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Virtual Environments for Mathematics and Geometry Education

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Abstract

Since ancient times mathematicians and geometricians have used visualisations to describe, discuss, study and teach mathematics. In mathematics education, visualisations are still used whenever possible to support teaching, to inspire students and feed their need to actually see abstract mathematical facts. In our times, virtual reality presents a fascinating, extremely motivating new tool in teachers' hands which allows students to see mathematics in three dimensions. This article gives an overview of various, mainly immersive, virtual environments that have been developed in the previous 10 years to support mathematics and geometry education. The focus lies on one advanced application for geometry education that has been used and evaluated with over 500 students throughout the years. Findings and teaching experiences are described and discussed.

Introduction

Mathematical knowledge is often fundamental when solving real life problems. Especially, problems situated in the two- or three-dimensional domain that require spatial skills are sometimes hard to understand for students. Many students have difficulties with spatial imagination and lack spatial abilities. Spatial abilities, in contrast, present an important component of human intelligence, as well as logical thinking. A number of studies have shown that spatial abilities can be improved by well-designed trainings (Souvignier, 2001; Waller, Hunt, & Knapp, 1998). Mathematics, specifically geometry education, has been proven as one powerful means of improving these skills (Gittler & Glück, 1998); recently, a number of training studies have shown the usefulness of virtual reality (VR) in training spatial ability (Durlach et al., 2000; Rizzo et al., 1998).

Virtual environments are inherently three-dimensional. They can provide interactive playgrounds with a degree of interactivity that goes far beyond what is possible in reality. If using VR as a tool for mathematics education, it ideally offers an added benefit to learning in a wide range of mathematical domains. Some topics that are included in most mathematics curricula world wide are predestined for being taught in VR environments. For students aged 10 to 18, such topics are, for instance, 3D geometry, vector algebra, graph visualisations in general and curve sketching, complex numbers (visualisations), and trigonometry, as well as other three-dimensional applications and problems. Students in primary school benefit from the high degree of interactivity and immersion throughout their first four years, not only when learning the four basic operations, but also when learning about fractions and having to solve real life problems. Examples are the SMILE project (Adamo-Villani, Carpenter, & Arns, 2006; Adamo-Villani & Wright, 2007) mentioned later and also the Virtual Playground (Roussou & Slater, 2005). VR is able to “bridge the gap between the concrete world of nature and the abstract world of concepts and models” (Bell & Fogler, 1995).

For higher education (university level), there is a wide range of potential for the use of VR in higher mathematics in domains such as analysis (e.g., complex functions), linear algebra, differential calculus and differential geometry, projective geometry, higher dimensional geometry and many more.

Since costs of a VR setup will always exceed costs of a standard desktop computer, justifications for the higher expenses must be given to those providing money. In addition, educators need to know which advantages VR technology provides before investing time and effort.

In the following, a few (immersive) virtual environments for mathematics education will be presented briefly. They all demonstrate unique advantages of using VR for mathematics education and provide insights into what the technology can offer. Construct3D – an application for geometry education - will be described in more detail. It was evaluated three times by over 500 students in total (Kaufmann & Dünser, 2007), and was continuously improved throughout the years. The article concludes with a discussion about lessons learned, analyzing factors that prevent usage of VR applications in mathematics education today and summarizing the strengths of the presented applications.

VR applications for mathematics education

Spatial algebra

One of the earliest publications about the use of virtual worlds in mathematics education is the paper by Winn and Bricken (1992). Having a strong background in educational technology and mathematics, they focus on how to improve students’ class-room experiences with a new emerging technology. Their domain of interest is teaching algebra. Frequently, students have difficulties with learning algebra, especially the symbol system. At their young age, they are often baffled by algebra's non-visual nature. Therefore, Winn and Bricken introduced a new concept called spatial algebra. It is a visual representation of algebraic concepts, and, in addition, it utilizes the three-dimensional nature of VR. This enables interaction with algebra in a 3D environment. Figure 1 shows two examples of spatial representations of algebraic terms and equations.

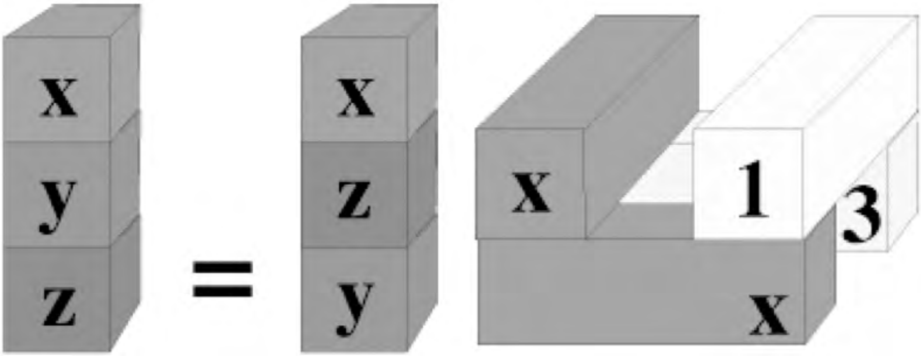


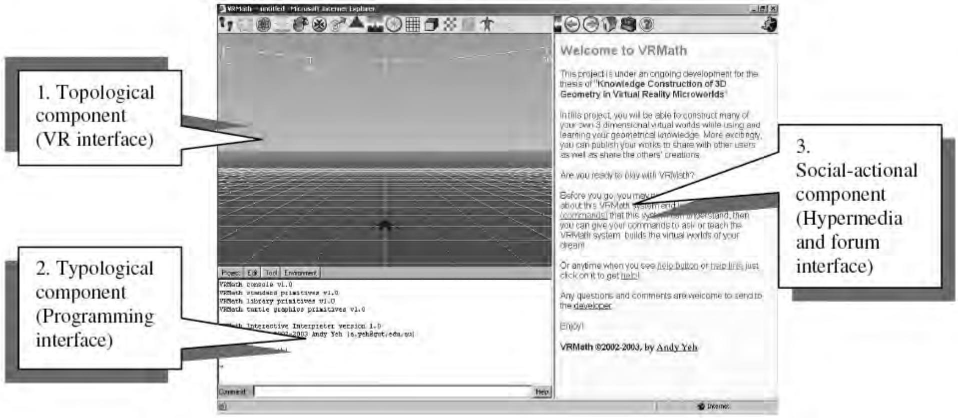
Figure 1. Left: The equation (x * y) * z = (x * z) * y is represented. Right: The term (x + 1) * (x + 3) is visualized in a three-dimensional manner. For a detailed description of the whole grammar, please refer Winn and Bricken (1992). Image courtesy of Dr. William Bricken.

Students are immersed and can build their own algebraic terms and equations in a natural way. It helps students to construct knowledge of algebra. Virtual (algebraic) objects can be manipulated in order to explore equations. As a conclusion, the authors state the great potential of VR for mathematics education.

Apart from this early work, the rest of the examples presented here have been published more than 10 years after Winn and Bricken's work and are sorted by the degree of immersion and complexity.

VRMath

The VRMath system (Yeh & Nason, 2004) is a non-immersive, online application that utilises desktop VR combined with the power of a Logo-like programming language to facilitate learning of 3D geometry concepts and processes. The design of the HCI components of VRMath was influenced by educational semiotics, which connect mathematical meanings with multiple semiotic resources (Figure 2). VRMath enables children to manipulate objects and write programs to create objects in a VRML environment.



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VRMath console v1.0
VRMath standard primitives v1.0
VRMath library primitives v1.0
VRMath turtle graphics primitives v1.0

th Interactive Interpreter version 1.0
2002-2003 Andy Yeh (a.yeh@qut.edu.au)
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Welcome to VRMath

This project is under an ongoing development for the thesis of "**Knowledge Construction of 3D Geometry in Virtual Reality Microworlds**"

In this project, you will be able to construct many of your own 3 dimensional virtual worlds while using and learning your geometrical knowledge. More excitingly, you can publish your works to share with other users as well as share the others' creations.

Are you ready to play with VRMath?

Before you go, you may about this VRMath system and (commands) that this sy understand, then you can give your commands to ask or teach the VRMath system builds the virtual worlds of your dream!

Or anytime when you see help button or help link, just click on it to get help!

Any questions and comments are welcome to send to the developer.

Enjoy!

VRMath ©2002-2003, by Andy Yeh

Figure 2. The VRMath environment with three interaction components – VR interface, scripting interface and a hypermedia forum interface. Image courtesy of Andy Yeh, Queensland University of Technology.

Six students (grades 4-5) evaluated VRMath by conducting nine learning activities designed for that environment. Primary findings regarding both the user interface and the learning experience were that VRMath is easy to learn, use and remember. It helps students to construct knowledge about the relationships between 3D geometry concepts and processes. As Yeh reports (Yeh & Nason, 2004), "in the collaborative construction of a spiral staircase, the students constructed their artefact by consolidating their geometric language (e.g., Logo-like programming language) with geometric concepts about scaling a cube in three dimensions (width, height and depth), positioning and rotating in 3D space (moving and turning the turtle), and collaborating in online discussion. The rich semiotic resources including topological (e.g., the spectrum of colours, continuous visualisation and navigation), typological (e.g., geometric terms and programming commands), and social-actional (e.g., discourse and online forum communication) enabled them to construct an effective and deep understanding of 3D geometrical knowledge".

In its approach to allow users to program or script a 3D scene, VRMath is similar to other educational CAD packages such as GAM (Podenstorfer, 2009). Having to use script code or commands to manipulate 3D objects is a slower way of interacting with a 3D scene than dragging-dropping or moving objects to their correct locations. However, the scripting approach is educationally much more rewarding. When using a protocol, script or command, students are forced to think about correct translations and rotations in 3D within a given coordinate system before doing the operation. If dragging and dropping is enabled, students frequently use trial and error methods to transform objects instead.

In this context, it is important to note that while geometry education software shares many aspects with conventional CAD software at a first glance, its aims and goals are fundamentally different. Geometry education software is not intended for generating polished results, but puts an emphasis on the construction process itself. While relatively simple geometric primitives and operations will suffice for the intended audience of age 6 to 18, the user interface must be both intuitive and instructive in terms of the provided visualizations and tools. Commercial CAD software offers an overwhelming variety of complex features and often has a steep learning curve. In contrast, educators are often interested in simple construction tools that expose the underlying process in a comprehensive way and encourage spatial thinking. In accordance to that, two applications will be presented – the AR 3D Geometry system and Construct3D. Their aim was not to create a professional 3D modeling package but a simple and intuitive to use 3D construction tool in a virtual environment for educational purposes.

Augmented reality 3D geometry system (ISMAR)

Augmented Reality (AR) (Azuma, 1997) is a variation of VR. VR technology completely immerses a user inside a synthetic environment. While immersed, the user cannot see the surrounding real world. In contrast, AR allows the user to see the real world, with virtual objects superimposed upon or composited with the real world. Therefore, AR supplements reality rather than completely replacing it. Ideally, it would appear to the user that the virtual and real objects coexist in the same space. In terms of used technology, AR can be said to require the following three characteristics: (1) combines real and virtual (2) is interactive in real time (3) objects as well as users are registered in 3D space.

Development of the ARToolkit 3D geometry system (Trien Do & Lee, 2007) was inspired by the real difficulties students have when studying descriptive geometry and interpreting technical drawings. The system utilizes ARToolkit (Billinghurst, Kato, & Poupyrev, 2001) for tracking and user interaction. ARToolkit is an open source software that uses a single ordinary web camera to track planar, quadratic markers usually produced on an ordinary office laser printer. It is an extremely cost effective tracking solution. A user-definable pattern inside the marker allows the software to distinguish between different targets, allowing users to build complex applications with multiple tracked interaction devices and artifacts.

The system provides users with basic functions such as constructing common 3D objects (Figure 3 right), customizing attributes, simulating object intersections by using two markers (Figure 3 left) and exporting results to VRML files. It is a useful tool for students to improve their study performance and a pedagogical device for lecturers to produce more intriguing and easily comprehensible lectures. The system was not evaluated with students yet.

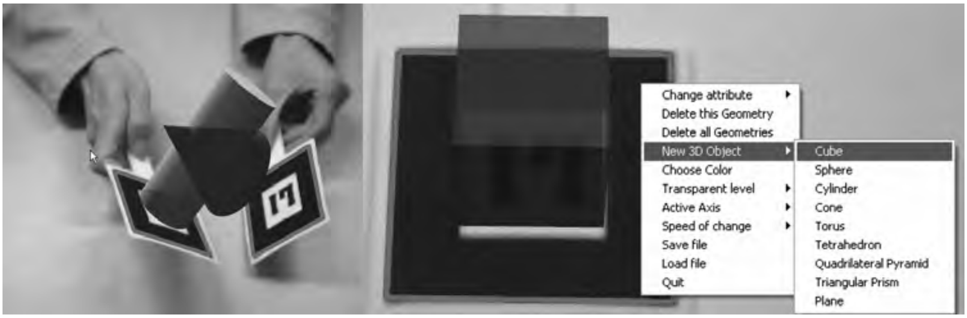

Figure 3. Left: Observing an intersection while moving the markers. Right: Selecting properties of 3D objects. Image courtesy of Van Trien Do, Sejong University.

Science and math in an immersive learning environment (SMILE)

The SMILE project (Adamo-Villani et al., 2006; Adamo-Villani & Wright, 2007) is an immersive learning game that employs a fantasy 3D virtual environment to engage deaf and hearing children in math and science-based educational tasks. SMILE is one of the first bilingual immersive virtual learning environments for deaf and hearing students, combining key elements of successful computer games, emotionally appealing graphics, and realistic real-time 3D signing, with goal-oriented, standards-based learning activities that are grounded in research on effective pedagogy. Adamo-Villani et al. report that historically it has been difficult for deaf individuals to gain entry into higher education for a number of reasons. There is a significant delay in deaf children's reading comprehension, hearing parents have difficulties conveying basic mathematical concepts to their children, and there is a lack of many sources (e.g., radio, conversations, etc) that hearing children take for granted. Consequently, some mathematical concepts that hearing children learn incidentally in everyday life have to be explicitly taught to deaf pupils.

Figure 4 shows two example scenes from SMILE. In the candy store, for instance, (Figure 4 right) "the child practices the association between the concept of number, the mathematical symbol, and the ASL representation. After walking into the candy store, the child uses the pinch glove to 'pick' a certain number of candies from various jars. After she releases them on the counter, the virtual store keeper (bunny) comes forward and signs the corresponding ASL number, while the number symbol appears on the cash register. The student can then practice addition and subtraction skills by asking the store keeper to put more (or remove some) candies on the counter. She does this by touching the 'MORE' or ('LESS') call bell with the pinch glove. The bunny adds (or removes) a random number of candies and signs the question: "how many candies are now on the counter?" The child signs the answer with the pinch glove and receives feedback in ASL and math symbols".

Special care has been taken to design the animations of characters – especially of fingers and hands that achieve most attention – as realistically as possible using motion capture. Signed communication is enabled with a gesture recognition system using pinch gloves. It is able to recognize a limited number of ASL signs. Regarding hardware setups, a CAVE has been used as the standard setup with wand or pinch gloves as interaction devices; desktop or head-mounted display (HMD) based setups are possible as well.

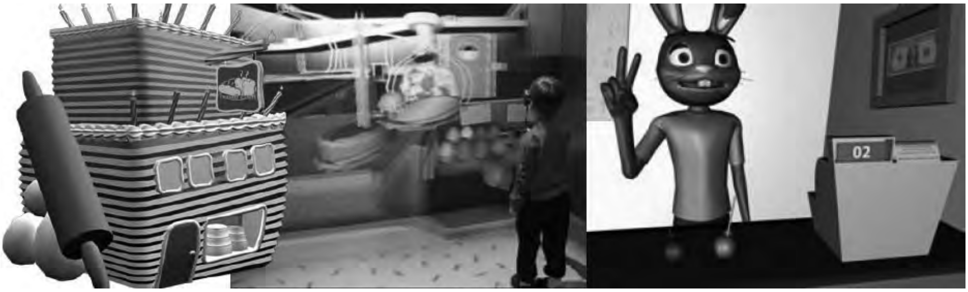

Figure 4. Left: Bakery building (outside view); Middle: A child interacting inside the bakery building in a CAVE environment. Right: The candy store shop owner (a bunny) using sign language. Image courtesy of Nicoletta Adamo-Villani, Purdue University.

SMILE is a game with the goal to make the people of Smileville smile again. To achieve this goal the user has to help the virtual characters by solving certain mathematical tasks. When successful, new meaningful objects are given to the virtual inhabitants of Smileville. The user takes a central role (active participation) during creation and modification of these new objects, which fosters engagement and motivation.

Usability evaluation of SMILE has been conducted following the methodology by Hix and Gabbard (Hix & Gabbard, 2002). An expert-guidelines based evaluation was followed by two formative evaluations so far. After a third formative evaluation, a summative evaluation is planned.

In the second formative evaluation, 21 children aged 6½ to 10 participated with 7 of them being ASL signers (Adamo-Villani & Wright, 2007). Results show that the children had high expectations but the reported experience surpassed them. The game was perceived to be more fun and easier to use than expected, and slightly more challenging. The learning outcomes of using SMILE have not been assessed yet. Future work involves production of additional content (for grades 4-5) and evaluation of learning and knowledge acquisition with children aged 5-10, in cooperation with a school for the deaf and elementary schools. A portable immersive system consisting of two projectors with polarisation filters and a tracking system will be used for testing in schools. SMILE is a very promising and exciting project which is still actively under development at this time.

Complex function graphs

Understanding the properties of functions over complex numbers is more difficult than working with functions in the real domain. The application by Robert Liebo (Liebo, 2006) provides an approach to visualize complex functions in augmented reality (AR) in order to gain insights into their properties.

Using multiuser AR for mathematical visualization enables sophisticated educational solutions for studies dealing with complex functions. A variety of visualization techniques can be used to see and understand a complex function through the location, the shape, the color, and even the animation of a resulting visual object. Proper usage of these visual mappings provides an intuitive graphical representation of the function graph and reveals the important features of a specific function. Figure 5 demonstrates two exemplary usages. On the left side, the function $f(z) = z^2 + c$ is displayed whereby z (a complex variable [light gray dot]) and c (dark gray dot) can be interactively moved in the Gaussian complex plane. The according surfaces are changed in real time. In this example, real (shown as gray) and imaginary (shown as black) part of f(z) are plotted separately. In the right image, the z-transformation of a FIR filter kernel is shown. The rendered surface uses the absolute value of f(z) as height information. The curve (shown as dark gray) is the image of the unit circle.

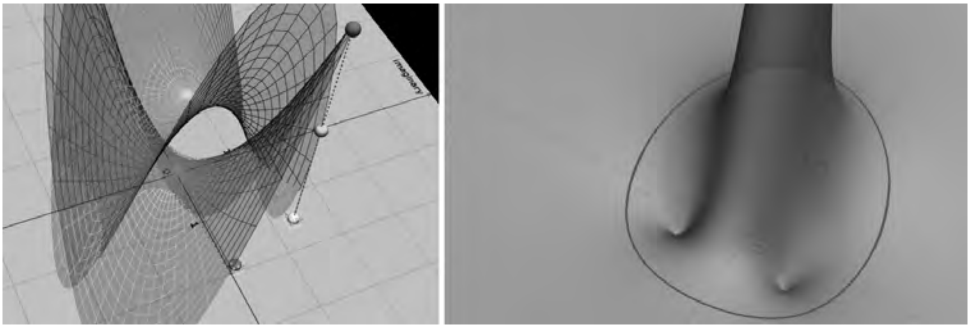

Figure 5. Left: $f(z) = z^2 + c$; Right: The z-transformation of a FIR filter kernel is shown. Image courtesy of Robert Liebo, Vienna University of Technology.

The standard hardware equipment for this application consists of one or two HMDs (for multiple users) as output and wireless pens as input devices, in addition to an accurate optical tracking system. It also runs as a desktop application. The system is widely applicable in higher education in multiple mathematical domains but has not been evaluated with students yet.

Cybermath

A technically advanced project is CyberMath (Taxén & Naeve, 2001). CyberMath is an avatar-based shared virtual environment aimed at improving mathematics education. It is suitable for exploring and teaching mathematics in situations where both the teacher and the students are co-present or physically separated (Figure 6). CyberMath is built like a museum with a virtual lecture hall in its center. Special care has been taken to make the environment as inviting as possible. Virtual mathematical objects can be manipulated and discussed in a realistic way. The first prototype was built on top of DIVE, a toolkit for building interactive shared distributed multi-user virtual environments.

Figure 6. Multiple users collaboratively using a distributed CyberMath environment, all represented by avatars. Image courtesy of Gustav Taxén and Ambjörn Naeve, The Royal Institute of Technology, Sweden.

CyberMath has been tested for distributed learning in CAVEs but is also running as a desktop VR application with no support of immersive displays. Two usability studies of the DIVE version of CyberMath have been performed with 15 participants in total. Teacher and students worked in two separate locations. The studies provided useful feedback for further improvement of the application and for increasing robustness of the distributed environment. The developers believe that "CyberMath in a networked CAVE environment holds the potential to provide a high-tech front end which is interesting enough to create public interest and contribute to a more positive attitude towards mathematics - especially among young people. It could also provide a useful platform for developing various forms of interactive problem solving games with an emphasis on cooperative problem solving skills".

Geometry education in virtual and augmented reality

Construct3D is a three-dimensional dynamic geometry construction tool that can be used in high school and university education. It uses Augmented Reality (AR) to provide a natural setting for face-to-face collaboration of teachers and students (Figure 7). The main advantage of using VR and AR is that students actually see three dimensional objects which they formerly had to calculate and construct with traditional (mostly pen and paper) methods. By working directly in 3D space, complex spatial problems and spatial relationships may be comprehended better and faster than with traditional methods.

Figure 7. Students working on a hyperboloid in Construct3D.

In a collaborative AR environment, multiple users may access a shared space populated by virtual objects, while remaining grounded in the real world. Because of an augmented real environment, users can see each other. This approach is particularly powerful for educational purposes when users are co-located and can use natural means of communication (speech, gestures etc.), but can also be mixed successfully with immersive VR (Billinghurst et al., 2001) or remote collaboration (Höllerer, Feiner, Terauchi, Rashid, & Hallaway, 1999). Supporting natural collaboration in the mathematics domain opens new possibilities to the educational process.

Before going into detail on Construct3D it might be interesting to deliver insight into state of the art in geometry education and the use of dynamic geometry in schools. The author is part of a group of Austrian geometry teachers developing innovative teaching content for high school geometry education and can report first hand on recent developments.

Geometry education in Austria

Geometry Education in Austria has a very long tradition. This tradition of a first-class geometry education has its roots in more than two centuries of world-known geometers and top researchers in the field of descriptive geometry in middle Europe in general. Researchers from Germany, Hungary and Austria are driving forces in traditional descriptive geometry today. Excellent examples of advanced and beautiful geometric constructions can be found in Brauner (1986) or Wunderlich (1967), and many other classical geometry books. Beautiful geometric constructions were not done to produce art but rather to give the drawing a structure for better readability. However, most geometry teachers are teaching geometry in a very different way nowadays. Descriptive geometry courses are offered in many Austrian high schools for students from grade 7-12 and are not integrated into mathematics courses.

Geometric constructions with pencil on paper play a minor role in modern geometry courses and are mainly used while teaching basic theory to quickly visualize geometric principles. In some cases, simple constructions on paper are just quicker to do than starting the CAD software of choice. The content in the geometry curriculum did not change much over the decades (until recently with the introduction of new surface classes such as extrusion and free form surfaces), but there were big changes to the way it is taught. Because of that, teachers are currently in a phase of experimenting with new media and new tools – seeking for optimal pedagogical tools to teach various kinds of geometric content in an optimal way.

However, a pedagogic tendency for systematic modern geometry education can already be seen. Studies show that students' spatial abilities benefit if geometric sketching (hand drawings) is integrated in geometry education, as is the case in US university courses (Leopold, Gorska, & Sorby, 2000, 2001). Computers are suited for solving more complex tasks that would take too much time to draw manually, and offer new possibilities to gain insight into geometric problems (i.e., see dynamic geometry software below). A useful combination of geometric hand drawings (sketching) and educational CAD programs as well as dynamic geometry software is suggested as a foundation to build a future methodology of geometry education (Husty, 2003).

In Austrian schools, the use of commercial 3D computer-aided design (CAD) software, such as AutoCAD™, Autodesk Mechanical Desktop™, Pro/Engineer™, MicroStation™ and others, is widespread in modern geometry education for teaching principles of 3D modeling. In addition, there are excellent educational 3D modeling programs such as CAD3D (Stachel, Wallner, & Pfeifer, 2007) or GAM (Podenstorfer, 2009) (developed by Austrian geometers specifically for students), which are frequently used. In recent years, a new category of educational geometry software emerged.

Dynamic 2D geometry software

Since a computer can record the way we construct geometric objects, the software is able to quickly redo the construction after changing some parameters. This is the key concept of dynamic geometry: pick a point, move it and see immediately how the construction changes. This dragging capability is the fundamental improvement versus drawings on paper or static CAD models.

Comprehensive work on dynamic geometry was done by Kortenkamp in "Foundations of Dynamic Geometry" (Kortenkamp, 1999) who explains "Much more important (for educational purposes) is the fact that you can explore the dynamic behavior of a construction by moving it. You can see what parts of the construction change and which remain the same. You get by far more insight into this particular construction and geometry in general if you can experience what happens under movements. More sophisticated software will also give you another dimension of understanding by supporting loci, the traces of objects under movement of other objects, that are adjusted dynamically as well".

The first software packages for dynamic geometry were Geometer's Sketchpad (Jackiw, 1995), which appeared first in 1989, and Cabri Géomètre (Laborde, 1993, 1995), dating back to 1988. Since then a lot of work has been done to discuss aspects of using dynamic geometry software in education. Today, there are more than 40 packages for dynamic geometry. The most popular ones are Cinderella (Richter-Gebert & Kortenkamp, 1999), Euklid (Mechling, 2008), Geometer's Sketchpad or Cabri Géomètre. All of them support two-dimensional geometry only.

Dynamic 3D geometry – parametric computer aided design

There have been two revolutions in the history of CAD. The first revolution was a shift from paper-based drafting to computer-aided drafting. And the second revolution was a shift from computer-aided drafting to computer-aided design. The leading CAD technology is called parametric CAD.

Variational or parametric CAD software behaves similarly to dynamic geometry software. Small parameter changes in a CAD construction should lead to slight changes of the construction. This can be used, for instance, to have a single prototype construction which can be customized quickly. Another use case for data compression is to store one template object with a database of parameters instead of storing a large number of similar objects. It is also used for easy and rapid construction of new models by starting with an approximate sketch that is made exact later. Not only is the situation similar to dynamic geometry, the problems (Hoffmann, 1996, 1997) are similar too. Parts of these problems are discussed and have been solved by (Kortenkamp, 1999).

Many CAD applications already support parametric construction to a certain extent. Amongst those are 3DStudio Max™, Maya™, Autodesk Mechanical Desktop™, SolidWorks™, and many others. There are also research systems such as the free parametric modeling tool VARKON (Sourceforge, 2007).

Before the author started working on Construct3D, there was little work (Qasem, 1997; Van Labeke, 1999) done towards developing educational parametric or dynamic 3D tools which were especially tailored to educational purposes. In recent years, Cabri 3D (Laborde, 2005) started to spread in schools. Up to date, none of the existing professional parametric CAD applications offers a similar degree of interactivity and dynamic flexibility as Construct3D does. In addition, their flexibility is limited by their desktop interfaces.

Construct3D

The aim with Construct3D was not to create a professional 3D modeling package but a simple and intuitive to use 3D construction tool in an immersive virtual environment for educational purposes. Construct3D (Kaufmann & Schmalstieg, 2003; Kaufmann, 2004) is based on the Studierstube Augmented Reality system (Schmalstieg et al., 2002). The main advantage of using AR is that students actually see three dimensional objects which they, up to now had to calculate and construct with traditional (mostly pen and paper) methods. By working directly in 3D space, complex spatial problems and spatial relationships may be comprehended better and faster than with traditional methods. We present a system that uses collaborative augmented reality as a medium for teaching, and uses 3D dynamic geometry to facilitate mathematics and geometry education. Both these aspects are novel to geometry education.

Construct3D supports generation of and operation on basic geometric object types: Points (either freely positioned in space or fixed on curves and surfaces), lines, planes, circles, ellipses, cuboids, spheres, cylinders, cones, B-Spline curves with an unlimited number of control points and variable degree, interpolated B-Spline curves, NURBS surfaces up to 8x8 control points and variable degree, interpolated NURBS surfaces and surfaces of revolution (rotational sweep surfaces). Regarding geometric operations, we implemented Boolean operations (intersection, union, difference); intersections between all types of 2D and 3D objects resulting in intersection points and curves; planar slicing of objects; rotational sweep around an axis, and many more. Translations, rotations and mirroring of objects are supported as well. In Figure 8, a few examples of teaching content created with Construct3D are given.

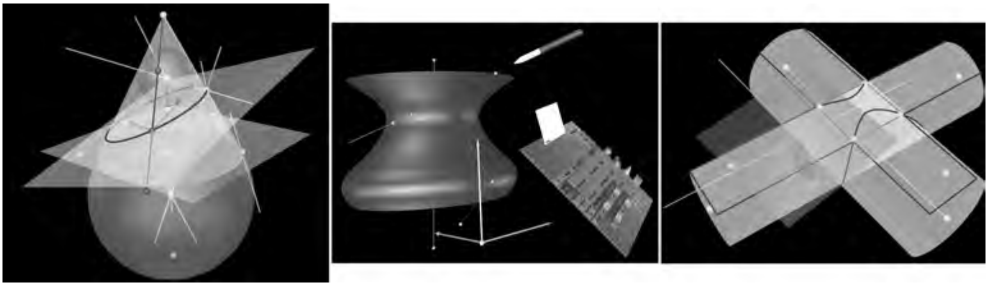

Figure 8. Geometric constructions in Construct3D. Left: Elliptic conic section - proof of Dandelin (1822). Middle: Surface of revolution with the user interface. Right: Intersection curve between two cylinders.

The menu system is mapped to a hand-held tracked panel called the personal interaction panel (PIP) (Szalavári & Gervautz, 1997). The PIP allows the straightforward integration of conventional 2D interface elements like buttons, sliders, dials, and so forth, as well as novel 3D interaction widgets (Figure 9 middle). Passive haptic feedback from the physical props guides the user when interacting with the PIP, while the overlaid graphics allows the props to be used as multi-functional tools. All construction steps are carried out via direct manipulation in 3D using a stylus tracked with six degrees of freedom. In order to generate a new point, the user clicks with his pen exactly at the location in 3D space where the point should appear. Users can switch between point mode (for setting new points) and selection mode (for selecting 3D objects).

A fundamental property of dynamic geometry software is that dynamic behavior of a construction can be explored by interactively moving individual defining elements, such as corner points of a rigid body. For example: moving a point lying on a sphere results in the change of the sphere's radius. Which parts of a construction change and which remain the same can be seen. The histories of constructions as well as dependencies between geometric objects are maintained. Experiencing what happens under movement allows better insight into a particular construction and geometry in general.

Collaborative, educational augmented reality setups

The standard setup used for Construct3D supports two collaborating users wearing stereoscopic see-through HMDs (Sony Glasstron or Emagin HMDs), providing a shared virtual space. The users interact with the system using pen and pad props

(Figure 9 left). Both users see the same virtual objects, as well as each others' pens and menu systems, which allows a student or teacher to help the other user, if necessary.

Other hardware setups such as desktop, projection based, mobile and hybrid setups have been tested during development of Construct3D (Kaufmann, 2004). All of them had pros and cons, but the big advantage of an HMD setup is that it allows users to actively "walk around" geometric objects which are fixed in space. Excited students sometimes lie down on the floor (Figure 9 right) to view objects from below or step on a chair to look down from above. This is a unique feature of an HMD setup which cannot be provided by monitor or projection screen based hardware configurations.

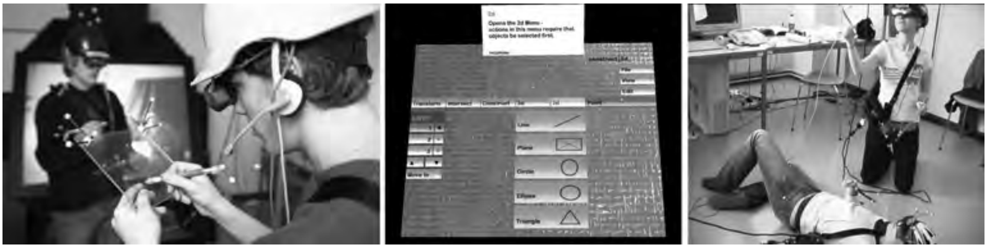

Figure 9. Left: Two students collaborate in our standard lab setup, wearing an HMD, holding a wireless pen and panel (PIP). All devices are optically tracked. Middle: The menu system which is displayed on the panel. Right: Students are actively engaged (outside view). The virtual surface floating between them is not visible in the picture.

The development process of Construct3D resembles the usability engineering methods of virtual environments suggested by Hix and Gabbard (Hix & Gabbard, 2002). The first informal evaluation in 2000 helped to compile a detailed user task analysis whereas expert guideline-based evaluations occurred numerous times during the development process. Visiting teachers and researchers evaluated the system and provided useful feedback. Two formative evaluations in 2003 and 2005 had a big impact on the design and development of Construct3D. All three usability evaluations and the lessons learned are summarized by Kaufmann and Dünser (2007).

Teaching experiences

In evaluations and discussions with teachers, three key strengths of Construct3D were identified:

1) First and foremost the construction of three-dimensional dynamic geometry is a major asset. 3D dynamic geometry cannot be realized with pencil on paper or with static modelling programs. Nearly haptic interaction with geometric objects supports explorative learning.

2) Students can actively walk around an object, which builds up a spatial relationship between the learner's body and object.
3) Teachers especially emphasized the strength to visualize abstract geometric problems. Therefore, the ideal content for this AR geometry learning environment exploits dynamic features, encourages modifications, and is a visualization of (abstract) geometric problems.

Students and teachers were thrilled by simple dynamic interaction such as moving tangential planes on a cylinder or on a surface of revolution. These are things they have never seen or done before.

In order to teach specific learning content, teachers usually pick the learning medium which is best suited for teaching the content. Since we know Construct3D's strengths, it should mainly be integrated in geometry education for teaching content which requires 3D dynamic geometry or requires the visualization of abstract problems (as shown in the learning objects below). In addition, these are exactly those areas that are hardly covered by other educational applications. Tasks that are well suited for Construct3D might be impossible to do in traditional courses, and those ideal for paper and pencil work might be impractical to do in Construct3D. Because each medium is different, different content should be used to utilise the strengths of each.

The advantages of working in an immersive AR setup are manifold. Users see their own bodies and hands, as well as the effects of their actions while working. The construction process physically involves students and resembles handcraft more than traditional computer operation. A spatial relationship between the user's body and the geometric object in 3D space is established. We believe these are key factors in the potential success of using AR for teaching geometry.

Construct3D was developed to a point where it can be productively used for educational purposes. Due to a wide range of features, useful content which utilizes dynamic functionality is easy to produce for teachers and students in a very short time. This tool allows novel ways of teaching. Dynamic 3D construction and visualization capabilities are its fundamental strengths.

Learning content

Content design must reflect these findings. It is obvious that doing simple, introductory geometric tasks such as cutting cubic blocks out of a static larger cube can be done faster and easier by hand drawing or simple desktop CAD programs, saving a lot of money for expensive VR hardware. Therefore, a challenge was to design interesting content for a 3D dynamic geometry application that justifies its use. Creating innovative content for 3D dynamic geometry is a new challenge for educators where profound geometric knowledge is needed. Dynamic 3D geometry is a new and emerging field of research.

In the following, two geometric examples and learning tasks are listed that are ideal for teaching geometry with Construct3D. Students of age 16-18 who have geometry classes in high school work collaboratively with a teacher and another student in an HMD based setup as described above. Beforehand, the problem description and a screenshot of the given elements that they will face later in the virtual world are given to the students. Web links are presented with additional information about the task; given images help to understand and translate the problem.

Flight route

The task is to construct the shortest flight route from Vienna to Sydney, which in an ideal case is an orthodrome. The given virtual scene in Construct3D (Figure 10) shows a model of earth (with texture) to help pupils find the correct places on earth and to immerse them further into the problem.

Figure 10. Left: Vienna is marked on the globe. Middle: Two students are collaborating to solve the problem. Right: Milling with a spherical cutting tool which is too big.

Milling and Curvature

Milling is frequently used to create rotation-symmetric objects – which are surfaces of revolution in geometric terms. If a spherical cutter is used as a cutting tool, the radius of the sphere must be chosen appropriately not to cut away too much of the material (Figure 10 right). The task is to construct an arbitrary surface of revolution and find a spherical cutting tool with the “right” size to mill the object. There must be no intersections of tool and object. A number of topics can be discussed while solving this task such as tangential properties, curvature of surfaces, surface normals and more. Students greatly enjoy this example which gives them the chance to play around with a surface of revolution, change it, walk into it, lie down or look into it from above. The principle of a tangential plane is easily explained and once students see it, they understand.

Obstacles

The question remains why applications such as Construct3D that seem to be successful and well developed are not used in classrooms yet. After many discussions during evaluations with teachers and students three main obstacles were identified. These are not only valid for Construct3D but for a wider range of educational VR applications.

Hardware costs

Current VR/AR hardware is expensive mainly because of no existing mass market for VR/AR solutions. It is unrealistic that an average high school can afford an immersive setup, which is most favored by teachers and students for use with Construct3D. Tracking position and orientation of multiple users and all their devices accurately is the most expensive part of an immersive system. This is the main reason why the author initiated development of a low cost optical tracking system iotracker to minimize the expenses for tracking, aiming at the integration of Construct3D in high school and university education (Pintaric & Kaufmann, 2007). Prices are still high compared to the available budget of schools, but movement can be seen in the rigid motion tracking market recently.

Support of limited number of users

Most of the applications presented here support only a very limited number of users. This narrows down the possibilities of usage for teachers. Integration into regular lessons becomes difficult if only a very few students are able to use advanced technology. Pedagogical concepts are needed to integrate high end technology in a meaningful way for all students to benefit. Another option is to use VR technology only in special courses where few students attend, for example, in courses for high or low achievers.

Technical complexity

Even if the hardware was cheap, teachers who were thrilled by the possibilities confirmed they would not use an immersive Construct3D setup in their school – given they had a spare room to set it up – because of its technical complexity. A technical setup consisting of multiple components is prone to errors and requires maintenance. Even a standard school PC lab with standard computers, each equipped with a monitor, mouse and keyboard, requires regular maintenance and work. No matter how simple additional technology is, it introduces new complexity. Adding a tracking system, interaction devices, additional display devices, wireless communication and more would require additional maintenance personnel for at least a few hours per week.

Last but not least, stability and reliability are very important criteria for educational software in general. Correct results are absolutely important and if the software crashes students can easily loose motivation. For learning, wrong results are worse than no results.

Conclusion

The applications presented in this article are all interesting in their own ways. They give an insight into the potential that the use of VR provides for mathematics and geometry education. They do not simply take traditional content and present it in a new form, but rather try to find innovative ways to teach mathematics in a new way. With the introduction of technology into classrooms - such as graphical calculators in the recent past - much more difficult and comprehensive problems can be studied in mathematics education that were far beyond reach for students in earlier days. Adapting traditional content to a new medium is one challenge, but in some cases VR technology enables us to teach completely new content. One such example is three-dimensional dynamic geometry which is impossible to teach by traditional - paper and pencil - means.

The key to achieve an improved learning experience is to find ways to utilize strengths of the new technology at hand. Since many of the technological problems have been solved in previous years and VR technology is becoming mature, the challenge is more and more becoming a didactic and pedagogic one. How do we as teachers unleash the power that is at our hands?

The presented examples give hints at how VR can be utilized for special target groups such as for the hearing impaired, and for specific mathematics domains such as complex numbers or dynamic geometry, and they show how content can be adapted to become tangible, visible and understandable, even in the abstract case of algebra. They all have one thing in common: They present mathematics in an extremely motivating way to students, in a way many will not forget for the rest of their lives. Even if there are obstacles that prevent the use of VR in mathematics education today, this technology exerts a fascination on students that should not be ignored. It would be a shame not to use the potential of the technology to raise new generations of mathematically skilled students.

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