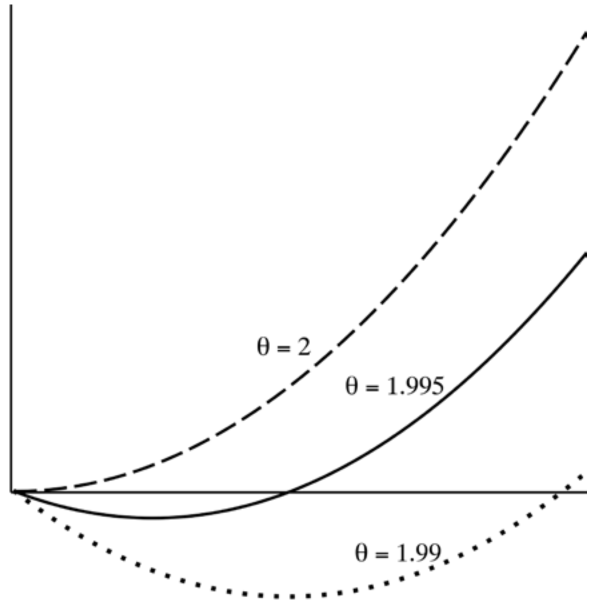

Figure 2. Graphical comparison of three profitability functions.

Now, let us look at the minimum amount invested that will ensure that the type II investment is more profitable than the type I investment right from the beginning. Put

$$\psi = \frac{c_2 \log \gamma}{c_1 \log \alpha} = \frac{\theta}{\left(\theta - 1 + \frac{b}{c}\right)} \cdot \frac{\log\left(1 + \frac{a}{100r}\right)}{\log\left(1 + \frac{c}{100r}\right)}$$

We see from the above, or from Theorem 1, that if $\psi \le 1$ the profitability period is 1, and that if $\psi > 1$ the profitability period is at least 1. Now, $\psi \le 1$ corresponds to having

$$\theta \ge \frac{1 - \frac{b}{c}}{1 - \frac{\log \gamma}{\log \alpha}} \qquad (23)$$

in which case the amount invested is at least

$$\left(\frac{1 - \frac{b}{c}}{1 - \frac{\log \gamma}{\log \alpha}} \right) u_0$$

and the type I investment is more profitable than the type II investment right from the beginning. Note that in (23), differentiating $\frac{\log x}{x}$ enables us to show that $\dfrac{\log x}{x}$ is decreasing between 0 and ∞, from which it can be shown that

$$\frac{1-\frac{b}{c}}{1-\frac{\log\gamma}{\log\alpha}}>1$$

Recalling that $\theta > 1$, we see that (23) imposes a genuine restriction on θ.

The profitability period may be calculated explicitly when $\alpha = \gamma^2$, a case we now investigate.

The quadratic case: The profitability period when $\alpha=\gamma^2$

When $\alpha = \gamma^2$, we have from the definition of the function *f* in (15) or Theorem 1 that

$$f(x)=u_0\left[\left(\theta-1+\frac{b}{c}\right)\left(\gamma^x\right)^2-\theta\gamma^x+1-\frac{b}{c}\right] \tag{24}$$

a quadratic expression in γ^x. Then, by using (2), (6) and (17), and some elementary manipulations, the condition in (ii) of Theorem 3 that

$$\frac{c_1}{c_2}<\frac{\log\gamma}{\log\alpha}$$

becomes, in terms of θ, *b*, *c*

$$\theta<2\left(1-\frac{b}{c}\right) \tag{25}$$

As $\theta > 1$, (25) imposes the condition

$$\frac{b}{c}<\frac{1}{2}$$

When (25) is satisfied by (iii) of Theorem 3, the minimum of *f* occurs at x_1, where

$$x_1=-\frac{\log\left(\left(\dfrac{\theta}{\theta-1+\frac{b}{c}}\right)\cdot\dfrac{\log\gamma}{\log(\gamma^2)}\right)}{\log\left(\dfrac{\gamma}{\gamma^2}\right)}=\frac{\log\left(\dfrac{\theta}{2\left(\theta-1+\frac{b}{c}\right)}\right)}{\log\left(1+\dfrac{a}{100r}\right)}.$$

Also, when $\alpha = \gamma^2$, when *f* is written as in (24), the equation $f(x) = 0$ becomes

$$c_1\left(\gamma^x\right)^2-c_2\gamma^x-c_1+c_2=0$$

Factorising, we have

$$c_1(\gamma^x)^2-c_2\gamma^x-c_1+c_2=c_1\left(\gamma^x-1\right)\left(\gamma^x-\frac{c_2-c_1}{c_1}\right)$$

Thus, $f(x) = 0$ is equivalent to having $\gamma^x = 1$ or $\gamma^x=\dfrac{c_2-c_1}{c_1}$. Thus, one solution of $f(x) = 0$ is $x = 0$, and the other is x_2, where

$$x_2 = \frac{\log\left(\frac{c_2 - c_1}{c_1}\right)}{\log\gamma} = \frac{\log\left(\frac{1-\frac{b}{c}}{\theta - 1 + \frac{b}{c}}\right)}{\log\left(1 + \frac{a}{100r}\right)} \tag{26}$$

Note that $x_2 > 0$, since we deduce from (25) that

$$\frac{1-\frac{b}{c}}{\theta - 1 + \frac{b}{c}} > 1$$

So when $\alpha = \gamma^2$, (26) gives an explicit solution for the positive zero of f, and from this we can find the profitability period.

The equation $\alpha = \gamma^2$, in terms of a, c and r is

$$1 + \frac{c}{100r} = \left(1 + \frac{a}{100r}\right)^2$$

So, with $r = 365$ we have equivalently

$$c = 36\,500\left(\left(1 + \frac{a}{36\,500}\right)^2 - 1\right) = 2a + \frac{a^2}{36\,500}$$

which gives that $2a$ approximates c to within 3 decimal points, provided that $a < 4$, say. But note that we must have $2a < c$. Thus, if we have $\alpha = \gamma^2$ and $a = 2.9$, we would have c is approximately 5.8. Now, one bank currently offers an interest rate of 2.9% for a type I investment. However, in the present financial environment, it is not likely that a bank will offer an interest rate of 5.8% for a type II investment. In general, but maybe depending on u_0, one would expect that a bank would make the interest rates a and c quite close, with the consequence that $\alpha \neq \gamma^2$.

Conclusions and further investigations

We can look on the analysis here as a comparison between two processes. In the one case, the response is constant over time while, in the other case, the response is muted for a period and then becomes more marked than in the first case. In the long run, the delayed but more marked response of the second case dominates over the first. Here, we carried out the analysis in a comparison of two types of investment, and with a view to establishing the time at which the second type of investment becomes superior to the first, with the aim of assisting in investment decision making. The fundamental equation (16) shows the precise difference in behaviour of the two investments, and how the outcome depends upon the parameters. The spirit of the analysis has an affinity with calculations in business of compound interest, the value of annuities and the present or future value of possible investments

(Hummelbrunner & Combes, 2012). However, complications arise from the 'delayed response' in the type II investment.

The dependence of the outcomes upon the parameters received some discussion, but there are other aspects not discussed which could be worthy of further thought with a view to encouraging mathematical thinking at a more general level. Some of these are mentioned below. A comment of N. Bourbaki is apposite: "every mathematician knows that a proof has not really been 'understood' if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed" (Bourbaki, 1950, p. 223).

1. The function f in (15) is given by $f(x) = c_1\alpha^x - c_2\gamma^x - c_1 + c_2$, and the zero of f is related to the profitability period of the type II investment over that of type I. It seems clear intuitively that if θ is increased in value—that is, if more money is invested—then the profitability period should decrease, unless it is already 1. Despite the difficulty in calculating x_2, provide a proper argument that confirms this intuition.
2. In the (equivalent) formulas (13) and (16), explain in words where possible what happens in each case if one of the following occurs: b becomes larger, b becomes smaller, u_0 becomes larger or smaller, a becomes larger or smaller.
3. In the case when $\alpha = \gamma^3$, can an analysis be carried out along the lines as described for when $\alpha = \gamma^2$?
4. Suppose that the bank offers a type I investment as described, but offers instead of the type II investment a type III investment where there is an annual interest rate of b% on the first \$$u_0$, c% on the amount between \$$u_0$ and \$$u_1$ (where $u_0 < u_1$), and d% on the amount over \$$u_1$. We assume $b < c < d$, and that \$$v_0$ is initially deposited into the type III account, where $v_0 > u_1$. We let v_n be the amount in the type III account after n periods. Show that, for $n \geq 1$,

$$v_n = \left(1 + \frac{d}{100r}\right)v_{n-1} - \left(\frac{c-b}{100r}\right)u_0 - \left(\frac{d-c}{100r}\right)u_1.$$

   Compare this with equations (6) and (7), and deduce that a type III investment is equivalent to a type II investment, when the parameters are suitably adjusted. At the time of writing, a prominent Australian bank offers a type III investment where there is 0.5% for the balance up to \$2000, 1.5% on the balance between \$2000 and \$48 600, and 3% on the remaining balance (if any) above \$48 600. This is a type III investment with $r = 365$, $u_0 = 2000$, $u_1 = 48\,600$, $b = 0.5$, $c = 1.5$, $d = 3.0$. The bank also offers a type I account with $r = 365$ and $a = 2.9$. So, an analysis along the preceding lines would apply to these investments.
5. In the quadratic case, we see from (24) that x_2 increases as θ decreases. However, as θ decreases to 1, x_2 increases to

$$\frac{\log\left(\frac{c}{b-1}\right)}{\log\left(1+\frac{a}{100r}\right)}$$

so there is an upper bound on the profitability period, regardless of the amount invested, as long as the latter is greater than u_0. Although we cannot always calculate x_2 in the non-quadratic case show that, nevertheless, there is always an upper bound on the profitability period, regardless of the amount invested, as long as the amount is greater than u_0. (This may need use of the generalised mean value theorem of calculus, but one might also consider what happens as θ approaches 1 with $\theta > 1$.)

6. The analysis here has emphasised the case of interest compounded daily —that is, when $r = 365$. However, how is the analysis affected when r is varied?
7. The analysis here assumed that the amount invested was larger than the 'threshold' amount u_0. This corresponded to having $\theta > 1$. But what happens when the amount invested is smaller than u_0—that is, when $\theta < 1$?
8. How is the analysis affected when the interest rates are varied, rather than the amount of the money invested?

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