Progressing from additive to multiplicative thinking is critical for the development of middle school students’ proportional reasoning abilities. Yet, many middle school mathematics teachers lack a thorough understanding of additive versus multiplicative situations. This article describes a sequence of instructional activities used to develop the proportional reasoning skills of middle school pre-service teachers. The activities could also be implemented in middle school classrooms.

Lily pad doubling: Proportional reasoning development

At the start of an undergraduate mathematics methods course, pre-service teachers (PSTs) were given the following Problem of the Day: The number of lily pads covering the campus lake doubles every day. If the lake is completely covered on day 28, on what day is the campus lake one-fourth covered? (Figure 1).

During the minutes that followed, the PSTs either immediately wrote a response or began meticulously calculating. Despite comments made that indicated they thought the Problem of the Day was “easy”, the majority of the PSTs incorrectly solved the problem and demonstrated faulty logic. When reviewing the responses, the instructor found that all of the PSTs answered with one of the following four answers:
1. “Day 7 because $28/4$ is 7”;
2. “Day 21 because one fourth of 28 is 7 and 28−7=21”;
3. “Day 28/4 is 7”;
4. “Day 28−7=21”.

Figure 1. Lily pad Problem of the Day.

The number of lily pads covering the campus lake doubles every day. If the lake is completely covered on day 28, on what day is the lake one-fourth covered?
3. “Day 26” with computations showing repeated doubling, and the assumption that the lake had one lily pad on Day 1 which yielded $134 \, 217 \, 728$ lily pads on Day 28 and then $\frac{134 \, 217 \, 728}{4} = 33 \, 554 \, 432$ which is Day 26; and,

4. “Day 26 because on Day 27 the lake would have been half full so the day before that it would be half of a half or one-fourth full”.

Clearly, only the few PSTs who responded with answer 4 were using appropriate multiplicative thinking. While the third response yielded a correct answer, the PSTs using that approach appeared dependent on exact quantities of lily pads, rather than on understanding the relative amount of coverage each day. Thus, the PSTs responding with answers 1, 2 or 3, appeared to lack the multiplicative thinking required to be effective teachers with regards to facilitating their potential future middle school students’ transition from additive to multiplicative thinking.

Progressing from additive to multiplicative thinking is one of the most crucial transitions in the development of middle school students’ mathematical abilities due to its role in later algebraic study (Langrall & Swafford, 2000). In order to reason proportionally, one must first recognise that ratio quantities are related multiplicatively rather than additively (Sowder et al., 1998). Middle school mathematics teachers must be able to differentiate between situations requiring additive or multiplicative thinking. They must also be able to solve problems in a variety of contexts which require varying levels of proportional reasoning so that they can facilitate the development of proportional reasoning within their students (Langrall & Swafford, 2000).

PSTs appear unable to use more conceptual instructional approaches when considering the relationships between and among quantities before determining an appropriate solution method (Riley, 2010; Shield & Dole, 2008). This could be the result of little attention being given to multiplicative reasoning and concepts embodied within multiplicative structures (i.e., proportions) provided in mathematics textbooks for pre-service elementary teachers (Sowder et al. 1998). To overcome this deficit, Sowder et al (1998) proposed four recommendations for teacher preparation programs (refer Figure 2). Based on these recommendations, the mathematics methods course instructor employed a sequence of three instructional tasks to develop the PSTs’ proportional reasoning skills, while simultaneously helping them to become aware of the transition from additive to multiplicative thinking that students should experience during middle school. Post-test results given at the end of the semester indicated that these PSTs were able to differentiate between additive and multiplicative situations, as well as, use a variety of approaches to solve proportional reasoning problems. This article describes the activities including excerpts of PSTs’ reasoning. The activities are also suitable for implementation with middle school students.

Figure 2. Recommendations for teacher preparation programs (Sowder, et al., 1998).

1. Pre-service teachers should be given opportunities to engage in quantitative reasoning with quantitative relationships.
2. Pre-service teachers should be provided with opportunities to distinguish between additive and multiplicative reasoning.
3. Pre-service teachers should be engaged in situations that allow them to reason about proportionality beyond symbolic manipulation.
4. Pre-service teachers should be provided with opportunities to make connections among the various types of rational numbers and with concepts of ratio and proportion.
Overview of the three activities

Understanding rational numbers is one of the most important prerequisites for success in solving multiplicative problems that require proportional reasoning (Behr, Lesh, Post, & Silver, 1983). Thus, the sequence of activities began with a focus on comparing rational numbers presented as ratios. The first activity, It’s Hip to Be Square, involved comparing rectangles in three real-world scenarios to determine which ones were “more square.” The three rectangle comparison scenarios were adapted from a proportional reasoning assessment (Bright, Joyner, & Wallis, 2003). Through completing this activity, the PSTs eventually realised that those rectangles that were “more square” were the ones whose lengths and widths had ratios that were closest to one.

The second activity, Magnified Pattern Blocks, focused on comparing the perimeter and areas of pattern block triangles, rhombi, and trapezoids using the ratio of the original pattern block to the magnified pattern block. Upon completion of this activity, the PSTs came to realise that changes in the perimeter and area were not additive, but were multiplicative. Specifically, a linear relationship existed between the side length and the perimeter, but a quadratic relationship existed between the side length and the area.\(^1\)

The third activity, What’s More?, presented comparison problems in the form of “then and now” scenarios that could either be interpreted additively using absolute comparisons or multiplicatively using relative comparisons (National Research Council, 2001; Van de Walle, Karp & Bay-Williams, 2013). In each scenario, two ratios, “then and now”, were compared to determine the answer to the problem; however, for those PSTs who viewed the scenarios additively, differences between “like quantities” were found. Collectively these activities facilitated PSTs in developing ratio and proportional reasoning skills as they analysed and solved problems appropriate for middle school students. A description of each activity follows.

The first activity completed was It’s Hip to Be Square (refer to Table 1). Since the difference between the two dimensions of the first scenario remained constant, as the dimensions increased, the ratio of the length to the width approached one. Given the constant difference, the use of additive thinking was easily assessed. Again in the second scenario, the difference between the dimensions was a constant to determine if additive thinking was again used or if a change in context might lead to a change in the type of thinking used. Because the instructor did not want the PSTs to conclude that the rectangle with the largest area is always the rectangle that is “most square”, the dimensions for the third scenario were chosen such that the rectangle with the largest area was not the most square; therefore, the differences between the dimensions were no longer constant.

Table 1. Activity 1: It’s Hip to Be Square.

<table>
<thead>
<tr>
<th>Scenario #1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mickey Mouse took a 4-inch by 6-inch photo of Cinderella’s castle. A week later, he took the photo to the Mouseketeer Photo Lab and enlarged it to an 8-inch by 10-inch photo.</td>
</tr>
<tr>
<td>• Which photo is “most square”, the original photo or the enlarged photo?</td>
</tr>
<tr>
<td>• How do you know this?</td>
</tr>
</tbody>
</table>

\(^1\) This paper describes activities developed in the USA; they use Imperial measurement. Australian teachers can substitute Metric figures.
To begin the activity, the PSTs discussed which rectangle in each scenario was the “most square” and why. After some debate, it was determined that there was no consensus as to which rectangles were “most square.” The instructor then distributed rulers, and cardstock rectangles that corresponded to each scenario to the PSTs to use as they wished. After the PSTs revisited each scenario, they shared their answers and provided explanations for why they selected each rectangle. Answers and explanations appeared to be based on the “look” of the cardstock rectangles; the use of additive thinking with regards to the length and width of each rectangle; or the use of multiplicative reasoning in which the ratios or equivalent ratios were compared (refer to Table 2). After discussing each scenario, the PSTs who relied on the “look” of the rectangles or on additive thinking began considering other approaches that “used fractions” to determine the solution.

**Building proportional reasoning through tangible manipulation: Magnified Pattern Blocks**

The second activity, Magnified Pattern Blocks, began with a focus on building shapes using pattern blocks that were “magnified” versions of the “original” pattern blocks; i.e., shapes that were proportional to the “original” pattern blocks (refer to Figure 3). While building “magnified pattern blocks”, the PSTs completed the charts on the recording sheet.
Table 2. Sample pre-service teacher responses for Activity 1: It’s Hip to Be Square

<table>
<thead>
<tr>
<th>Nature of response</th>
<th>Sample pre-service teacher responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent on the appearance of the rectangles</td>
<td>“The 4 × 6 photo because it has shorter sides making it look smaller and the 8 × 10 has longer sides that make it look more like a rectangle.” “19 ft by 22 ft looks pretty close but looking at all of them together, 26 ft by 29 ft looks the ‘most square’.”</td>
</tr>
<tr>
<td>Demonstrated additive thinking</td>
<td>“The first photo was only a two inch difference and so was the second photo. They are both about the same, neither photo is more square.” “They are equally square because of the 3 foot difference in each.” “The 5 × 7 garden and the 6 × 8 garden are equally square because the shorter side only needs 2 more yards to equal the other side and be a square.”</td>
</tr>
<tr>
<td>Some multiplicative reasoning, although not necessarily a correct solution</td>
<td>“The 8 × 10 because ( \frac{4}{5} ) is larger than ( \frac{4}{6} ). I compared the two fractions/ratios to see which was bigger.” “I made the 26 feet by 29 feet into a fraction and I knew that is ( \frac{26}{29} ) the closest to a whole.” “The 5 × 7 garden and the 6 × 8 garden are both two yards from being square, but the 5 × 7 garden would be most square because it would only need 14 units of area added to it to make it a square, rather than 16 like the 6 × 8.”</td>
</tr>
</tbody>
</table>

Building the triangles posed no cognitive conflict in terms of additive or multiplicative thinking since no other triangles could be built with the green pattern block triangles, other than proportional equilateral triangles. The PSTs were quick to notice the growing patterns formed with respect to perimeter, area, and number of triangles used to build each triangle. However, both additive and multiplicative thinking were apparent when the PSTs were building “magnified” rhombi. Those thinking additively initially did not keep the ratio of the side lengths equal to one; instead, they increased only one dimension of the rhombus. Consequently, their rhombi were growing in one dimension, not in two dimensions. The PSTs were guided to notice the dimensions of the triangles previously made and determine if the length of both dimensions changed with each successive triangle or if only one dimension changed. Once the PSTs determined this, they began to rethink what “magnified” meant and began reconstructing the growing rhombi such that the ratio of the two dimensions remained constant. After building their magnified rhombi, the PSTs noticed the growing patterns of the perimeter, area and number of “original” rhombi used to build each rhombus. When building the “magnified” trapezoids, the PSTs initially thought it was impossible to magnify the trapezoid since “all the sides aren’t the same length to begin with” and claimed there was no way to build a magnified trapezoid using only trapezoids. After some discussion about how the magnified triangles were built, the PSTs considered the possibility of turning or flipping the trapezoids in order to make magnified trapezoids, rather than simply adding vertical and horizontal “slides” of the original trapezoid, as was the case for the magnified rhombi. Subsequently, most of the PSTs were able to construct magnified trapezoids and keep the ratio of the side lengths constant. Given the patterns within the previous two charts, the PSTs were expecting similar patterns in the trapezoid chart and looked for this as justification of correctly building magnified trapezoids; i.e., growing perimeters that were multiples of the original perimeter, and growing areas that were perfect squares.
Task 1: Magnified shapes

a. Can you build “magnified” pattern block triangles, rhombi, and trapezoids as described below?

b. As you build each shape, record it on triangle grid paper in the appropriate color.

<table>
<thead>
<tr>
<th>Green triangles</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of a side</td>
<td>Perimeter of triangle</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice in the above chart?

<table>
<thead>
<tr>
<th>Blue rhombi</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of a side</td>
<td>Perimeter of rhombus</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice in the above chart?

<table>
<thead>
<tr>
<th>Red trapezoids</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of short base</td>
<td>Perimeter of trapezoid</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice in the above chart?

Upon completion of the triangle, rhombus, and trapezoid charts, the PSTs individually completed the second page of the magnified pattern block designs recording sheet (refer Figure 4). This was used as a formative assessment to determine the extent to which the PSTs were now thinking multiplicatively in terms of: (1) the number of each pattern block that would be needed in their magnified design; and, (2) the perimeter and area of their magnified designs compared to their original designs. Not all of the PSTs were successful on their first attempt, but eventually, after multiple attempts, all were able to do so.
Task 2: Magnified shape design

a. Using an assortment of 3 to 10 pattern blocks (at least, one triangle, one blue rhombus, and one trapezoid), create a design and record it on triangle grid paper.
b. Next, magnify your design so that it is either twice as large or three times as large as your original design.
c. Record your magnified design on triangle grid paper.

How many of each pattern block did you use in your original design?
Triangles: ...................  Rhombi: ..................  Trapezoids:..................

How many of each pattern block did you use in your magnified design?
Triangles: ...................  Rhombi: ..................  Trapezoids:..................

What is the perimeter of your original design? ...............................
What is the perimeter of your magnified design? ...............................
Explain the relationship between these two perimeters.
..........................................................................................................
..........................................................................................................
What is the area in terms of green triangles of your original design? ..................................
What is the area in terms of green triangles of your magnified design?
..........................................................................................................
Explain the relationship between these two areas.
..........................................................................................................
..........................................................................................................
What is the ratio of your original design to your magnified design?
..........................................................................................................
How do you know this?
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Figure 4. Magnified Pattern Block designs recording sheet (page 2).

Differentiating between additive and multiplicative reasoning:
What’s more?

The final activity, What’s More? targeted the PSTs’ ability to distinguish between additive reasoning or absolute comparisons and multiplicative reasoning or relative comparisons through “then and now” situations. For this activity, they were given four comparison scenarios (refer to Figure 5) to determine which situation in each scenario had the greater gain using a model of their choice (Misailadou & Williams, 2003). To create their models, the PSTs were given square grid paper, rulers, string, colour tiles, adding machine tape, and coloured pencils.
For each scenario, determine which of the two situations had a greater gain. Use any of the materials provided to model this and provide a written explanation of each answer. Then create your own scenario with one situation having a greater gain than the other.

Scenario #1
Daisy purchased two bean plants for her garden. On the day she planted them, one bean stalk was 8 inches tall and the other bean stack was 12 inches tall. After one week, she measured the two bean stalks again and found that the 8-inch bean stalk was now 12 inches and the 12-inch bean stalk was now 16 inches. Which bean stalk has a greater gain? Explain how you determined your answer using a model to support your written explanation.

Scenario #2
At the midpoint of the first nine-weeks of school, Chip had an average in math of 75% and Dale had an average in math of 60%. At the end of the first nine weeks, Chip’s average was 90% and Dale’s average was 75%. During the second half of the nine weeks, which student had the greater gain? Explain how you determined your answer using a model to support your written explanation.

Scenario #3
Sam and Drew joined the local fitness club. On their first day to exercise, Sam ran 3 miles in 39 minutes while Drew ran 3 miles in 45 minutes. After steadily exercising for one month, Sam ran 3 miles in 26 minutes and Drew ran 3 miles in 30 minutes. Who had the greater gain in speed? Explain how you determined your answer using a model to support your written explanation.

Scenario #4
On the first day of spring, Marion’s pond had 10 lilies blooming, while her niece Robin’s pond had 8 lilies blooming. Four days later, Marion’s pond had 15 lilies blooming, while Robin’s pond had 13 lilies blooming. Which pond had a greater gain of blooming lilies? Explain how you determined your answer using a model to support your written explanation.

Your turn
Create your own scenario with two situations, one of which has a greater gain than the other. Write your scenario on the front of your index card and the solution on the back of the index card.

Figure 5. What’s More?

Although most of the PSTs recognised that ratios and multiplicative reasoning should be used to solve each scenario, they were reluctant to create models for the problems as a means of determining the solution. Rather, they seemed to rely on more computational solution methods first and then constructed a model once they had found a solution (refer Figure 6).
“The way we solved this was to find what fraction of each original bean stalk 4 is and then we compared those fractions; 4 is one half of 8 and is one third of 12, so the beanstalk that grew from 8 to 12, grew the most, because $\frac{1}{2}$ is greater than $\frac{1}{3}$. Then we modeled this on graph paper using 1 square for each inch of growth.”

Figure 6. What’s More? scenario #1 model.

After being encouraged to first model and then use the model to help solve the problem, a majority of the PSTs began constructing their models first and then allowed the models to guide their thinking and subsequent solutions. The PSTs discussed in small groups how the model for the second scenario should be scaled (refer Figure 9). Very few of the PSTs solved the scenarios additively and those that did were quick to adjust their thinking once their peers’ explanations were shared. The PSTs then created a comparison scenario of their own, traded their scenarios with each other, and solved each other’s scenarios. Those who were still thinking additively struggled with this part of the activity; however, they were aware of this and continued to work towards more multiplicative thinking. One PST remarked, “I know I’m adding the same amount of growth which makes me want to say the growth is the same, but since the starting amounts differ, I have to think about it differently.”
Final thoughts

Each of the tasks presented authentic “problems”, as defined by Heibert et al (1997); that is, a problem is any task for which those solving the problem have no given or memorised procedure to use, nor is there the belief by those solving the problem that a specific “correct” solution method should be used. Since each of the tasks had no given or possibly memorised solution method, the PSTs were truly problem solving (NCTM, 2000), and by the end of the semester had begun to consider multiplicative approaches to situations requiring such approaches when engaged in any problem solving task. If PSTs are going to be successful in facilitating their future students in successfully transitioning from additive to multiplicative thinking, they will need to possess a deeper understanding of the conceptual field of multiplicative structures (Vergnaud, 1988), particularly those concepts which lead to proportional reasoning. As Fennema and Franke (1992, p 147) stated, “one cannot teach what one does not know.” Furthermore, responses to the lily pad problem, such as “Day 7 because $\frac{2^8}{4}$ is 7”, provide evidence that “one does not know” and therefore, one would not be able to teach the underlying multiplicative concepts. The importance of the concepts that lead to proportional reasoning demands that mathematics educators continue to facilitate a conceptual and lasting understanding of multiplicative concepts within their students.
Editor’s note
Whilst the above article provides examples from a North American context, and uses Imperial units of measurement, it provides innovative and insightful ways which can be directly linked to, The Australian Curriculum: Mathematics (ACARA, 2014), in terms of how multiplicative thinking is critical for the development of middle school students’ proportional reasoning abilities. It also provides an interesting sub-activity of conversions for students. For example, turning 6 inches into centimetres, and miles into kilometres.

References