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MATHEMATICAL MODELLING IN ENGINEERING: A PROPOSAL TO INTRODUCE LINEAR ALGEBRA CONCEPTS

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Abstract

The modern dynamic world requires that basic science courses for engineering, including linear algebra, emphasise the development of mathematical abilities primarily associated with modelling and interpreting, which are not exclusively calculus abilities. Considering this, an instructional design was created based on mathematical modelling and emerging heuristic models for the construction of specific linear algebra concepts: span and spanning set. This was applied to first year engineering students. Results suggest that this type of instructional design contributes to the construction of these mathematical concepts and can also improve first year engineering students' understanding of key linear algebra concepts and enhance the development of higher order skills.

Keywords **–** Mathematical modelling, Emerging models, Instructional design, Linear algebra, Spanning set, Span.

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1 INTRODUCTION

Linear algebra is among the subjects first taken by students in the engineering area and is considered one of the fundamentals in the field. This is due in part to the essential role it plays later on in the development of other subjects, given its unifying and generalizing nature (Dorier, 2002). In addition, it is also a powerful tool for resolving problems in different fields (Carlson, Johnson, Lay & Porter, 1993). However, despite its relevance, teaching linear algebra at university level is considered a frustrating experience both for teachers and students (Hillel, 2000), and independently of how it is taught, it is a hard subject for students both cognitively and conceptually (Dorier & Sierpinska, 2001).

In order to search for alternative methods for teaching linear algebra, experiments have been designed and executed. Said experiments include variations in lectures, incorporating the use of technology, group work, and the creation of a collaborative environment where teachers create discussions with students after explaining a new topic (Day & Kalman, 1999). Gómez and Fortuny (2002) describe the mathematical modelling process as an innovative tool for teaching linear algebra, indicating that it is an effective method that works as a means for channelling information, providing knowledge acquisition while at the same time establishing a relationship between mathematics and reality.

Specifically, referring to modelling and its applications, Kaiser (2010) proposes that in the last decades, both the learning and teaching of this subject have become an important topic, not only in schools but also in universities, due to the growing world demand for the use of mathematics in science, technology and daily life. According to Alsina (2007), at the university level, research in mathematical education has emphasised that the focus on mathematical modelling has been successful. To verify this, there is scientific evidence that shows students learn better in context, either because it provides motivation and interest or because it involves solving real world problems.

However, the importance of mathematical modelling in learning mathematics at universities has been slow coming, since traditional mathematics teaching still predominates, regardless of the fact that said teaching is aimed at students whose primary interest is precisely the application of mathematics and not mathematics itself (Trigueros, 2009). Considering this, actions should be taken to help maths teachers to realise the power of mathematical modelling (Kadijevich, 2007). Vanegas and Henao (2013) indicate that a plausible intervention is the consideration of contexts such as those used within realistic mathematics education. That is to say, to promote the mathematical modelling processes at the same time as creating connections to pass from the definite to the abstract. According to Gravemeijer (2007), students begin modelling their own informal mathematical activity and, over time, the character of the model gradually changes for the student, evolving into a more formal model of mathematical reasoning, while still rooted in the experiential knowledge of the student.

In particular, what promotes students to progress from informal mathematical models to formal mathematical reasoning are the so-called emerging models (Gravemeijer, 1999), which correspond to four types or activity levels: situational (interpretation and solution of the problem in a particular setting), referential (involving models, descriptions, concepts and procedures, which address the problem of situational activity), general (developed through exploration, reflection and generalisation shown in the previous level but with a mathematical focus on the strategy without making reference to the problem) and formal (working with conventional methods and notations).

Due to this new development, interest has arisen in designing and applying an instructional design for linear algebra that includes mathematical modelling and emerging models. In particular, for this teaching experience, concepts of spanning set and span are considered, since their understanding is relevant as they form a part of special vectors which according to Kolman and Hill (2006) are used in many mathematical, science and engineering applications. With this in mind, the following research question emerged: what does an instructional design that incorporates mathematical modelling and emerging construction models of spanning set and span contribute?

2 METHODOLOGY

The objective of this research is to comprehend what mathematical modelling and emerging models contribute to the construction of specific linear algebra content (spanning set and span). This involves creating an instructional design and research as to how this supports students in their transition from their preconceptions to formal mathematical reasoning. Given these considerations, the chosen methodology for this study is design research since it corresponds to a family of methodological approaches in which the instructional design and research are interdependent (Cobb & Gravemeijer, 2008). This type of research consists in three phase cycles: design, the teaching experiment and the retrospective analysis (Gravemeijer & Cobb, 2013).

2.1 Design

In the first phase, we develop the hypothetical learning trajectory (Simon, 1995). The construction of learning activities is based on mathematical modelling from the realistic mathematics education perspective and emerging models (Gravemeijer, 1999). In Table 1, a summary is shown of how the above-mentioned tasks that form part of the instructional design are manifested.

Task description	***What activity levels are demonstrated in the task***
Task 1: Generating passwords with vectors. Information is given on the importance of safe passwords, the characteristics which they should have and examples of how to create passwords with Excel. Subsequently, they invent a secure password generator using vectors.	*Situational level. Students use strategies together with their knowledge of maths and passwords to develop a secure password generator.*
Task 2: Relating the password generator with the spanning set and span. With the resolution of the password generator they are asked for two sets: one which contains all the numeric passwords of the password generator and another which has numeric vectors which when combined linearly produce a generic vector that generates numeric passwords. The teacher introduces the concepts, then makes an analogy between these and their password generator.	*Referential level. Students making reference to the proposed solution of their password generator present two sets with determined characteristics. The teacher then formally defines the concepts, links this new mathematical reality with the real problem, making an analogy between them.*
Task 3: Applying the lessons learned. Using spanning set and span concepts and deepening the understanding of them using conventional mathematical notation.	*General and formal level. Students use their work from the previous task to explore the spanning set and span using their knowledge and mathematical notations.*

Table 1. Description of tasks from the instructional design and relationship with activity levels

2.2 Teaching experiment

The experiment was conducted during the 2013-2014 period in L'Escola Politècnica Superior d'Enginyeria of Vilanova i la Geltrú (Univesitat Politècnica de Catalunya) located in Spain with a group of students that was heterogeneous in terms of its members' previous study backgrounds (from high school and polytechnic), but homogeneous with regard to the fact that all were first year engineering students who were studying mathematical fundamentals and did not have previous experience with problems that involved mathematical modelling, nor had they previously studied spanning set and span concepts.

The study area was the classroom and the materials used were: written learning activities, sheets for registering calculations and other elements used in a normal class. Each class session started with the introduction of a new problem or the continuation of the activities from the previous day. The remainder of the class consisted in cycles of group work (3 to 5 students) and whole class discussion. The experiment was conducted in 5 hours, distributed over 3 class sessions.

The data collected in this experiment were: written protocols of the learning activities developed by the students, videos and audio recordings of each session, and individual interviews at the end of the experiment.

2.3 Retrospective analysis

Analysis of the experimental data began with organisation and categorisation. Next, the task developed by the students and the recordings were analysed from the perspective of the research question: What does an instructional design based on mathematical modelling and emerging construction models of spanning set and span provide? Subsequently, data was analysed identifying examples that demonstrate some change from informal to formal reasoning with regard to studied concepts. For this, emerging models were used as an interpretive framework. From this analysis, a story was created which reconstructs the learning process that the students followed.

3 RESULTS

The results of the study were presented through the story of the learning process that the students followed in relation to the instructional design (Doorman, Drijvers, Gravemeijer, Boon & Reed, 2013).

3.1 Task 1: Generating passwords with vectors

The purpose of this task is, firstly, that the students activate their preconceptions about vectors, and secondly, that students use mathematical modelling as a tool to solve the problems that they face.

The chosen context for task one was the generation of passwords. The information provided to the students is a news story about hacked social networks and how to generate passwords using Excel. With this background, it is proposed that, in groups, they make a password generator that takes into account the use of vectors. The groups, supervised by the teacher, follow the mathematical modelling process proposed by Blum and Leiss (2007). The phases of mathematical modelling that the groups followed to create a proposal for the question raised are summarised in Figure 1. The students did not necessarily follow the order of the outlined steps.

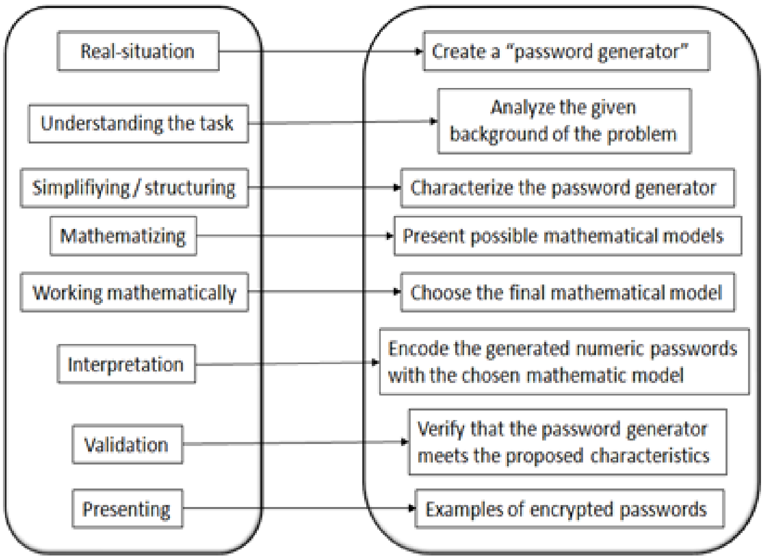
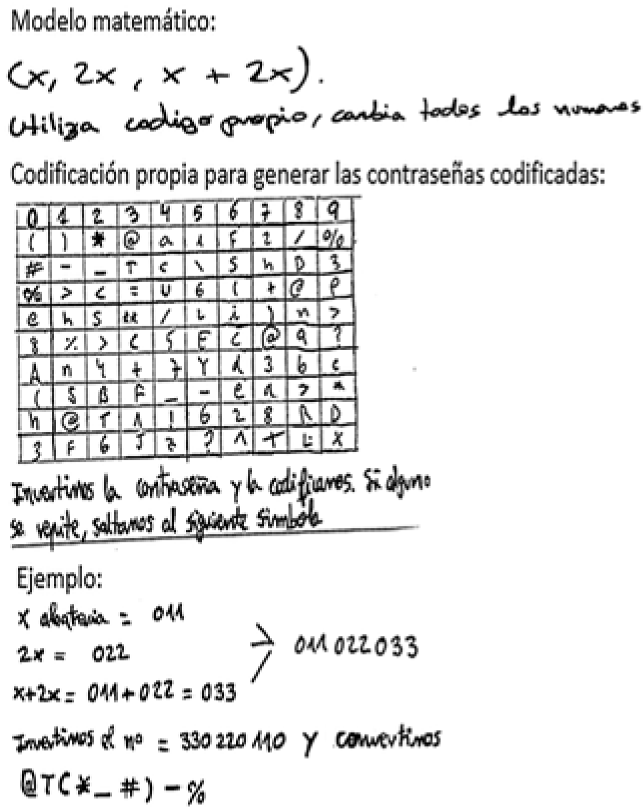


Figure 1. Mathematical modelling process followed by the students to solve the problem

The groups propose different solutions, although all are similar to what is seen in Figure 2. In other words, they propose a model to generate passwords which correspond to a generic vector and to show its usefulness, they give a specific number to their variable(s) and immediately apply their codifying process to interpret mathematically, within the context of the problem, obtaining a password created by their password generator.



Mathematical model:

(x, 2x , x+2x)

Use your own code, change all the numbers

Coding to generate encrypted passwords:

0	1	2	3	4	5	6	7	8	9
(	)	*	@	a	1	F	2	/	%
#	–	_	T	c	\	S	n	D	3
%	>	<	=	U	6	(	+	@	P
e	h	S	**	/	L	i	)	n	>
8	%.	>	c	5	E	c	@	9	?
A	n	4	+	7	Y	d	3	b	c
(	S	β	F	_	–	e	≈	>	*
h	@	T	^	!	6	2	8	R	D
3	F	6	J	Z	?	^	+	L	X

We invert and encrypt the password. If a number is repeated, we skip to the next symbol

Example:

random x = 011

2x = 022 → 011 022 033

x + 2x = 011 + 022 = 033

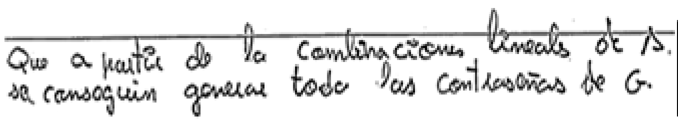
Invert the nº=330220110 and convert it

*@ T (* _ #) – %*

Figure 2. Proposed solution by one group for the problem of creating a password generator

3.2 Task 2: Relating the password generator with the spanning set and span

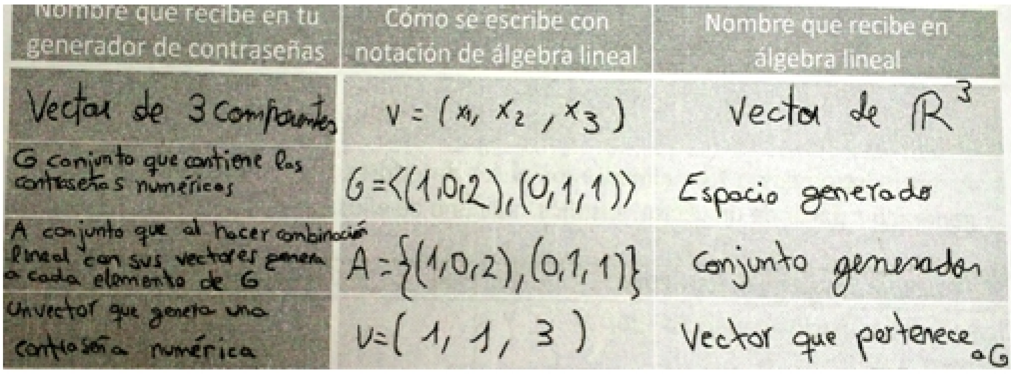
Task 2 is composed of two parts. In part I, students are asked for two sets: one, G, which contains all the numeric passwords of their password generator and another, A, which has numeric vectors that when made into a linear combination, a generic vector is obtained that creates numeric passwords. The groups answer appropriately to this, although some presented errors in mathematical annotations. Therefore, once they wrote down sets A and G they were asked what the relationship between these is. The predominant response is that the relationship is “A generates the G elements through the linear combination of its vectors”, meaning that the students made a connection between the sets. However, various groups had difficulty using mathematical language when expressing their answer, since when they recorded “A combinations”, it is possible that what they meant to say was “the linear combinations of A vectors”, as observed in Figure 3.



From the lineal combinations of A all the passwords of G can be generated.

Figure 3. Example of an answer of part 1 of task 2

Next, when the teacher formally defines the spanning set and span, they perform part II, which consists of making an analogy between these concepts and their password generator. In general, all the groups make the analogy according to their password generator as seen in Figure 4, except for some which designated another name to the spanning set, calling it generator vector(s) o set vectors.



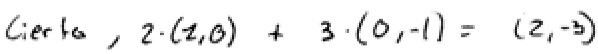
Nombre que recibe en tu generador de contraseñas	Cómo se escribe con notación de álgebra lineal	Nombre que recibe en álgebra lineal
Vector de 3 componentes	$v=(x_1, x_2, x_3)$	Vector de $\mathbb{R}^3$
G conjunto que contiene las contraseñas numéricas	$G=\langle(1,0,2),(0,1,1)\rangle$	Espacio generado
A conjunto que al hacer combinación lineal con sus vectores genera a cada elemento de G	$A=\{(1,0,2),(0,1,1)\}$	Conjunto generador
Un vector que genera una contraseña numérica	$v=(1,1,3)$	Vector que pertenece a G

Name of your password generator	As written in linear algebra notation	Given name in linear algebra
3 component vector	$v=(x_1,x_2,x_3)$	*Vector of R^3*
G set which contains the numeric passwords	$G=\langle(1,0,2),(0,1,1)\rangle$	*Span*
Set A when making the lineal combination of its vectors creates each G element	$A=((1,0,2),(0,1,1))$	*Spanning set*
Vector which generates a numeric password	$v=(1,1,3)$	*Vector belonging to G*

Figure 4. Example of an answer of part II of task 2

3.3 Task 3: Applying the lessons learned

In this task, spanning set and span were explored, but in problems which involve conventional mathematical annotations. The answers to this task show that a large number of the groups managed to differentiate between the concepts that vectors of spanning sets create span, recognising in set notations that one of these has a finite number of vectors and the other an infinite number. Also, the groups were able to verify if a vector belongs to a certain span by making a linear combination of vectors from this spanning set as seen in Figure 5. However, only some groups managed to represent the span for a determined spanning set and others observed related difficulties graphically and analytically: the lack of rigour in the use of mathematical language, the use of terms interchangeably in mathematical notations and the different forms of representing the studied concepts.



True, $2\cdot(1,0)+3\cdot(0,-1)=(2,-3)$

Figure 5. Example of an answer for the question from task 3 to establish whether it is true or false that the vector belongs to the span

The results from the teaching experiment show that the students solved the problem of creating a password generator, drawing from their previous knowledge of vectors. And from this, they worked on the next task, describing sets that contain finite and infinite vectors related to the initial context, which then links with the spanning set and span. The latter allows them to visualise concepts both in a real context and in a mathematical one. As a result, they identify some characteristics of the spanning set and span, such as the inclusion relationship between them. However, some groups presented difficulties with the different forms of representing these, especially geometrically. Additionally, in the development of the task, the use of mathematical language is an obstacle.

This suggests that the instructional design favours the comprehension of spanning set and span, since a large part of the students managed to make the transition from their informal mathematical knowledge to a more formal comprehension.

4 CONCLUSIONS

4.1 Contributions

The main contribution of this study is to provide a first approximation of the use of mathematical modelling and the emerging models for learning about spanning set and span for engineering students, since currently no studies exist on this subject, which makes this research innovative and original.

This study posed the question of what the instructional design, which incorporates mathematical modelling in the construction of spanning set and span, contributes. From the data analysis, it was observed that the following characteristics of the instructional design contribute to the comprehension in the following ways:

- The students, through the creation of a password generator, put their previous vector concepts into use, which helped them connect these with the following task that sought a first approximation on spanning set and span.
- The analogy table between the context of generating numeric passwords and the concepts in the study helped students visualize both of them in a real and a mathematical context. At the same time, they were offered the possibility to differentiate between them when used in a real situation.
- Task 3 strengthened the notations of both concepts when deepening students' knowledge of them in problems within a mathematical context. The results show that the majority can identify the inclusion relationship between these and also recognise the conditions that a vector should meet to belong to a span.

The results of this study provide evidence of the contributions of instructional design for students when constructing a spanning set and span, since through the use of mathematical modelling in a real context, as Gravemeijer (2007) indicates, they model their own informal mathematical activity, and then continue with other tasks which lead to a more formal understanding of what they are studying.

On the other hand, we can indicate that mathematical modelling contributes to the construction of the spanning set and span because this permits students to: give a sense of real context to what they are learning, discuss both new situations and mathematical content with peers (decreasing their dependence on teachers) and enhance the development of mathematical abilities linked mainly to modelling and interpreting.

4.2 Discussion

This study shows the results of a first experiment with the instructional design and future studies intended to create new experimental cycles to continue refining it. Accordingly, the results suggest the following modification to the instructional design: incorporate formative evaluation in the development of the instructional design with the purpose of helping students in their difficulties, and rethinking task 3 with the aim that questions exist both about the properties and the applications of the spanning set and span.

Finally, it indicates that the emerging models not only orient the instructional design, but also promote the cognitive development of the students, using implicit concepts of spanning set and span in the context of generating passwords for their later use in conventional mathematical problems, meaning that it allows the transition from one model to another as Gravemeijer (1999) proposes. From this point of view, this instructional design proved suitable.

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