A developmental perspective on mathematics teaching and learning:
The case of multiplicative thinking

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Abstract: This paper looks at the issue of mathematics learning from a developmental perspective. It begins by focusing on the importance for teachers of understanding how mathematical thinking develops.

The New Zealand Number Framework is used as an example of a developmental progression that is of particular relevance to the teaching of mathematics. The paper examines data from the Numeracy Development Project gathered by teachers using the diagnostic interview – the Numeracy Project Assessment (NumPA). The data comes from almost a quarter of a million students whose teachers have participated in the Numeracy Development Project over the past three years.

Two major developmental progressions are explored in the paper – the transition from counting to part-whole strategies, and the move from additive to multiplicative part-whole thinking.

The final section of the paper looks at some practical ways of fostering multiplicative thinking using structured materials.

The importance for teachers of taking a developmental perspective towards their teaching is increasingly being recognised. A recent publication from the Committee on Teacher Education of the US National Academy of Education presents what is currently considered to be key foundational knowledge for teaching in a series of 12 state-of-the-art papers (see Darling-Hammond & Baratz-Snowden, 2005; Darling-Hammond & Bransford, 2005). The framework for understanding teaching and learning has three major components: knowledge of learners and their development in social contexts, knowledge of subject matter and curriculum goals, and knowledge of teaching. Knowledge of learners comprises: understanding learners and learning, understanding human development, and understanding the development and use of language. In New Zealand, the Teachers' Council has drafted a document outlining the graduating standards for teacher education qualifications leading to provisional registration as a teacher. As in the US, New Zealand teachers are expected to have professional knowledge (including theory and current research) of "human development, learning and pedagogy which contributes to understanding the learning needs of a diverse range of learners" (ref needed).

Such is the importance placed on understanding human development, that the US material includes an entire chapter devoted to the topic, aptly titled 'Educating teachers for developmentally appropriate practice' (Horowitz, Darling-Hammond & Bransford, 2005). In this chapter, it is asserted that "novice teachers should understand that knowing about development is central to being an effective teacher... [and] understanding developmental pathways and progressions is extremely important for teaching in ways that are optimal for each child" (p. 92).

Developmental Perspectives on Students' Thinking

Many people associate developmental perspectives on students' thinking with the work of Jean Piaget, the Swiss psychologist whose theory about the development of children's thinking is well known to those working in the fields of education and social sciences. Although Piaget's theory has been criticised by many writers, there are key aspects of his work that are still very powerful in helping to explain how students learn. Piaget's idea that children learn to make sense of the world as part of a process of *adaptation,* makes a valuable contribution to our understanding of learning processes. According to Piaget, *adaptation* happens as a result of the twin processes of *assimilation* and *accommodation*. Initially, people make sense of or interpret the world according to their existing conceptual frameworks (assimilation). However, often some aspect of a new object or experience does not quite fit with that existing knowledge and an extension or elaboration of the model of understanding is required in order to take account of that new feature of the experience (accommodation). Once we become aware of these processes, it is easy to see how assimilation and accommodation happen over and over again as we encounter new experiences or information in our everyday lives. These processes explain the way that new information builds on prior knowledge, and the importance for teachers of finding out about students' existing knowledge and understanding *before* attempting to teach something new. The processes of assimilation and accommodation also explain the way learning is usually a fairly gradual process, with students continually constructing and reconstructing their understanding in response to engagement with their environments. Piaget's theory has been credited by many writers as a forerunner of Constructivism, a theoretical approach to teaching and learning that is widely accepted within the education profession (Barker, 2000/2001; Biddulph & Carr, 1999/2000).

Developmental pathways or progressions consist of ordered sequences of steps or stages that describe increasingly sophisticated ways of responding to particular learning tasks. They provide the basis for expectations about what the next milestone in a sequence might be. They have the potential to enhance learning by informing teachers about the likely next step in terms of learning goals, so that energy is not wasted on trying to teach something that is much further along the pathway than is realistically achievable for a particular student. On the other hand, a potential disadvantage is that they may constrain learning by imposing limits on what a particular student is perceived as being able to learn. Having a good understanding of developmental pathways and progressions should enable teachers to tailor their teaching to meet the developmental learning needs of their students.

Some writers have used a staircase metaphor to characterise the way Piaget's theory explains student learning, contrasting it with the network metaphor of constructivist approaches (Biddulph & Carr, 1999/2000). Others have contrasted a ladder metaphor with a jungle-gym metaphor (Begg, 2004). Socio-cultural approaches to learning, such as Vygotsky's, have become popular in recent times because of the way that they acknowledge the social and cultural aspects of learning as a person participates in their communities. Although Piaget's theory tends to emphasise the individual acquisition of knowledge and understanding, it sits comfortably alongside socio-cultural approaches to learning. As Sfard (1998; 2003) has pointed out, the so-called acquisitionist metaphor (reflected in Piaget's theory) complements the participationist metaphor (evident in the work of socio-cultural theorists). The value of both approaches is reflected in the title of Sfard's (1998) paper: 'Two metaphors for learning and the dangers of choosing just one'.

A Developmental Pathway for Numeracy Learning

Like other frameworks developed elsewhere, the NZ number framework is a research-based framework that describes progressions in understanding number (see Figure 1) (Bobis, Clarke, Clarke, Thomas, Wright, Young-Loveridge, & Gould, 2005; NSW Department of Education & Training, 2003). The number framework was developed to give teachers a way of describing students' attainment on the basis of their number knowledge and problem-solving strategies. It needs to be considered alongside the individual task-based interview (NumPA) designed to help teachers assess children's mathematical thinking and make judgements about the stage on the number framework a student's response best fits. Teachers are introduced to the number framework and the diagnostic interview as part of a comprehensive programme of professional development (for more information, see Ministry of Education, 2005a; 2005b; 2005c; n.d.).

Figure 1:

Overview of the New Zealand Number Framework

0 **Emergent** (cannot yet count)
1 **One-to-one counting** (can count a single collection only)
2 **Count from one with materials** (counts all for two collections of materials)
3 **Count from one using imaging** (counts all for two screened collections)
4 **Advanced Counting** (counts on from one of two collections)
5 **Early Additive Part-Whole** (uses simple partitioning & recombining)
6 **Advanced Additive P-W** (uses a range of additive part-whole strategies)
7 **Adv'd Multiplicative P-W** (uses a range of multiplicative p-w strategies)
8 **Advanced Proportional P-W** (uses a range of proportional p-w strategies)

The framework has two main components: Strategy and Knowledge. The Strategy component focuses on how students solve number problems, and the extent to which they use mental processes as part of their solution strategies. The Knowledge component encompasses key items of knowledge about the number system, including the identification and ordering of whole numbers and fractions, as well as place value and basic facts. The two components are seen as interdependent, with Strategy creating new knowledge through use, and Knowledge providing the foundation upon which more sophisticated strategies are built. Strategies include the domains of addition/subtraction, multiplication/division, and proportion/ratio. The domain of addition/subtraction provides a core component on which all students are assessed. Students' responses to addition/subtraction tasks are then used to determine whether students should be assessed using the easiest form (A), the hardest form (C), or the one in between (B). The choice of form affects the opportunities students are given to show what they understand and can do. For example, only Forms B and C include tasks to assess multiplication/division and proportion/ratio, and only Form C includes the most challenging tasks that allow students to show the highest levels of reasoning in these two domains.

Progression from Counting to Part-Whole Thinking

One of the challenges for teachers of students at lower framework stages (0 to 4) is to help them move beyond counting to part-whole strategies (Young-Loveridge, 2001; 2002a). Data from the Numeracy Development Project shows that even at the year 7 and 8 level, a notable group of students continue to rely on counting strategies to solve problems, even after they have been given extra help (31% & 24% before the project, and 15% & 9% after the project, for years 7 and 8, respectively; see Young-Loveridge, 2005).

The NZ Numeracy Project credits students with Stage 5, Early Additive Part-Whole Thinking, if they can use any part-whole strategy to solve addition or subtraction problems mentally by reasoning the answer from basic facts and/or place value knowledge. For example, when given 9 + 8, they partition 8 into 1 and 7 so that the 1 can be put with the 9 to make 10, then the remaining 7 is added on to 10 to make 17. Alternatively, they might use knowledge of doubles with an adjustment (9 + 9 – 1 or 8 + 8 + 1), or add 10 to 8 then take 1 off. Students can be credited with Stage 6, Advanced Additive Part-Whole Thinking, if they are able to use at least two different mental strategies to solve addition or subtraction problems with multi-digit numbers. The three problems which allow students to show part-whole thinking include the bus problem: 53 people on the bus, 26 get off, How many people left on the bus; 394 stamps plus another 79 stamps, How many stamps then?; and $403 in the bank account, $97 used to buy a new skateboard, How much money left in the account?

Closely connected to the transition to part-whole thinking are the two different ways of thinking about the number system (Young-Loveridge, 2004). These different conceptions of number have been identified as underpinning children's solution strategies for problems involving addition, subtraction, multiplication, or division (Yackel, 2001). *Counting-based* (or sequence-based) solutions are based on the number line. They begin with one of the numbers in the problem and involve jumping along the number line, either forwards (in the case of addition or multiplication) or backwards (in the case of subtraction or division). Even when there is no evidence of counting per se, it is assumed that abbreviated counting (e.g. counting on or skip counting) is the basis for the solution (Yackel, 2001). The empty number line developed by the Dutch is a good example of a counting-based model (Beishuizen, 1999; Carr, 1998; Klein, Beishuizen & Treffers, 1998).

Collections-based solutions involve the partitioning of numbers into component parts and the subsequent joining (in the case of addition or multiplication) or separating (in subtraction or division) of the parts to get the answer (Yackel, 2001). Standard place-value partitioning (breaking up numbers according to the value of the units, such as hundreds, tens & ones) is just one way of partitioning numbers. A different partitioning strategy could be based on doubling or halving. For example, a small but notable group of Year 6 students used their knowledge that two groups of 25 make 50 to solve a problem involving 53 – 26; that is, they halved 50 to get 25, took one more off to get 24, then put on 3 to arrive at an answer of 27 (Young-Loveridge, 2002a).

The number frameworks used in various numeracy initiatives usually begin with counting at lower levels and progress to the use of derived number facts at higher levels, implying that collections-based ways of thinking about numbers are more sophisticated than counting-based approaches (e.g. Ministry of Education, 2005a; NSW Department of Education & Training, 2003). These two different conceptions complement each other. According to Yackel (2001) it is "important for children to have both a collections-based and a counting-based conception of number," (p. 25) because of the flexibility that it gives them in terms of possible solution strategies.

Progression from Additive to Multiplicative Thinking

Much of the writing on the importance of acquiring part-whole thinking focuses almost exclusively on getting students to use part-whole thinking for problems involving addition and subtraction. However, additive thinking is just the first step towards part-whole thinking. It also provides the foundation on which multiplicative thinking can build (Young-Loveridge & Wright, 2002). Multiplicative thinking has increasingly been the focus of research and writing in recent times (Clark & Kamii, 1996; Fuson, 2003; Mulligan & Mitchelmore, 1997). Several academics have written about the importance of distinguishing between additive and multiplicative reasoning (eg, Clark & Kamii, 1996; Jacob & Willis, 2001). Sowder (2002) provides a simple but powerful example of the crucial difference between additive and multiplicative thinking using an investment scenario: one person invests $2 and gets back $8; while the other person invests $6 and gets back $12. Typically additive thinkers simply calculate the difference between the investment and the profit, and conclude that the deals are the same because both people make a $6 profit. Multiplicative thinkers, on the other hand, can appreciate that the first investment quadrupled, whereas the second investment only doubled, so the first investment is a better deal.

The New Zealand (NZ) Number Framework recognises the way that multiplicative thinking builds on additive thinking, and multiplicative thinking in turn provides the foundation on which proportional reasoning can be built (see Figure 1 and Ministry of Education, 2005a; n.d.). Research evidence supports the idea that students have difficulty reasoning proportionally unless they can use multiplicative part-whole strategies (Young-Loveridge & Wright, 2002). Likewise, students find it difficult to reason multiplicatively unless they have a good grasp of additive part-whole strategies (Young-Loveridge & Wright, 2002). The NZ Numeracy Project credits students with Stage 7, Advanced Multiplicative Thinking, if they can use at least two different multiplicative part-whole strategies to solve problems such as finding the total muffins in six baskets each with 24 muffins, and/or finding how many 4-wheeled cars could be made from 72 wheels. Examples of multiplicative strategies for the muffins problem include: 6 x 20 = 120, 6 x 4 = 24, 120 + 24 = 144 (standard place-value partitioning); or 6 x 25 = 150, 150 – 6 = 144 (compensation); or 6 x 24 = 12 x 12 = 144 (doubling and halving). Possible multiplicative part-whole strategies for the wheels problem include: 80 ÷ 4 = 20 so 72 ÷ 4 = 20 – (8 ÷ 4) = 18 (compensation); or 10 x 4 = 40, 72 – 40 = 32, 8 x 4 = 32, 10 + 8 = 18, so 18 x 4 = 72 (reversibility and place-value partitioning); or 9 x 8 = 72 so 18 x 4 = 72 (doubling and halving) (Ministry of Education, 2005a).

Recently, I have become more aware of the importance of multiplicative thinking and the need for children to progress beyond additive part-whole thinking. My interest in this issue was sparked by curiosity about the way that the current Numeracy Project materials present the New Zealand Number Framework using number-line models to illustrate each of the framework stages (see Ministry of Education, 2005a). In an earlier version of the materials (Ministry of Education, 2001), multiplicative thinking was illustrated using array-based models. The advantage of using arrays is that the distributive property can be very nicely shown (see Young-Loveridge, in press). Both number-line and array-based models are iconic representations of the multiplicative process. In other words, they provide visual images that can act as a bridge between concrete materials and abstract representations (see Ministry of Education, 2005c).

A big advantage of array-based models is that they allow the two-dimensionality of multiplication to be shown clearly. Number-line models, by contrast, are uni-dimensional as they present multiplication as a series of (equal-sized) jumps along a number line. The unfortunate implication of this way of presenting multiplication is that multiplication is conveyed as a process of repeated addition. However, research literature indicates that repeated addition is a relatively unsophisticated strategy for solving multiplication problems that does not take account of the complexity of multiplication processes (see Mulligan & Mitchelmore, 1997). Working with arrays – partitioning and recombining the parts – enables students to develop a solid understanding of the way multiplication works and an appreciation of the distributive property of number. Having this deeper and more flexible understanding of multiplication is part of what enables multiplicative thinkers to use more powerful strategies than their additive peers.

Evidence from the Numeracy Project Data on Developmental Progressions

Analysis of the data gathered over the past three years by teachers using the diagnostic interview NumPA (Numeracy Project Diagnostic Assessment) individually with each of their students shows the importance of moving from counting on (Stage 4) to part-whole thinking (Stage 5 and above). Table 1 shows the percentages of students at the end of the project who were at Stages 4 to 6 on Addition/Subtraction and were also at Stage 5 or higher on other domains. Table 2 shows comparable data for Multiplication/Division. Data from almost a quarter of a million students is included in these two tables. The patterns, which were reasonably consistent across the three cohorts that participated in the project between 2002 and 2004, have been averaged over the three years for the purposes of this paper.

TABLE 1:
*Percentages of students at Stages 4-6 on **Addition/Subtraction** who were at Stage 5 or higher on other domains averaged across 2002-2004*

ADDITION/SUBTRACTION	Stage 6 Advanced Additive n=59929	Stage 5 Early Additive n=104296	Stage 4 Advanced Counting n=73620
Mult/Division			
5 Early Additive	10.7	41.0	12.3
6 Early Multiplicative/Adv Additive	40.6	29.2	2.7
7 Advanced Multiplicative	46.8	1.5	0.1
Total Add Part-Whole	***98.1***	***71.7***	***15.0***
Total Mult Part-Whole	***87.4***	***30.7***	***2.7***
Proportion/Ratio			
5 Early Additive	17.5	37.9	11.4
6 Advanced Additive	30.5	19.2	1.9
7 Early Proportional/Adv Mult've	35.6	2.0	0.1
8 Advanced Proportional	10.8	0.2	0.0
Total Add Part-Whole	***94.3***	***59.3***	***13.4***
Total Mult Part-Whole	***76.9***	***21.4***	***2.0***
Total Prop Part-Whole	***46.4***	***2.2***	***0.1***
Fractions			
5 Orders units fractions	26.5	44.4	20.1
6 Coordinates numerators & denoms	28.1	15.5	2.2
7 Recognises equivalent fractions	18.7	1.0	0.0
8 Orders fractions w unlike num/den	15.1	0.4	0.1
Total	***88.3***	***61.4***	***22.3***
Decimals & Percentages			
5 Identifies decimals to 3 places	24.7	3.8	0.6
6 Orders decimals to 3 places	17.0	1.6	0.1
7 Rounds to nearest whole, tenth, hth	13.3	0.6	0.1
8 Converts decimal to percentage	11.4	0.3	0.1
Total	***66.3***	***6.2***	***0.8***
Grouping & Place Value			
5 Knows tens in 100	25.6	42.4	14.0
6 Knows tens & hund in whole nos	23.2	11.4	1.6
7 Knows tens, hun, thou in whole nos	23.0	1.3	0.1
8 Knows tenths, hths, thths in decimals	18.7	0.7	0.0
Total	***90.5***	***55.7***	***15.7***

Table 2:

*Percentages of students at Stages 4-7 on **Multiplication/Division** who were at Stage 5 or higher on other domains averaged across 2002-2004*

MULT'N/DIVISION	Stage 7 Advanced Mult've n=29626	Stage 6 Early Mult've n=56856	Stage 5 Repeated Addition n=58156	Stage 4 Skip Counting n=65499
Addition/Subtraction				
5 Early Additive	5.3	53.3	73.2	40.5
6 Advanced Additive	94.5	43.2	11.2	1.3
Total Add Part-Whole	***99.8***	***96.5***	***84.4***	***41.8***
Proportion/Ratio				
5 Early Additive	5.7	31.0	47.6	16.6
6 Advanced Additive	21.1	43.9	12.5	1.8
7 Early Prop'al/ Adv Mult've	51.0	13.2	1.6	0.1
8 Advanced Proportional	20.8	0.8	0.0	0.0
Total Add Part-Whole	***98.5***	***89.0***	***61.7***	***18.5***
Total Mult Part-Whole	***92.8***	***57.9***	***14.1***	***1.9***
Total Prop Part-Whole	***71.7***	***14.0***	***1.6***	***0.1***
Fractions				
5 Orders units fractions	15.7	41.8	47.8	28.6
6 Coordinates num's & denom's	25.4	32.0	12.0	2.7
7 Recog equivalent fractions	26.7	6.8	0.8	0.1
8 Orders fractions w unlike n/d	26.7	2.5	0.3	0.0
Total	***94.5***	***83.0***	***60.8***	***31.5***
Decimals & Percentages				
5 Identifies decimal to 3 places	22.0	15.6	5.2	1.0
6 Orders decimals to 3 places	19.5	8.7	1.8	0.3
7 rounds nearest whole, tenth, hth	20.1	4.1	0.5	0.1
8 converts decimal to percent	20.6	1.5	0.2	0.0
Total	***82.2***	***29.9***	***7.6***	***1.3***
Grouping & Place Value				
5 Knows tens in 100	13.9	42.9	44.6	21.9
6 tens & hund in whole nos	19.1	26.1	9.3	1.6
7 tens, hun, thou in whole nos	30.6	9.4	1.2	0.1
8 tth, hth, thth in decimals	32.3	3.7	0.3	0.1
Total	***95.8***	***82.0***	***55.3***	***23.6***

The first column in Table 1 shows the results for students who had been judged by their teachers to be at Stage 6 Advanced Additive Part-Whole. To the right are students at Stage 5 Early Additive Part-Whole, and then those at Stage 4 Advanced Counting. More than three-quarters (87.4%) of the Stage 6 students were able to use multiplicative thinking to work out that, if 3 x 20 = 60, then 3 x 18 = 60 – (3 x 2) = 54, or if 5 x 8 = 40 then 5 x 16 = 80. In contrast, fewer than a third (30.7%) of students at Stage 5 were able to do this, and virtually no students at Stage 4 (2.7%). Just under half (46.8%) of students at Stage 6 were able to use a multiplicative strategy to solve the muffins problem (6 x 24) and/or the wheels problem (72 ÷ 4). This was virtually the same as the number (46.4%) who were able to use proportional reasoning to work out that, if 12 is 2/3 of a number, that number is 18. Very few of the students at Stage 4 or 5 were able to solve that problem (2.2% & 0.1%, respectively). Likewise, substantially more students at Stage 6 were able to work flexibly with fractions and decimals than at lower stages. Students who continued to use counting on/back to solve addition and subtraction problems (Stage 4) were extremely limited in their work with fractions and decimals. Fewer than a quarter (20.1%) of students at Stage 4 could order unit fractions (Stage 5), and only a tiny proportion were able to convert an improper fraction (e.g. 8/6) into a mixed fraction (Stage 6), recognise the equivalence of 2/3 and 6/9 (Stage 7), or order a series of fractions with unlike numerators and denominators (Stage 8).

A similar pattern is shown in Table 2, except that the first column shows students who were at Stage 7 on Multiplication/Division, and each column to the right of this shows students at the stage below. It is evident from Table 2 that only students at Stage 7 showed strong evidence of proportional reasoning (51.0% & 20.8% at Stages 7 & 8 on Proportion/Ratio, respectively). These Stage 7 Advanced Multiplicative students did substantially better than those at lower stages on fractions, decimals, and percentages, underlining the importance of multiplicative thinking for these domains. Evidence from Table 1 and Table 2 supports the idea of a developmental progression from additive thinking to multiplicative thinking, and from multiplicative thinking to proportional reasoning.

Practical Ways of Developing Multiplicative Thinking

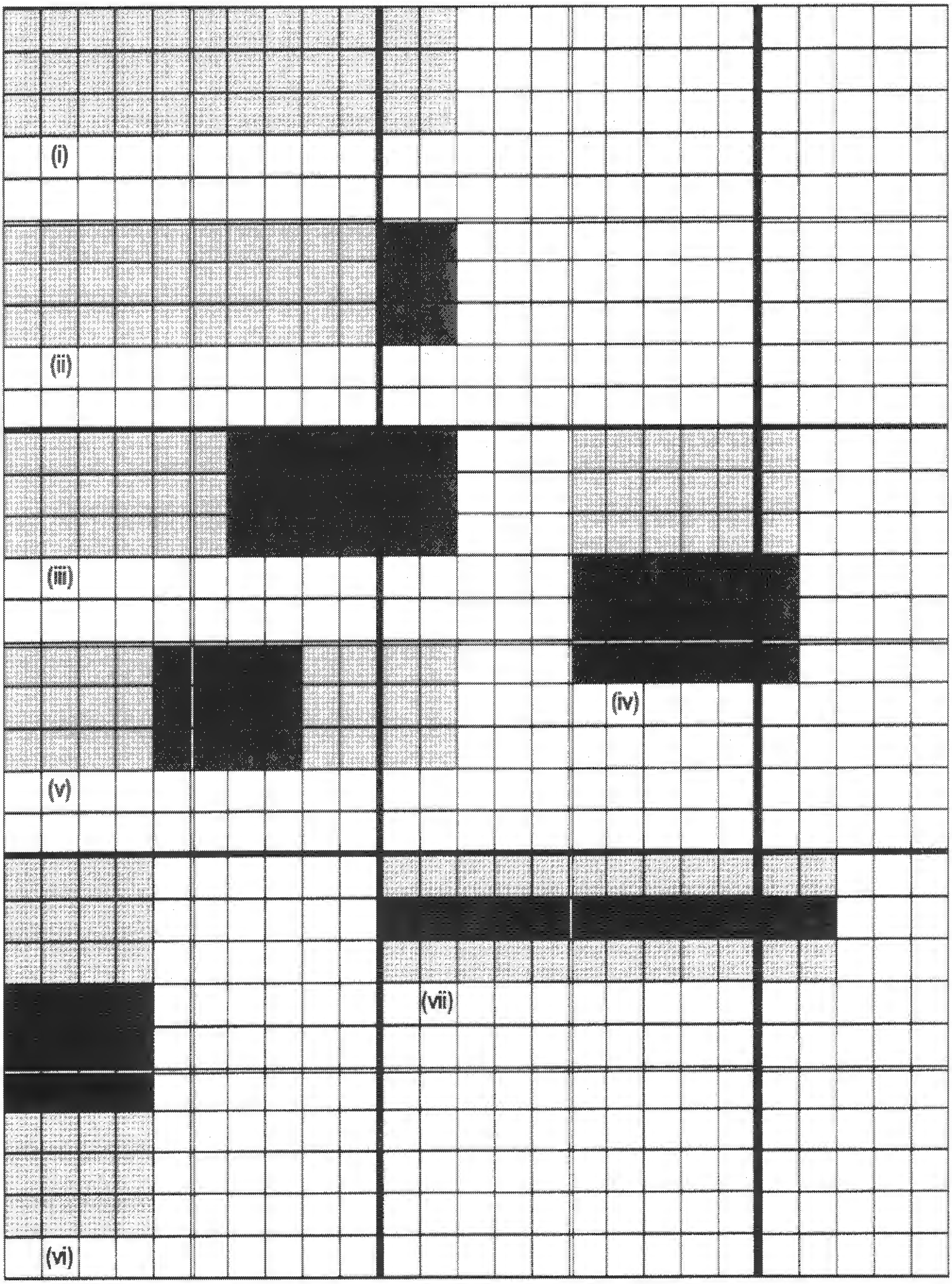
A variety of structured materials are available to support and encourage children's transition to using part-whole strategies rather than counting strategies. Ten-frames are particularly useful for helping students use groupings of five and ten to learn about partitioning and recombining quantities to solve problems. A three-dimensional[1] version of the ten-frame, constructed from recycled egg cartons trimmed down to just ten compartments, allows students to manipulate materials without worrying about the possibility that the objects might slide off the surface of the ten-frame (Young-Loveridge, 2001; 2002b). They have the added advantage of appealing to young children's acquisitive tendencies and are large enough to hold a range of small treasures.

Recently, I became intrigued with a piece of computer software (Maddy the Multiplier) that allows students to partition a grid showing a multi-digit multiplication problem by starting with their existing knowledge of multiplication (the known), and building onto this the remaining part/s of the array (the unknown) (for further details, see The Learning Federation, n.d.). A computer mouse is used to partition the array into parts, by sliding a horizontal dividing line up and down and a vertical dividing line from left to right (by clicking & dragging), to stretch or shrink the known array (shown in one colour). The remainder of the array can then be further partitioned into two or three smaller arrays (each shown by a different colour). Although there are textbooks that include software with them that allow the reader to model the operations and manipulate fractional numbers on a computer, Maddy requires the user to be simultaneously logged into a website (for authentication purposes), and this may not be possible in some classrooms. I developed a paper version of the software using a structured grid and different-coloured highlighter pens to show the smaller arrays within the overall larger array (see Appendix A and Young-Loveridge, in press). The grid is structured to show sections of 10 by 10 (heavy black lines), and the 5 by 5 sub-sections within these (fine double lines). A multi-digit multiplication problem up to 35 x 25 can be shown using the grid. The base-ten structure of the grid makes it easy to see the six blocks of 100, the five blocks of 50 (5 x 10) that together make another 250, and a block of 5 x 5 that completes the array. Summing the partial products: 600 + 250 + 25, gives a total of 875, the product of 35 by 25.

Students start by colouring the array for a multiplication fact they do know. They then use different colours for the remaining array/s. Adding the partial products of each smaller array yields the overall product. We can see how the system works with a simple multiplication problem like 3 x 12 (see Figure 2 i). For example, colouring/shading a 3 by 10 array in one colour/shade leaves an array of 3 by 2, which accounts for the remaining 6 (see Figure 2 ii). The array can also be made from two identical blocks of 3 by 6 that are side by side (see Figure 2 iii). Moving one of those blocks directly below the other reveals the equivalence of a 6 by 6 array to the original 3 by 12 array (see Figure 2 iv). An alternative way of partitioning the 3 by 12 array is to distinguish three identical blocks of 3 by 4 that are side by side (see Figure 2 v). Moving the second and third blocks directly below the first block reveals the equivalence of a 9 by 4 array to the original 3 by 12 array (see Figure 2 vi). Three strips of 12, each a different colour/shade, shows the repeated addition strategy of 12 + 12 + 12 (see Figure 2 vii). This final model differs from the other six in being a linear (one-dimensional) additive model, composed of three single rows of each colour. The other six examples are two-dimensional multiplicative models, each composed of different combinations of smaller arrays within the larger array. One of the advantages of this paper version of Maddy Multiplier with different colouring/shading is that a permanent record can be made of the alternative partitionings, and this can be shared with other students and with the teacher.

Figure 2:

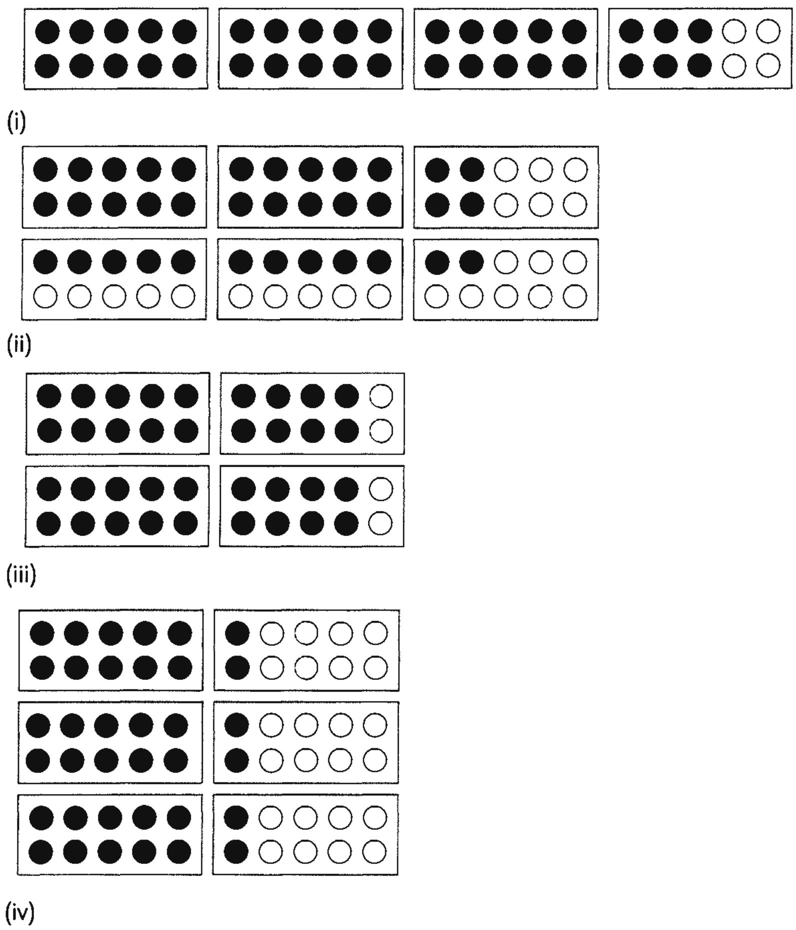
A range of ways to partition 3 x 12 = 36 using multiplicative (array-based) partitioning (i to vi) and additive (linear) partitioning (vii), shown with different colouring/shading



For some students, the paper grids may be a little too abstract to help them make sense of multiplication initially. A more recent development builds on the use of the egg carton ten-frames to help students experience more directly multiple partitionings of quantities and involves repackaging. Using coloured ping pong balls for eggs, students are asked to think about other possible sizes and shapes for a new design of egg carton by exploring as many different ways they can come up with of arranging a collection of eggs, using the three-dimensional ten-frames as convenient grouping containers. For example, 36 eggs can be arranged in two rows of 18, three rows of 12, four rows of 9, or six rows of 6 (see Figure 3). By directly experiencing the manipulation and transformation of different arrays for a relatively small quantity, students should come to appreciate the two-dimensionality of multiplication. For example, the first arrangement (2 x 18) can be cut in half and the right-hand half re-positioned below the left-hand half to create the third arrangement (4 x 9) via halving and doubling processes. The same halving and doubling process can be used with the second arrangement (3 x 12) to create the fourth arrangement (6 x 6). Working with arrays in these ways also provides a more gradual introduction to the idea of multiplication as groups of groups, and can provide an introduction to the use of the paper grids to help solve multi-digit multiplication problems.

Figure 3:

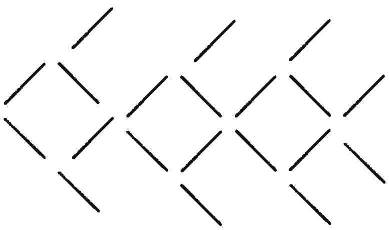
Different ways of arranging 36 objects in ten-frame containers, including: 2 rows of 18, 3 rows of 12, 4 rows of 9, and 6 rows of 6.



Although Piaget's theory proposed that concrete operations precede formal operations, there is a major question about whether or not all adults reach a stage where their thinking is universally formal and abstract. Unfortunately, many students at the upper primary level seem to regard the use of equipment as beneath them, considering it appropriate only for young children (Year 5/6 children told us "That's just for babies"). In my experience, concrete materials can be very useful in the early stages of learning even quite challenging concepts, and this is just as true for adult students as it is for children. These materials help students to engage with complex ideas more directly and allow them to explore and experiment with the ideas. A nice example is the use of stick patterns to model simple algebraic patterns. For example, a row of fish can be constructed whereby the tail of one fish doubles as the nose of the next fish (see Figure 4). The challenge for students is to work out how many sticks are required to make the row of three fish, and then how many would be required to make a longer row, of say ten fish, 100 fish, or any number of fish. Students soon work out that six sticks are needed for each fish that is connected to a fish in front and another fish behind, but the final fish in the row needs two extra sticks to make its tail. By working out a formula for this (6 x the number of fish + 2), they can generalise the pattern to any number of fish.

Figure 4:

An example of concrete materials used to illustrate an algebraic pattern (number of sticks = 6 x number of fish + 2).

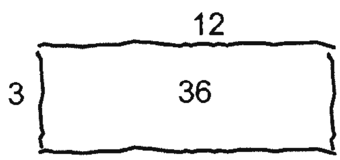


Modelling with materials like this can help provide a bridge to more abstract concepts or processes. According to many writers, students' thinking moves from a dependence on materials, through a transitional stage where imaging is used, to a stage where understanding of number properties can be used to solve problems in mathematics (Fuson, 2003; Hughes, 2002; Ministry of Education, 2005c; Pirie & Kieran, 1994). Structured materials such as the three-dimensional ten-frames or paper grids can provide a bridge to solving multiplication problems. Eventually students may be able to leave paper grids behind and simply do rough free-hand sketches of arrays, conveniently partitioning an array into the known and unknown parts in order to work out the partial products and thus solve the problem (Figure 5).

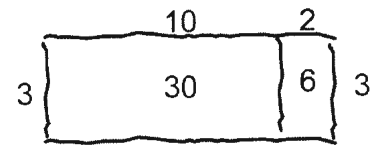
Figure 5:

Free-hand sketches showing ways to partition 3 x 12 = 36 using multiplicative (array-based) partitioning

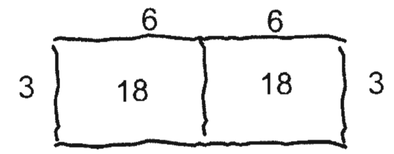
(i)



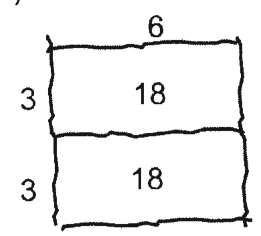
(ii)



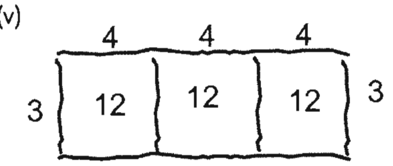
(iii)



(iv)



(v)



Conclusion

Developmental approaches to teaching and learning can make a valuable contribution to our understanding of students' mathematics learning. A particularly useful aspect of developmental theory is the idea that new learning builds on existing knowledge and understanding, leading to the idea that it is possible to identify developmental progressions in learning. There seems to be reasonable agreement within the mathematics education research community about the kind of developmental sequence involved in coming to understand the number system. The transition from counting to part-whole thinking, and from additive to multiplicative part-whole strategies, are two significant developmental progressions for mathematics learners. These transitions can be supported through the use of materials as well as diagrammatic models of the processes. In concept development, having a range of images, words, symbolic expressions, and concrete representations is valuable to ensure that a particular concept is fully developed. For this reason, exclusive reliance on *either* a number-line *or* an array-based model may limit the opportunities for supporting students' learning. This is a particularly important issue for multiplication processes, which can be represented very effectively using array-based diagrams. Although a number-line model can be useful for showing some aspects of multiplication (e.g. the equivalence of doubling and halving), it tends to reinforce a linear repeated-addition approach to multiplication. The complexity and power of multiplication can only really be conveyed by an array-based model because it captures the two-dimensional nature of the process.

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Footnotes

[1] Although the egg carton is "three-dimensional" in that it has height (in addition to length and width) to prevent objects in the compartments from moving, the model it provides is actually two-dimensional, just like a flat ten-frame.