Ghosts of mathematicians past: Paolo Ruffini

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Paolo Ruffini (1765–1822) may be something of an unknown in high school mathematics; however his contributions to the world of mathematics are a rich source of inspiration. Ruffini's rule (often known as *synthetic division*) is an efficient method of dividing a polynomial by a linear factor, with or without a remainder. Although not described by Ruffini, the process can be generalised to non-linear divisors. Ruffini's rule can be further generalised to evaluation of derivatives at a given point, does not require technology and, most importantly, it is not beyond the reach of high school mathematics students to prove why the rule works for polynomials of a specific degree.

Introduction

This article does not propose to discuss the origins of the method for long division of polynomials known to many as synthetic division, nor to discuss whether it should be known as *Horner's method* as ascribed to William Horner by Augustus De Morgan (Robertson & O'Connor, 2005) or as some cousin of the *Chinese remainder theorem* developed by Qin Jiushao (Joseph, 2011). It is known however that Paolo Ruffini was aware of the technique for dividing a polynomial dividend by a linear divisor with greater efficiency than algebraic long division.

Whether or not Ruffini knew that the technique could be extended to polynomial divisors of higher degree is largely irrelevant to this article, because the focus of this discussion is on the implications of the technique for high school mathematics.

Linear divisor where the coefficient of x is 1

As an example, we will try dividing $x^3 + x^2 - 4x + 3$ by $x - 2$ using Ruffini's rule:

Draw two lines as shown and write the coefficients of the polynomial (in descending powers of x) on the top line as shown:

$$\begin{array}{r|rrrr} & 1 & 1 & -4 & 3 \\ & & & & \\ \hline \end{array}$$

Now, testing to see if $x - 2$ is a factor is equivalent to testing to see if $x = 2$ is a solution. Write 2 to the left of the vertical line as shown:

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -4 & 3 \\ & & & & \\ \hline \end{array}$$

Write the first number inside the box immediately beneath the line as shown (the reasoning here is very simple, $1x^3 \div 1x = 1x^2$, so we know the first term in the quotient will have coefficient 1.

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -4 & 3 \\ & \downarrow \textit{copy} & & & \\ \hline & 1 & & & \end{array}$$

Multiply the number to the left of the vertical line by the first number beneath the line. Write this product below the second number inside the box as shown.

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -4 & 3 \\ \textit{multiply} & & 2 & & \\ & & \textit{write the result here} & & \\ \hline & 1 & & & \end{array}$$

The second column now contains two numbers. Add the column and write the result below the line:

$$
\begin{array}{c|cccc}
2 & 1 & 1 & -4 & 3 \\
 & & 2 & & \\
\hline
 & 1 & 3 & &
\end{array}
$$

Repeat the previous two steps until there are four numbers below the line:

$$
\begin{array}{c|cccc}
2 & 1 & 1 & -4 & 3 \\
 & & 2 & 6 & 4 \\
\hline
 & 1 & 3 & 2 & 7
\end{array}
$$

The last line (if read properly) now gives the quotient and remainder as $x^2 + 3x + 2$ and remainder 7. This answer can be checked by working backwards:

$(x - 2)\,(x^3 + 3x + 2) + 7 = x^3 - 2x^2 + 3x^2 - 6x + 2x - 4 = x^3 + x^2 - 4x + 3$ as required.

Had this division been done by traditional polynomial long-division, the calculation would have appeared thus:

$$
\begin{array}{r|ccccccccc}
 & & & x^2 & + & 3x & + & 2 \\
\cline{2-8}
x-2 & x^3 & + & x^2 & - & 4x & + & 3 \\
 & -(x^3 & - & 2x^2) & & & & \\
 & & & 3x^2 & - & 4x & & \\
 & & & -(3x^2 & - & 6x) & & \\
 & & & & & 2x & + & 3 \\
 & & & & & -(2x & - & 4) \\
 & & & & & & & 7
\end{array}
$$

There can be no doubt that Ruffini's rule is a simpler path to the solution, but the scope of its usefulness has not yet been tested. Before we look at more unusual linear factors, here is a proof of the cubic dividend case:

Proof

According to the factor theorem, when cubic $ax^3 + bx^2 + cx + d$ is divided by $(x - e)$ the remainder is $ae^3 + be^2 + ce + d$.

If we use Ruffini's rule to find the remainder, we produce the following working out:

$$
\begin{array}{c|cccc}
e & a & b & c & d \\
 & & ae & ae^2 + be & ae^3 + be^2 + ce \\
\hline
 & a & ae + b & ae^2 + be + c & ae^3 + be^2 + ce + d
\end{array}
$$

The result can be checked by working backwards:

$(x–e)\,(ax^2 + (ae + b)\,x + ae^2 + be + c) + ae^3 + be^2 + ce + d$
$= ax^3 + (ae+b)\,x^2 + (ae^2+be+c)\,x - aex^2 - (ae^2+be)\,x - ae^3 - be^2 - ce + ae^3 + be^2 + ce + d$
$= ax^3 + bx^2 + cx + d$
QED

Linear divisor where the coefficient of x is not 1

As an example, we will divide $x^3 - 2x^2 + 5x - 4$ by $2x - 1$. By polynomial long division the working out would be:

$$
\begin{array}{r|ccccccc}
 & & & x^2 & + & 3x & + & 2 \\
\cline{2-8}
2x - 1 & x^3 & - & 2x^2 & + & 5x & - & 4 \\
 & -(x^3 & - & \frac{1}{2}x^2) & & & & \\
\cline{2-4}
 & & - & \frac{3}{2}x^2 & + & 5x & & \\
 & & & -(-\frac{3}{2}x^2 & + & \frac{3}{4}x) & & \\
\cline{4-6}
 & & & & & \frac{17}{4}x & - & 4 \\
 & & & & & -(\frac{17}{4}x & - & \frac{17}{8}) \\
\cline{6-8}
 & & & & & & & -\frac{15}{8}
\end{array}
$$

Ruffini’s rule requires the coefficient of the leading term in the divisor (in this case x) to be 1. Where this is not the case, as in the example being considered, we could first divide both the dividend and the divisor by the coefficient of the lead term in the divisor. This will make our dividend $\frac{1}{2}x^3 - x^2 + \frac{5}{2}x - 2$ and our divisor $x - \frac{1}{2}$.

This can then be calculated using Ruffini’s rule:

$$\begin{array}{r|rrrr} \frac{1}{2} & \frac{1}{2} & -1 & \frac{5}{2} & -2 \\ & & \frac{1}{4} & -\frac{3}{8} & \frac{17}{16} \\ \hline & \frac{1}{2} & -\frac{3}{4} & \frac{17}{8} & -\frac{15}{16} \end{array}$$

This gives the quotient as $\frac{1}{2}x^2 - \frac{3}{4}x + \frac{17}{8}$. The remainder comes out as $-\frac{15}{16}$ but since we divided both the dividend and divisor in the original problem by 2, the remainder has been accidentally divided by 2. Instead of solving the equation $P(x) = (2x - a)Q(x) + R(x)$ for the quotient and the remainder, we have solved

$$P(x) = 2(x-b)Q(x) + R(x) = 2\left[(x-b)Q(x) + \frac{1}{2}R(x)\right] \text{ where } a = 2b$$

Multiplying $R(x)$ by 2 gives a remainder of $-\frac{15}{16}$ which is the same as found using polynomial long division.

So, if care is taken with the fractions and you remember to multiply the remainder (if there is one) by the factor removed from both the dividend and divisor, Ruffini's rule can be used with linear divisors in which the coefficient of x is not 1.

Quadratic divisor with no linear term

For the sake of simplicity in calculations, we will assume that the coefficient of the lead term in the divisor is 1. In this first example, we will make the additional assumption that there is no linear term so the divisor will have the form $x^2 + a$. For example, divide $x^4 - 9x^3 + 25x^2 - 8x - 2$ by $x^2 - 2$. By polynomial long division, the working out would be:

$$\begin{array}{r|rrrrrrrrr}
\multicolumn{1}{r}{} & & & x^2 & + & 3x & + & 2 & & \\ \cline{2-10}
x^2-2 & x^4 & - & 9x^3 & + & 25x^2 & - & 8x & - & 2 \\
\multicolumn{1}{r}{} & -(x^4 & & & - & 2x^2) & & & & \\ \cline{2-6}
\multicolumn{1}{r}{} & & - & 9x^3 & + & 27x^2 & - & 8x & & \\
\multicolumn{1}{r}{} & & - & (-9x^3 & & & + & 18x) & & \\ \cline{3-8}
\multicolumn{1}{r}{} & & & & & 27x^2 & - & 26x & - & 2 \\
\multicolumn{1}{r}{} & & & & - & (27x^2 & & & - & 54) \\ \cline{5-10}
\multicolumn{1}{r}{} & & & & & & & -26x & + & 52
\end{array}$$

A variation of Ruffini's rule has been published online by American mathematics teacher Pat Bellew (Bellew, 2009) and in print by researcher Lianghuo Fan (Fan, 2003). As the divisor is quadratic (and therefore we are not expecting a quartic nor cubic term in the quotient), the middle row of numbers will be offset one place to the *right* as shown (this means that the first *two* coefficients of the dividend are written below the line as a first step):

$$
\begin{array}{r|rrrrr}
2 & 1 & -9 & 25 & -8 & -2 \\
 & & & 2 & -18 & \\
\hline
 & 1 & -9 & & &
\end{array}
$$

multiply

write the results here

Then proceed as though using a linear divisor:

$$
\begin{array}{r|rrrrr}
2 & 1 & -9 & 25 & -8 & -2 \\
 & & & 2 & -18 & 54 \\
\hline
 & 1 & -9 & 27 & -26 & 52
\end{array}
$$

If you remember that when the divisor is $x^2 - a$ then a is written on the *left* of the vertical line, this variation on Ruffini's rule will allow for division by a quadratic with no linear term.

Quadratic divisor

When the divisor is a quadratic with a linear and constant term of the form $x^2 + ax + b$, Ruffini's rule can be further modified to make the division possible.

As an example, we will divide $x^4 - x^3 + 2x^2 - 3x + 1$ by $x^2 - 2x + 1$.

To see the result we are aiming for, the problem is done here by polynomial long division:

$$
\begin{array}{r|ccccccccc}
\multicolumn{1}{r}{} & & & & & x^2 & + & x & + & 3 \\
\cline{2-10}
x^2 - 2x + 1 & x^4 & - & x^3 & + & 2x^2 & - & 3x & + & 1 \\
\multicolumn{1}{r}{-} & (x^4 & - & 2x^3 & + & x^2) & & & & \\
\cline{2-6}
\multicolumn{1}{r}{} & & & x^3 & + & x^2 & - & 3x & & \\
\multicolumn{1}{r}{} & & - & (x^3 & - & 2x^2 & + & x) & & \\
\cline{3-8}
\multicolumn{1}{r}{} & & & & & 3x^2 & - & 4x & + & 1 \\
\multicolumn{1}{r}{} & & & & - & (3x^2 & - & 6x & + & 3) \\
\cline{5-10}
\multicolumn{1}{r}{} & & & & & & & 2x & - & 2
\end{array}
$$

When we use Ruffini's rule, we have always changed the signs in the divisors, so in this when our divisor is $x^2 - 2x + 1$ we will write 2 and –1 (remember that when we use Ruffini's rule, we change the sign in the divisor) on the *left* of the coefficients in the dividend as shown (and bring down the first coefficient as normal):

	1	–1	2	–3	1
2					
–1					
	1				

Multiply the top number on the left by the first number under the line and place it in the *second* column as shown, then multiply the bottom number on the left by the number below the line and place it in the *third* column as shown (where the number was placed in the example of a quadratic divisor with no linear term). The reasons for the different positions makes sense when thought through: the linear part of the divisor will affect the quartic, cubic, quadratic and linear terms in the dividend, whereas the constant part of the divisor will affect the cubic, quadratic, linear and constant terms in the dividend.

	1	–1	2	–3	1
2		2			
–1			–1		
	1				

the third line should always be offset 1 place to the right of the second line

Add the second column.

	1	–1	2	–3	1
2		2			
–1			–1		
	1	1			

Now repeat the first step with this new number below the bottom line (this will result in three numbers in the third column).

	1	–1	2	–3	1
2		2	2		
–1			–1	–1	
	1	1			

only add the column once you know it is complete

Add the third column:

	1	–1	2	–3	1
2		2			
–1			–1		
	1	1			

Multiply again and place the products in the appropriate spaces:

	1	–1	2	–3	1
2		2	2	6	
–1			–1	–1	–3
	1	1	3		

Add the fourth and fifth columns:

	1	–1	2	–3	1
2		2	2	6	
–1			–1	–1	–3
	1	1	3	2	–2

Which means the quotient is $x^2 + x + 3$ and the remainder is $2x - 2$ (which is the same as determined by the polynomial long division).

Note: One of the keys to the success of this technique is ensuring the second and third rows of the working out contain the same number of terms which means each part of the divisor has been used the same number of times (which makes logical sense as it is one divisor with two parts), hence the gap in the middle of the final column.

Calculus applications

In the previous example, we divided a quartic by $x^2 - 2x + 1$. It is no coincidence that $x^2 - 2x + 1 = (x - 1)^2$. Kermond (2013) demonstrated that the remainder when a polynomial is divided by $(x - a)^2$ is the equation of a tangent to that polynomial at $x = a$.

Whilst this is an interesting application in its own right, it is a little known fact that Ruffini's rule can be used to find the gradient of a polynomial at a given point.

According to the remainder theorem, the remainder when a polynomial $p(x)$ is divided by a linear divisor $(x - a)$ is equal to the value of the function evaluated at that point, $p(a)$.

Consider again the first example in this article when $p(x) = x^3 + x^2 - 4x + 3$ was divided by $d(x) = x - 2$.

$$
\begin{array}{c|cccc}
2 & 1 & 1 & -4 & 3 \\
 & & 2 & 6 & 4 \\
\hline
 & 1 & 3 & 2 & 7
\end{array}
$$

We have already established that $p(2) = 7$. Repeating the process (which Ballew (2009) suggests may have been identified by the Indian mathematician Bramagupta) with the new coefficients gives the following result (remember that the first derivative polynomial will have degree 1 less than the original polynomial, so the final line should have one less number):

$$
\begin{array}{c|cccc}
2 & 1 & 1 & -4 & 3 \\
 & & 2 & 6 & 4 \\
\hline
 & 1 & 3 & 2 & 7 \\
 & & 2 & 10 & \\
\hline
 & 1 & 5 & 12 &
\end{array}
$$

$p'(x) = 3x^2 + 2x - 4$ and $p'(2) = 12$.

This result can be proven for a general cubic dividend and linear divisor with the coefficient of x equal to 1 as follows:
Let the general cubic be $p(x) = x^3 + ax^2 + bx + c$ and the divisor be $d(x) = x - d$. Then $p'(d) = 3d^2 + 2ad + b$. Using Ruffini's rule:

$$
\begin{array}{c|cccc}
d & 1 & a & b & c \\
 & & d & d^2 + ad & d^3 + ad^2 + bd \\
\hline
 & 1 & d + a & d^2 + ad + b & d^3 + ad^2 + bd + c \\
 & & d & 2d^2 + ad & \\
\hline
 & 1 & 2d + a & 3d^2 + 2ad + b &
\end{array}
$$

Which gives $p'(d) = 3d^2 + 2ad + b$ as found previously using calculus.

Conclusion

There may be some in the teaching profession who question the benefits of teaching Ruffini's rule at all. However, it could be argued that there are a number of benefits:

1. It brings a sense of history into the mathematics classroom. Whilst not all teachers would suggest that this is a good idea it does lend a sense of relevance and purpose to the ideas. This also supports the curriculum rationale found in the current study guide, "to provide access to worthwhile and challenging mathematical learning which takes into

account the interests, needs, dispositions and aspirations of a wide range of students" (VCAA, 2015).

2. It is less confusing to many students than polynomial long-division and in the case of linear divisors where the coefficient of x is 1, it is notably more efficient. If the remainder is zero, Ruffini's rule can be used to find multiple factors quickly, which could prove to be a highly efficient method of factorising a quartic or higher degree polynomial.
3. The rule can be proved for specific degrees of polynomial using mathematical skills which should not be beyond the grasp of a medium-strong late high school mathematics student. Fan (2003) proves the rule for the general case but the mathematics required is beyond the scope of high school mathematics.
4. There are applications to be found such as tangents and gradients without calculus which could be an interesting diversion for the more able students of senior mathematics classes before calculus has been formally introduced as a topic.
5. Finally (and most importantly to some) Ruffini's rule requires no technology (algebraic solver enabled calculators) for any of these explorations.

At no point is it being suggested that Ruffini's rule should become a compulsory part of the senior Mathematics curriculum in high schools, but it should not be easily passed over without consideration either.

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