

Equality matters

The critical implications of precisely defining equality



Valerie Faulkner
NC State University, USA
<valerie_faulkner@ncsu.edu>



Temple Walkowiak
NC State University, USA
<temple_walkowiak@ncsu.edu>



Chris Cain
Mars Hill University, USA
<ccain@mhu.edu>



Carrie Lee
NC State University, USA
<cwlee5@ncsu.edu>

Equality is such an important concept for children to develop. In this article it is argued that a precise definition is needed to ensure that students are provided with a consistent 'picture' of what it is that equality really means.

How would you finish this sentence: "Equality means". What did you think to yourself or write down? Many of you may have thought or written "Equality means the same as". Others of you may have said "Equality means balance". Still others may have said "same value as". If you answered "same as", you are not alone.

We have found that it is a common cultural practice or habit to define equality for students as 'the same as'. In fact, that definition has been utilised in curriculum documents (ACARA, 2012) and other resources (Mann, 2004), likely due to the lack of explicit attention to the problematic nature of it. For example, the *Australian Curriculum: Mathematics* (ACARA, 2012) states that students in the Foundation Year are to compare and order "items of like and unlike characteristics using the words 'more', 'less', 'same as' and 'not the same as' and giving reasons for these answers". The intent of the latter part of this learning goal seems to be that students explain how the given items are the same as each other, but this goal seems to warrant attention to the very aims of this article.

The primary purpose of this article is to explain how a simple, consistent change in language to 'same value as' will create greater precision, understanding, and discussion regarding the concept of equality within your classroom. First, we describe the change in language, explain its importance, and explore the concept of balance as an important access point for students regarding equality (Warren & Cooper, 2005). Next, we present two classroom scenarios during which the idea of equality is presented differently. Then, we share both symbolic and contextual examples involving equality across the primary grades.

We also present an example of how defining equality precisely has implications for children's understanding of the properties of the operations. Finally, we consider how a precise definition of equality creates curricular connections with measurement and geometry.

Defining equality with precision

Equality is a critical concept that impacts student understanding of number and operations (Groves, Mousley, & Forgasz, 2004; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Knuth, Stephens, McNeil, & Alibali, 2006); therefore, precision in mathematical language is critical when discussing equality in mathematics classrooms. Consider the graphic in Figure 1. Are the elephant and the two trucks equal? How so? Are they the same as each other? As you can see, the two trucks and the elephant have the same weight as each other. In this single attribute—their weight—they are the same. In other words, they are equal in weight, but they are otherwise very different. They are not the same size or the same colour. They are not even both animals or both vehicles. In other words, they are most definitely not the same as each other.

Why is this particular opportunity to be precise with language ('same as' versus 'same value as') so important? Consider the work we do supporting young students in reasoning with shapes and attributes. They have learned to see that triangles are not the same as rectangles because squares have four sides and triangles have three sides. They also work to understand that circles of different colors look different, but they are still circles. In other words they are not the same as each other with regard to the attribute of colour,

but they are alike with regard to the attribute of shape. Therefore, some things about the two objects are the same, and some things are not the same. Depending on our mathematical question, we need to focus on different aspects, or attributes, of certain objects or numbers. In one situation we might want to know if things are equal in weight, but in another situation if they are equal in length. Two objects, then, can be equal in one numerical attribute but not in another. This distinction matters, and precise language supports students who are learning to pay attention to different attributes and their implications.

Notice in Figure 1 that the idea of balance helps us to access this idea of equality. Using a balance scale or other classroom devices such as a balance beam can deeply communicate this idea of equal weight. It also allows us to clearly make the point that two things do not have to be the same, or look the same, to be balanced. This is a terrific starting point, but there are situations regarding equality that do not imply this type of balance.

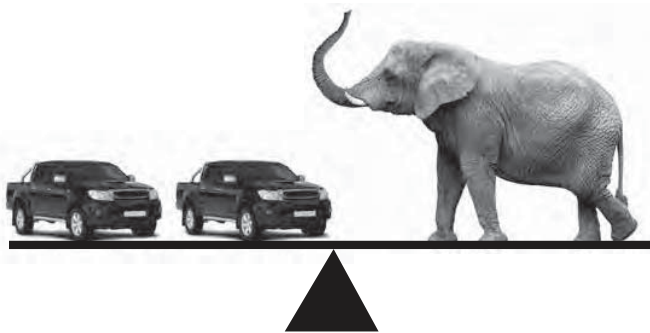


Figure 1. Equality is not 'the same as'.

One such situation is demonstrated in Figure 2. Perhaps I want to weigh these two screwdrivers to see if they are equal in weight. Here again, they do not have to look the same to be 'balanced' in the sense that they have the same weight, but can they be equal in other ways that do not imply a balance? For example, could they be equal in length or girth around the handle? What does it mean to be equal in any given context? It means that you can find one particular numerical trait about two objects that have the same numerical value. Here we have two screwdrivers that might be equal in weight (balanced) and equal in the measure around their handle but not equal in length. Balance is an important concept, and we certainly continue to discuss balancing equations as a metaphor; however, it is important that we convey to students that there are many ways that we can use numerical characteristics to compare things. The idea of balance can be tricky for students if they do not begin to

understand that things can look unequal in weight but still be equal in some way.

When we slow down our discussions of equality, we also have the opportunity to discuss the whole idea of quantifying something versus valuing things in some other way. For instance, we can have a discussion about non-numerical attributes like beauty. In Figure 2, we might discuss which screwdriver you think is more aesthetically pleasing, or more ergonomic, simply to make the distinction between a qualitative attribute and a quantitative attribute. This supports students' later understandings of quantitative and qualitative attributes in statistics and helps them to parse out what it means to measure and compare numerical attributes.



Figure 2. Different aspects of equality.

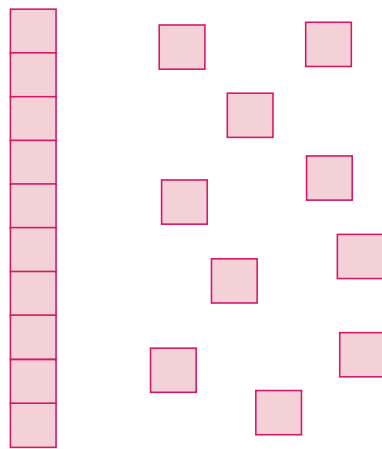
Two classroom scenarios

Once we begin to define equality more precisely, we may notice the need to shift our language in other discussions as well. Consider these two classroom examples (Table 1) that the authors have themselves experienced in their own instruction. What do you notice?

In classroom example A, we know what this teacher is intending to say; she is trying to imply that the value of the two sets of manipulatives are equal. However, the student sees one large object in one hand and ten small objects in the other. Clearly to the student these are not the same as each other because well, they are not the same as each other (Figure 3). Besides, if they were exactly the same, then why would we need to make the exchange?

Table 1. Teacher discussions of equality.

Classroom Example A	Classroom Example B
<p>A teacher is working on decomposing tens in order to subtract with regrouping. The teacher holds the 10-rod in one hand and the group of ten ones in the other and asks the student “are these the same?” The student proudly answers “No”. The teacher looks at the student and says “No, look again, they are the same”. It is clear from the student’s face that there is a misunderstanding.</p>	<p>A young student was arguing with the teacher that one five-dollar note was the same as 50 ten-cent coins. The teacher understood that the student was missing a key point as it related to equality. The teacher tried her best to explain to the students that they are equal in value but they are not the same thing. It was not until the teacher asked the student to carry a five-dollar note in his hand during the morning and later asked the student to carry 50 ten-cent coins during the afternoon that the student agreed that, yes they would both buy the same amount of something, but no, they are not the same thing.</p>



The same? No, but the same value, yes.

Figure 3. Different forms of ten.

The exchange occurs precisely to ‘cash in’ on the difference in form between a composed ten and a decomposed set of ten. This conversation is greatly aided by the simple tweak in language from same as to the same in value. We can now agree that ten ones-cubes are not the same thing as one ten-rod, but they are equal in value so we can exchange them if needed.

In Classroom Example B, we see a teacher reinforcing the correct meaning and language of equality. This teacher is helping the student understand that these two forms of the five dollars are equal in value, but they are not the same thing. Fifty ten-cent coins have the unfortunate attribute of being quite heavy and difficult to lug around all day. A five-dollar note is light and convenient (Figure 4). Which would you rather carry around? Why? Paying attention to the precise idea of equality opens up this sort of discussion on a regular basis. Also consider questions such as “If you wanted to get something from a vending machine would you rather have a \$10 note or five \$2 coins? Why?”



Which would you rather carry around with you all day?

Figure 4. Different forms of money.

Here students will begin to understand, as mathematicians do, that different forms of numbers have different assets and weaknesses to the user.

In our quantitative world, the different representations of a value imply different assets—sometimes we like our value compact and composed into a ten or a five banknote, and sometimes we want it decomposed into ones or into cents (Faulkner, 2009). For instance, if we want a soft drink from a soft-drink machine, most of us would rather have a five banknote in the form of \$2 and \$1 coins!

Symbolic and contextual equality examples

While perhaps not as practical as wanting access to a drink, the same idea is true when we consider numbers at the more abstract level of symbolic, numerical representations. Even though two expressions may be equal, they may be very different in other ways. They may tell a different story, they may be in entirely different forms, they may be unsimplified or in simplest form. From the mathematician’s perspective, different forms have different assets. For instance noticing that

$(x - x)$ is indeed equal to 0 may be the key to explaining something through a proof. And at other times it is just helpful to know that number expressions can look complicated, yet actually represent a very simple whole number for instance, $\sqrt{(100-91)}$ versus the simplified 3. The form of the number matters and we begin to have extremely productive discussions about this idea if we understand that two equal values are *not* necessarily the same. Three is not the same as $\sqrt{(100-91)}$. They have the same value, but they do not look the same, and they are certainly not equally handy in all situations.

These nuances can be discussed starting with very young students. Story problems support students' understanding of how two things might be equal, but not the same (Roche, 2013). These literal examples again allow us to see the maths from the students' perspectives. Let's say "Senga has three red apples and picks one green apple more". In another story, "Senga has two red apples and picks two green apples more". Those are two distinctly different stories. It is true that, in both instances, Senga ends up with four apples, but the stories are clearly not the same as each other.

Consider the equations $4 + 4 = 8$ and $7 + 1 = 8$. As noted above, the left expression in both equations equals 8, but the equations are different stories. Let's say, for instance, there is a store that sells a box of eight chocolates for \$8. One day Paola and Rusty walk into the store to buy a box of chocolates; Paola pays \$4, and Rusty pays \$4. Splitting the chocolates fairly is a relatively easy affair. But what if on another day Paola pays \$7 and Rusty pays \$1. Here again, the two will walk out of the store with a new box of chocolates, but will they split them in the same way? The equation and the situation each have different implications, even though in both cases eight chocolates have been bought.

From an instructional perspective simply stating that one value equals another value is not enough. At all times we want to consider with students, who are learning to think like mathematicians, why do we want to change the form of a number in a given situation? From a student's perspective there has to be some reason behind the need to change numerals into different forms such as when we are adding fractions with related denominators (Year 6 of the *Australian Curriculum: Mathematics*). In the problem $\frac{1}{2} + \frac{3}{4}$, if $\frac{1}{2}$ is the same thing as $\frac{2}{4}$, then why did we bother putting that numeral into a different form? If $\frac{3}{4}$ is the same thing as 0.75, then why do we bother making conversions to decimals? We do so because these are the same

values in different forms, and the different forms allow us different access into the values and allow us to manipulate them in different ways.

Understanding the assets of different forms of numbers becomes increasingly important as the curriculum becomes more integrated. In the upper primary grades, stories become even more complicated when we consider decimal and percentage forms within the same story. Consider a percentage form of a rate, versus fractional form of a rate. If a basketball coach knows that a recruit is shooting 40% from the three-point line, is that enough information? Does coach not need that information in fractional form as well? Perhaps this recruit has only taken five shots and made two of them. That is very different than if the recruit took $\frac{200}{80}$ shots and made 80 of them! 40% and $\frac{2}{5}$ and $\frac{80}{200}$ are all equal rates, but they are clearly not the same thing. By holding students accountable to a more precise definition of equality, our discussions immediately improve. In this scenario, a class of children could discuss why the coach might favor one player over the other even though both players are hitting 40% of their 3-point shots. This deepens students' understanding of equivalent fractions—they are equivalent in their rate, but not in their fundamental meaning or description of a situation.

Equality and properties

Precisely and consistently defining equality also has a clear impact on our discussions of mathematical principles such as the commutative property. Take for example a classroom talking point of $4 + 3 = 3 + 4$. By pointing out that these two expressions are *not* the same, we can then have a discussion about what *is* the same about the two expressions. The values of the expressions are the same. The order of the addends within the expressions is not the same so we can have a discussion about how, with addition, the sum of the values does not depend on the order of the parts. This conversation gets to the heart of the commutative property as something to consider rather than a definition to memorise (Warren, 2003). The students are now primed to expand the discussion into considering what else does or does not behave in this way: "What operations are commutative?" These discussions stem readily from a consistently precise definition of equality.

Extending to geometry and measurement: Equality versus congruence

A clear understanding of equality is critical. When we define equality as ‘same as’ we are led down a path of incorrect conclusions. Think about how this ‘same as’ definition might cloud students’ future understanding of congruence. If equal means ‘same as’ then what does congruence mean and why do I need another vocabulary word that means the ‘same as.’ Consider the theorem you learned in middle or high school: If two angles are equal in measure, then they are congruent. If I understand equality precisely, I understand that this is not always the case: Just because two things are equal in some way does not mean that they are congruent, or the same as. Consider also that all triangles are equal in total angle measure—they all have a total angle measure of 180° . This does not make them all congruent. Most things we consider have many attributes, like the aforementioned screwdrivers or money or triangles. But in the specific case of angles, they really only have one attribute—their angle measure *defines* them. So, here it is true that just knowing that the angle measures are equal is enough to know that the two angles are congruent (Figure 5). The key here is that this is actually quite unusual and not business as usual.

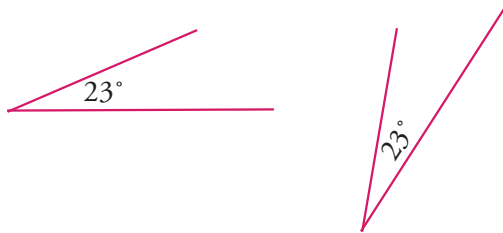


Figure 5. Angle measures define angles.

Another example is the case of two rectangles that have the same area. Students might erroneously state that an 8×3 rectangle is congruent to a 6×4 rectangle. Perhaps this confusion is due to hearing ‘same as’ as a definition for both equality and for congruence. When the areas are both 24 square units, the student recognises they are the ‘same’ and mistakenly concludes they are congruent. With attention to precision in defining equality, we can discuss with our students how they are equal in area, but they are not the same and they are not congruent (or identical when superimposed on each other).

Conclusion

The accuracy of definitions we explicitly and unconsciously use has an impact on the mathematical development of students. Carpenter, Franke, and Levi (2003) stress the importance by stating, “unless we start to address directly children’s concepts of equality, children’s misconceptions will persist” (p. 23). Teachers and teacher educators must be aware of their own language about equality. Furthermore, we know that truly proficient mathematics students are able to construct viable arguments and attend to precision. If the language we use to describe equality is faulty, we are not modelling these practices for our students. Teaching equality is more about the language you use consistently than with a particular unit, lesson, or activity. It has been described as the ‘soul’ of arithmetic (Ma, 1999). Considering and adjusting our habitual presentation of equality in these ways will help us to emphasise this most important ‘soul’.

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