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MATHEMATICS TEACHERS' KNOWLEDGE OF STUDENT THINKING AND ITS EVIDENCES IN THEIR INSTRUCTION

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Abstract

The aim of this case study is to examine mathematics teachers' knowledge of students' thinking and its evidences in their teaching. The participants were three secondary mathematics teachers. Data were gathered from interviews and observations. While analyzing the data, the framework about teachers' knowledge of students' thinking was used. The findings showed that each teacher mainly considered the knowledge of students' thinking as knowing students' prior knowledge. They expressed that they benefited from the questions to reveal students' ideas, encouraged their students to use different solution ways for the problems, and had ideas on misconceptions and difficulties their students might be confronted. The participants also considered students' prior knowledge in their lessons, but they did not tackle their difficulties, errors and misconceptions unless students asked questions to them. They had the limited approaches for building on students' mathematical ideas, promoting students thinking mathematics, triggering and considering divergent thoughts, engaging students in mathematical learning, and motivating students learning.

Keywords: Knowledge of Content and Student, Knowledge of Students' Thinking, Mathematics Teachers

Abstrak

Tujuan dari studi kasus ini adalah untuk menguji pengetahuan siswa guru matematika berpikir dan bukti-bukti dalam pengajaran mereka. Para peserta tiga guru matematika sekunder. Data dikumpulkan dari wawancara dan observasi. Sementara menganalisis data, kerangka tentang pengetahuan siswa guru berpikir digunakan. Temuan menunjukkan bahwa setiap guru terutama dianggap sebagai pengetahuan siswa berpikir dengan mengetahui siswa pengetahuan sebelumnya. Mereka menyatakan bahwa mereka mendapat manfaat dari pertanyaan-pertanyaan untuk mengungkapkan ide-ide siswa, mendorong siswa mereka menggunakan cara-cara solusi yang berbeda untuk masalah, dan memiliki ide-ide tentang kesalahpahaman dan kesulitan siswa mereka mungkin dihadapi. Para peserta juga dianggap pengetahuan siswa sebelumnya dalam pelajaran mereka, tetapi mereka tidak mengatasi kesulitan mereka, kesalahan dan kesalahpahaman kecuali siswa mengajukan pertanyaan kepada mereka. Mereka memiliki pendekatan yang terbatas untuk membangun ide-ide matematika siswa, mempromosikan siswa berpikir matematika, memicu dan mengingat pengalaman yang berbeda, melibatkan siswa dalam belajar matematika, dan memotivasi siswa belajar.

Kata kunci: Pengetahuan tentang Konten dan Mahasiswa, Pengetahuan Pemikiran Siswa, Matematika Guru

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The mathematics teachers' content knowledge and also the interaction between their knowledge about students and the knowledge of their learning ways affect students' conceptual learning (Hill & Ball, 2004; Loong, 2014). Hill, Ball, and Schilling (2008) stated that there was an agreement about the fact that the mathematics teachers who realize effective teaching have the knowledge of students' thinking. Noticing and focusing students' thinking has inherently complex structure. Student-centered instruction can be implemented if the teachers reflect their "knowledge of students' thinking (KoST)" in their teaching. Anwar, Budayasa, Amin, and Haan (2014) explained that the mathematics teachers should to change their instructions to include the activities that will allow the students to develop mathematical thinking. Kung and Speer (2009) expressed that KoST was one of the fundamental components of pedagogical content knowledge. KoST provides that the mathematics teachers recognize their students' mistakes, misunderstanding, ideas and thoughts (Empson & Junk, 2004). The researches handled KoST and tried to explain it with different components. For example, An, Kulm, and Wu (2004) stated four components as addressing students' misconception, building on students' mathematical idea, engaging students in mathematics learning, promoting students' thinking mathematically. Based on the research of An, Kulm, and Wu (2004), Lee (2006) added some components to their frame. These components were triggering different ideas, motivating students learning, evaluating students' understanding and using prior knowledge. When considered KoST and its component, it is thought to be important that teachers reflect their KoST in their lessons for students to learn mathematical concepts accurately, to gain different perspectives related to concepts to use them in practice. In this study, firstly it was tried to be determined what mathematics teachers think about KoST, then to be examined the evidences of teachers' KoST in lessons in a detail way. While we were examining mathematics teachers' KoST, we considered Özaltun (2014) KoST framework included the nine components and their contents. The components were named as building on students' mathematical ideas, promoting students thinking mathematics, triggering and considering divergent thoughts, engaging students in mathematical learning, evaluating students' understanding, motivating students learning, considering students' misconceptions and errors, considering students' difficulties and estimating students' possible ideas and approaches.

The aim of this study is to examine three mathematics teachers' KoST and its evidence in their teaching. In this direction, we tried to answer the questions of "What do the mathematics teachers think about the KoST?" and "What are the evidences that indicate the existence of the mathematics teachers' KoST in their teaching?".

METHOD

The study was based on a qualitative case study design because we aimed to examine the teachers' KoST and its evidences in their teaching with KoST framework in a detail way. We handled the mathematics teachers' in-class teaching and actions as cases. By the framework, we could determine the selection criteria for the situations that be needed to examine from huge amount of data sources that were collected in different ways. Also, we could explain the indicators of the framework with the emergent situations from the cases. The participants of the study were three mathematics teachers [Ali (male), Ozden (female), Serin female)] working at a high school.

The data were gathered from the interviews and observations regarding two-hour lessons for each teacher. Firstly, we conducted the semi-structured interviews for revealing teachers' ideas about planning, KoST and the content of KoST. After the interviews, we observed their lessons to determine how they reflected their existing KoST. A video camera was used during the interviews and two video cameras were used during the lessons. Then, all the video records were verbatim transcribed. We also took the field notes while observing the lessons. These notes included especially the teachers' actions for evidence of teachers' KoST. While analyzing the data, we independently examined the transcripts based on the KoST framework. We considered which approaches can be related to KoST on the sections of the lessons and which practices performed by the participants while considering student thinking. We examined the sections that could be related to student thinking. Then, we compared our analysis with each other and arrived at a consensus.

RESULTS AND DISCUSSION

In the context of the findings, the lessons of the three teachers were presented by identifying with the content of KoST and were supported with the excerpts taken from the interviews and their lessons.

Ali's Knowledge of Student Thinking

Ali expressed the necessity of determining the students' prior knowledge related to the concepts. Especially, he stated that he asked questions about prior knowledge while introducing new topic, he tried to tackle students' prior knowledge while he was providing for students to understand the definitions of the concepts. An excerpt of the Ali's interview was as follows:

> Especially, at the beginning of the lesson and when we need more definitions, namely, when we want to make mathematical definitions of the concepts, I relate the new concept or subject to prior knowledge and then, if it is necessary, I try to support their understanding related to the prior knowledge with several questions. It is definitely necessary to know what they learnt before. [Ali-interview]

Ali stated that he could estimate students' possible difficulties in the context of the subject by depending on his experience. Also, he stated that he dealt with difficulties in a detail way, gave many examples andhe was mostly aware of students' misconceptions and errors about the concepts and he asked questions about them. Also, he emphasized the importance of knowing students, and mentioned that he could estimate his students' thoughts and their errors. When the students made an error, he asked those students to do their solutions on the board and then by asking questions to them he provided for them to be able to overcome the incorrect solutions. An excerpt from Ali's interview was given below:

> It is evident in which each student will make errors and even, which student can make errors… each student tried to solve the problems in different ways, some of them try to remember, for example, a student can make an error because he did not remember the knowledge, another student can solve by interpreting the questions. As we can understand students' thinking and strategies, we need more experience. [Ali-interview]

As he believed that it was important to present different solutions in geometry, he included different solutions in his lessons. An excerpt from Ali's interview was given below:

> I absolutely want to tackle different solutions on the board. For example, asking a question about triangle, maybe area or lengths. When a student solved a question and other one said that she/he solved differently, I definitely ask her/him to make her/his solution on the board. Especially in

geometry, I believe that the more we solve different questions, the more the students easily understand. [Ali-interview]

According to the interview with Ali, it was apparent that Ali considered the contexts of knowing and considering students' prior knowledge to reflect KoST on his teaching; asking questions to reveal students' thoughts and increase their existing understanding; knowing students' difficulties, misconceptions and errors, presenting approaches to overcome these; predicting students' thoughts and possible solutions and encouraging them to produce different ways of solutions.

Ali conducted two hour lessons about "finding the greatest common divisor (GCD) and the least common multiple (LCM) of two or more polynomial functions". When Ali's lessons were examined in terms of building on students' mathematical ideas, it was seen that Ali determined his students' prior knowledge by asking questions. When his students' responses were insufficient or wrong, he reminded the related definitions and concepts and gave additional information about process and he encouraged the students to find GCD and LCM in real numbers and used different representations for finding them.

Ali:	A= 24 B= 18 Who could find the LCM and GCD of these numbers? You taught them at Grade 9. Did you remember that? We can write: $A = 2^3.3$ $B = 3^2.2$
Students:	Yes
Ali:	How do you find the GCD?
Student 1:	We consider the smallest exponent. GCD=$2^1.3^1$
Ali:	The smallest one from common exponents. What is the GCD? What is the LCM? GCD=$2^1.3^1$ LCM=$3^2.2^3$ Ok, what are the LCM and the GCD of these numbers? $A = 2^3.3^4.5^2$ $B = 2.3^3.7.5$
Student 2:	GCD=$2^1.3^3.1$ LCM=$2^3.3^4.5^2.7$
Student 3:	Why do we have to consider the biggest exponents to find LCM? Why do not we multiply these numbers?
Ali:	$\begin{array}{ll\|l} 24 & 18 & 2 \\ 12 & 9 & 2 \\ 6 & 9 & 2 \\ 3 & 9 & 3 \\ 1 & 3 & 3 \\ & 1 & \end{array}$ We found LCM and GCD of two real numbers in this way, do you remember?
Student 3:	Yes.
Ali:	Today, we study the LCM and the GCD of polynomials. I write a polynomial as $P(x) = x^2 - 2x - 3$ and the other polynomial as $Q(x) = x^2 - 9$ What do you think the LCM and the GCD of these polynomials? You can think what you did while finding the LCM and the GCD of real numbers.
Student 4:	At that case, we will factorize these polynomials.
Ali:	Yes, you are right. We factorize the polynomials. $P(x) = (x + 1)(x - 3)$ $Q(x) = (x - 3)(x + 3)$

	For finding the GCD, we consider only common factors. As the common factor is only (x-3) , GCD$[P(x), Q(x)] = (x-3)$ For finding the LCM, we consider the biggest exponential of the common factors and the others. So, LCM$[P(x), Q(x)] = (x+1)(x-3)(x+3)$
Student 5:	Why did not we write $(x-3)^2$ for the LCM?
Ali:	$\begin{array}{ll\|l} (x-3)(x+3) & (x-3)(x+1) & (x-3) \\ (x+3) & (x+1) & (x+3) \\ 1 & (x+1) & (x+1) \\ & 1 & \end{array}$ Can I find the LCM by using this way? Can you understand?
Student 5:	Yes, this way is better.

While he promoted his students thinking mathematics, he frequently gave examples and asked questions to the students. After the students worked on examples related to GCD-LCM in polynomials, Ali made his students estimate about a question containing the degrees of polynomials taught in previous lessons. He asked many questions to improve students' understanding. He also encouraged the students to make predictions while expressing their thoughts.

To triggering and considering students' ideas, Ali asked questions to uncover students' ideas and requested his students to express what they thought. After getting their thoughts, he provided feedback regarding correct or incorrect aspects of their ideas but did not explain or expand these ideas by himself. In parallel to this, Ali neither asked the students to explain his/her own expressions nor did he ensure that students clarified each other's thoughts. In the beginning of the lesson, he stated that students might obtain result by using different methods for a question about finding GCD and LCM and encouraged them to use different solution ways. But, in another example, Ali warned his student who performed a different solution way from what he thought. The below excerpt indicates this;

Board:	$P(x) = x^3 - 6x^2 + 9x$ $Q(x) = x^4 - 9x^2$. Find the LCM and the GCD of these polynomials
Student on the board:	$P(x) = x(x^2 - 6x + 9)$. There is x below.
Students:	$(x^2 - 3x)(x^2 + 3x)$
Ali:	It won't be that way. It is wrong, think one more time and correct it.

Ali did not continuously ask the reasons underlying students' ideas. He did not make students make give contradictory examples. Although he represented various solution, in certain cases, when students proceeded on a different solution way and he thought there was a more appropriate, he interfered the students' solutions.

Ali directly corrected students' mistakes occurred in solving examples/questions on the board. He realized a student's mistake related to the GCD and LCM of polynomials upon another student's question and he tried to eliminate by explaining the solution directly. As can be seen in the excerpt below, Ali mentioned the rules about finding LCM to make correct the solution of the student who incorrectly found the LCM of two polynomials.

The student on the board:	$P(x) = (x-3)(x+1)$ $Q(x) = (x-3)(x+3)$ LCM= (x+1).(x+3)

	GCD=(x-3)
Student 2:	Why do not we include (x-3) in the expression of LCM? [He is comparing his ideas to those on the board]
Ali:	We have to write (x-3) for LCM. Which one is bigger? The degrees of both of them are one.

He introduced concepts to eliminate students' errors, mentioned the rules where students made mistakes and gave different explanations to overcome the students' difficulties. Ali, instead of directly explaining, sometimes helped students overcome on their own difficulties by giving the clues.

Ozden's Knowledge of Student Thinking

In the context of the interview, Ozden stated that KoST contained knowing students' prior knowledge and misconceptions as a response to the question of what KoST includes. Ozden expressed that she questioned her students' prior knowledge and explained the deficient points about their prior knowledge again. Also, she said that she understood their misconceptions or difficulties through the help of homework and exams, but she realized their misunderstanding when they asked questions during the lessons. An excerpt from Ozden's interview is given below:

> I can understand the misconceptions only if any student ask question, but if they do not ask questions, I can't understand… In other words, I can only understand that there is a problem or difficulty by their questions such as "I didn't understand this", "Does it mean this here?". [Ozden-interview]

In addition, she stated that she utilized additional questions when her students made errors and changed the way of teaching when she thought to lead to difficulties. An excerpt from Ozden's interview is as follows:

> When a student has a difficulty, I try to explain it in a more simplified way to eliminate it; I change the teaching methods or the examples. I generally simplify it or I think that the reason of the not understanding depends probably on prior knowledge, so then I explain the content and give the information. [Ozden-interview]

Ozden stated that she encouraged her students to think by asking questions. Also, she expressed that she gave chance for the students who solved in a different way to explain their solutions. She stated that she utilized daily life examples and told theorems proposed to be given in the textbooks but she proved these theorems herself instead of asking her students to do. An excerpt from Ozden's interview was give below:

> I usually ask questions to encourage students to think, I think that this is the most important thing for learning. I can ask questions to have them think before teaching the subject. I enable them to think with the questions which will prompt them to interpret. I may represent different solutions on the board, too. That is, when one student solves the question and then another student said "I solved it like this", I consider that solution way, too. … Also, I use daily life examples from the textbook. [Ozden-interview]

Ozden carried out two-hour lessons about "finding solution set of first-degree equation or inequation" and "noticing features about the absolute value of a real number" and "finding solution set of first-degree equation or inequation including absolute expression value". Ozden taught the concept depending on textbooks rather than students' thinking. She was not flexible for arranging the instruction according to the students' thinking. She asked only the students who could give correct answer.

Ozden began her lessons by reminding the properties of first degree inequalities containing one variable in the set of real numbers which were studied during the previous lessons. She determined the students' prior knowledge by asking questions. After determining the students' prior knowledge, she asked problems including the solution sets of the first degree equations and inequalities with one variable in various numbers sets. Generally, she asked the students who gave the correct answer to come to the board and to tell how he/she solved the problem or the question. While a student was solving on the board, she expressed his/her actions in terms of mathematical rules and ideas to support students' mathematical ideas. When students could not understand solution of absolute value example, Ozden explained them by using the number line. So, she tried for the students to understand by means of different representations. She determined some real numbers on the number line and asked her students finding their absolute value. Ozden's approach regarding using different representation was as below:

Board: $|1-\sqrt{3}| + |3-\sqrt{3}| = ?$
Ooo

Ozden: When the value is negative in the absolute sign, if you do not make it positive, the result will not be right; so, it will be negative still if you do not consider the value. If the value is positive, I write it as positive number. If it is negative, my purpose is to make it positive, so I multiply it with -1. Is it correct? For example, this is the absolute value of -7. To make -7 to be positive, 7, I multiply it with -1.

Student: We'll try to make it positive.

Ozden: My purpose is to make it positive, because the distance value will not be negative, the distance between a number and 0 is positive value, it is correct?

Ozden gave a real world example from the textbook regarding the equations and inequalities including absolute values and ensured that the students construct a relation between the concept and the daily life. In the context of the real world example, she evaluated with the students for the temperature to be equal to a certain value and then she asked the students to interpret this situation mathematically. Thus, she prompted the students to consider explanations related to their actions while solving the problem and clarified their actions by expanding them as well.

Ozden determined that her students made an error because they did not consider the domains given in the question about finding the biggest or smallest whole number which an expression can be equal. The excerpt which presented her approach related to considering and correcting the students' misunderstanding by taking into consideration the reason for this error was given below.

Board: $x, y \in \mathbb{R}$
If x and y were defined as $-4 < x < 1$ and $-3 < y < -1$, what is the biggest whole number value of $2x^2 - y^3$?

Ozden:	Yes, this is the question, How can you interpret this question? You remember how to square and cube. What will I do for solving this question?
	ooo
Student:	I found 30
Ozden:	Ok, how did you find?
Student:	I squared it. The square of -4 will be positive, starting from 0.
Ozden:	For this inequalities, $0 < x^2$, Is there equity also or only inequality?
Student:	Yes, there is. The square of 4 is 16, so it will be 16.
	ooo
Student:	The biggest whole number value of x^2 is 15
Ozden:	But, does the values of x^2 have to whole number? It doesn't have to be a whole number [She is showing the condition of $x, y \in \mathbb{R}$]. But, x and y don't have to be whole numbers, is there any necessity to give a whole number value?
	ooo
Ozden:	$1 < 2x^2 - y^3 < 59$ So, what is the biggest value that you can find?
Students:	58.
Ozden:	58. What did you find when you assigned a whole number to x^2? 30. So, now it has a bigger value.

She did not want students to make estimations about the questions/examples/problems; instead, she asked the students who solved correctly to show the solution on the board. During her lessons, Ozden continuously verbally rewarded her students for their correct solutions and divergent thoughts and motivated them. Ozden frequently repeated the mathematical rules throughout the lesson, thus she supported that students could improve their mathematical thoughts. She gave enough time for students to think the problems and questions and to solve them.

Serin's Knowledge of Student Thinking

As a response to the question of what KoST includes, Serin replied that it contained knowing students' prior knowledge and eliminating deficiencies of their prior knowledge. Serin stated that she did not try to determine the students' prior knowledge and that only when she observed any problematic points of their prior knowledge, she gave information about them and so provided for the students to remember. Serin expressed that she considered the students who solved the questions correctly, but did not prefer to consider the students who made mistakes to not change her lesson's flow. She stated that she was aware of the parts in which students would have difficulties for understanding based on her experiences. An excerpt from Serin's statements is given below:

> I do not make extra study to determine the students' prior knowledge, but they are understood during the lesson, If I noticed the problems, then I gave several information about them. [Serin – interview]

Serin stated that she considered what the students had difficulties and she explained the important and necessary properties about the content by considering the student who had difficulties because all class might have the same difficulties even though a student explained that. She told that she encouraged the students to write the solutions step by step and she asked the students' ideas and thoughts in critical steps to prompt them to think. Also, she expressed that she directed the students to use the solution ways which were easy for her and she thought to be easy. An excerpt representing this approach from Serin's interview is given below:

> For example, I tell the students "it will be easy for you if you solve in this way"…I try to prompt them to find the solution by enabling them to think. For example, I ask "what can you do after this step". [Serin – interview]

It is deduced from Serin's statements that she considered students' prior knowledge and eliminating their deficiencies during her teaching, simplifying the things which were difficult for the students and asking questions to reveal students' ideas.

Serin realized two-hour lessons about "factoring polynomial with real valued coefficients". Serin traditionally taught the content of the lesson and could not encourage her students to think. She gave definitions and properties about the concept and then she selected one of the students who correctly responded the question. She ignored students' incorrect solutions and did not consider them. She frequently mentioned mathematical rules regarding the identities when solving these problems. If students had difficulties, she repeated mathematical rules and explanations. As can be seen in the following excerpt, she gave an example with an analogy from daily life to define the concept before proceeding to factorization applications by adding or subtracting a term from a given polynomial.

Serin:	We will study on factorization by adding and subtracting a term from a given polynomial. ooo In old times, a man had 17 camels and three children as younger, middle and bigger. He wanted to share the camels among his children by giving the half of the camels to the bigger child, one third of them to the middle child and one ninth of them to younger child. It was seen that he could not share by this way, because he could not get the half of 17 or $\frac{1}{3}$ of 17 or $\frac{1}{9}$ of 17. As he did not know how he would solve this problem, he wanted for a wise man to help. The wise man found the solution to the problem by giving a camel to the man. When he gave a camel, the man had 18 camels. The half of 18 was 9 camels; he gave these camels to the biggest children. The one third of 18 was 6 and the one ninth of 18 was 2 camels. The shared camels were 17 in total and the number of remained camel was 1. He gave back this camel to the wise man. That is, he could share the camels and then the wise man took back his own camel. When the man could not find a solution for sharing, adding 1 became a solution way for him, was not it? There were 18 camels by adding one camel and he easily shared them. When we cannot directly factorize a polynomial, we can use from adding and subtracting a term.

Serin did not encourage students to think with the questions and did not prompt them to estimate about the questions, Also, she did not support for students to improve their understanding and did not establish the class discussion at any moment. In other words, she exhibited relatively insufficient approaches regarding the triggering and considering divergent thoughts.

When she determined that a student did not understand a question related to the identities, she explained the solution process again and tried to eliminate her students' difficulties. However, Serin focused only on mathematical rules and processes to improve students' mathematical knowledge during the lesson for considering students' misconceptions and errors. Serin lectured directly without thinking her students and thus, she generally did not give chance to students to think and ignored her students. But, she was able to assess how they understood and realized the instructions from their statements or solutions on their notebooks.

The aim of this study is to examine three mathematics teachers' KoST and its evidence in their teaching and the findings were evaluated in the context of this aim. According to the findings from the interviews, each teacher mainly considered KoST as knowing students' prior knowledge. Parallel to this finding, Carpenter, Fennema and Franke (1996) stated KoST as knowing the knowledge of students' prior knowledge, misunderstandings and concepts that can support students' learning. In addition, teachers expressed that they benefitted from questions to understand students' ideas, they promoted students to use different solution ways for solving problems and they had ideas about which errors their students could make and the difficulties their students could have. Taker and Subramaniam (2012) emphasized that KoST includes knowing students' understanding, conceptual difficulties, possible ways of learning and developing sensitivity to what they think and do in mathematics lessons.

The findings obtained from their lessons presented that the participants considered students' prior knowledge in parallel with their statements expressed in the interviews. On the other hand, the participants did not consider their students' difficulties, mistakes and misconceptions unless students asked a question. All three teachers displayed insufficient approaches to the component of triggering and considering divergent thoughts. Van Zoest, Stockero, and Kratky (2010) regard the teachers' approaches to mentioning different thoughts and methods, revealing different opinions, comparing students' ideas and encouraging them to question one another as the components of understanding students' thinking. It is very important that this component of KoST to reflect on teaching for the development of students' thinking. Besides, we can say that the teachers' views about mathematics and teaching mathematics were effective for the approaches both promoting students thinking mathematics and motivating them to learn such as relation the concepts with daily life.

In the study, the teachers generally repeated the rules to eliminate the students' error but they did not prompt the students to notice the reasons of their errors or misconceptions and did not create a discussion environment. An and Wu (2012) expressed that KoST includes that teachers should know how well the students understand mathematical concepts, should understand their possible misconceptions and examples about misconceptions and should develop proper strategies to eliminate misconceptions. In this context, the participant teachers did not have effective actions.

Even though Ali stated that he could understand in which his students had difficulties and which students what they could do, we did not see an action which supported this view. When he asked questions, he selected the students who always gave correct answers or whom he thought that they would correctly respond instead of selecting any students and asking them to respond. This approach led him to notice the students who had difficulties. Also, he only considered the incorrect solutions or responses when the student on the board had errors. He did not examine different errors which other students in the classroom could have. Although he stated during the interview that he asked questions to determine students' errors, difficulties or misconceptions, he did not realize questioning in the context of his teaching. Besides, Ali expressed that he revealed different thoughts and solutions but he did not use questions such as "Is there anyone who solved in a different way?"

or "Is there anyone who thought differently?" in the lessons for considering alternative ideas. It was revealed that there were inconsistencies between his explanations during the interview and his actions in the lessons.

When considered Ozden's instruction, the prominent aspect was that she generally asked the students who correctly responded to explain their solutions. Additionally, she considered specific students and their thinking. Even though she stated during the interview that she recognized the students who solved in a different way, she never asked to students questions such as "Is there anyone who made a different solution?". That is, Ozden also did not have any approach triggering different solution or thinking like Ali. However, she simplified the concepts or solution steps and controlled whether the students understood with questions such as "Is there any problem until this point?", so she supported them to understand. Also, when the students had difficulties, she used figural representations such as the number line. This approach was also an appropriate approach in the context of KoST.

Serin stated that she considered the students' prior knowledge and determined incorrect ideas in their prior knowledge during her teaching. But, she did not pay attention to the students' prior knowledge which was important for leading in the subject. In the interview, she expressed that she discussed the thoughts of the student who had difficulties or responded incorrectly in the classroom. In accordance with these statements, she eliminated the students' errors by communicating with them one-to-one while walking among their desks. Her explanations were generally rule-oriented for both all students' difficulties and students' individual difficulties. Her approach regarding eliminating the errors or difficulties was to directly emphasize the rules instead of trying to understand their reasons. Thus, she could not realize what the students thought. Besides, she did not try to reveal possible different thoughts and did not regard them.

In brief, the expressions of three mathematics teachers from the interviews were inconsistencies with their implementations during their teaching. It is possible that they expressed opinions about what needed to be done but they did not reflect these opinions in their instructions because they may not adopt these approaches or their routines do not include them.

CONCLUSION

This study which we analyzed the teachers' actions and thoughts regarding KoST in a detail way has an important place for the mathematics education different from many research which examined mathematics teachers' pedagogical content knowledge. We shared and discussed the findings with the participant teachers, thus they gained a critical point of view to their instructions and their awareness were improved. In this context, we can say that sharing the findings of the research with the participants is an important issue.

In the following studies, a specific subject-oriented research can be done by determining subject or concept and by examining the teachers' KoST of this subject or concept. These studies enable the teachers to understand the detailed approaches which they can integrate in their lessons and can implement by supporting their knowledge of teaching, strategy and content. It is necessary for the mathematics educators who have responsibilities for the teaching processes to become more effective to realize the content-oriented studies about KoST and to support the mathematics teachers' professional development regarding KoST.

Besides, the issues which the mathematics teachers have to consider during planning can be understood from the KoST framework. When it is thought that the lesson plans are important for teaching, it is expected for the mathematic teachers to consider their students' thinking while planning and to have the idea that they plan the lessons by regarding their students. Similarly, the researchers can realize extensive and developmental studies about the lesson plans which include the students and their thinking. Thus, it is possible to say that the connection between the research about mathematics education and school implementations can be stronger.

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