Opening the door on triangular numbers


Kimberley McMartin
Passfield Park School
<kimberley.mcmartin@det.nsw.edu.au>

Heather McMaster
Sydney University
<heather@workingmaths.net>

As an alternative to looking solely at linear functions, a three-lesson learning progression developed for Year 6 students that incorporates triangular numbers to develop children's algebraic thinking is described and evaluated.

Triangular numbers provide many wonderful contexts for mathematical thinking and problem solving. Triangular numbers are figurate numbers because they represent counting numbers as a geometric configuration of equally spaced points. This makes triangular number problems easy to present in a physically and visually engaging way that supports children to find generalisations and make connections with other figurate numbers such as square numbers and rectangular numbers (Samson, 2004).

Triangular numbers and problem-solving

Triangular numbers are involved in the solutions of many mathematical problem types (Szetela, 1999). They appear for example in solutions to problems requiring one to find the number of matches needed in a round robin competition for a given number of teams; the number of different handshakes possible given the size of the group of people; the maximum number of chords that can be drawn between a given number of points on the circumference of a circle; the number of dominoes in a set given the maximum number of dots on each half; and the number of different double-scoop ice creams possible given the number of different ice cream flavours available. These problems provide children with a wealth of opportunities to make conjectures, carry out investigations, and find and generalise patterns and relationships.

Triangular numbers and algebraic thinking

A triangular figure formed by an arrangement of equally spaced counters is a spatial structure pattern (Papic, Mulligan & Mitchelmore, 2011). When these figures are placed in a sequence with the number of counters along each side increasing by one, they become elements of a growing pattern. A pattern made from a sequence of geometric structures that change from one figure to the next in a predictable way is called a pictorial growing pattern or a geometric growing pattern (Billings, Tiedt & Slater, 2007).

Geometric growing patterns have been commonly advocated by researchers as a valuable tool for developing children's algebraic thinking, in particular their functional thinking, because they are able to provide meaning to the numerical calculations (Booker & Windsor, 2010; Hourigan & Leavy, 2015; Markworth, 2012; Muir, Bragg & Livy, 2015; Rivera & Becker, 2005; Switzer, 2013; Wilkie, 2014). Papic, Mulligan and Mitchelmore (2011) found that given appropriate opportunities, even pre-schoolers can continue a pattern of triangular figures made from dots or counters and make generalisations that reflect both their geometric and numerical structures. Working with children in Years 2 and 3 in the United States, Billings, Tiedt and Slater (2007) also found that children who had no previous experience with geometric growing patterns could successfully extend a geometric figure to other figures in the sequence. Furthermore, provided that they could physically construct the sequence, these young children could begin to think about the relationship between a geometric figure and its position in the sequence. An understanding of this relationship is powerful because it enables one to determine what geometric figure will be in any position down the sequence, without needing to construct all the figures in-between.

Table 1. Language used by researchers to distinguish between two types of generalisations.

Generalisation about what changes from one figure to the next	Generalisation about a relationship between a figure & its position in the sequence	Reference
A co-variational analysis of change	A correspondence analysis of change	Billings, Tiedt and Slater (2007)
Near generalisation	Far generalisation	Markworth (2012)
Recursive rule	Explicit rule	Wilkie (2014)
Recursive rule	Generalised rule	Hourigan and Leavy (2015)

Two types of generalising about growing patterns

Generalisation is the process of finding common features across representations, problems or situations so the same ideas can be applied to new representations, problems or situations if they possess the same features (Hill, Lanin & van Garderen, 2015). For a geometric growing pattern, the generalisation that children first make is usually a rule that describes the change from one figure in the sequence to the next. This rule can dominate their thinking so they fail to see the more powerful generalisation concerning the relationship of a figure to its position in the sequence (Billings, Tiedt & Slater, 2007; Hourigan & Leavy, 2015; Markworth, 2012; Wilkie, 2014). Table 1 lists the language used by researchers to distinguish between these two types of generalisations.

Number patterns and triangular numbers in the *Australian Curriculum: Mathematics*

The *Australian Curriculum: Mathematic*s acknowledges that number and algebra are developed together, with the creation and exploration of number patterns playing a central role in the development of children's algebraic thinking (*Australian Curriculum: Mathematics*, Assessment and Reporting Authority [ACARA], 2015). At every year level in primary school, there is an outcome involving the creation and/or exploration of number patterns (Table 2). The emphasis is on children articulating verbal descriptions of these patterns in preparation for them making concise descriptions using algebraic symbols when they reach high school (Booker & Windsor, 2010).

Notice in Table 2 that from Year 1 to Year 5, the number patterns specifically mandated in the *Australian Curriculum: Mathematics* are sequences created by repeated equal addition (multiplication) or repeated subtraction, in other words, sequences in which the difference between consecutive terms is constant. This focus is understandable because these sequences provide an entry into multiplicative thinking and an understanding of linear relationships (the simplest type of algebraic function). A popular activity to promote functional thinking in the primary school years is the use of a 'function machine', i.e. a picture of a machine or the use of a box as a machine which converts every numerical input to an output by applying the same 'rule' (Muir, Bragg & Livy, 2015).

Table 2. Outcomes involving the creation and/or exploration of number patterns at each year level.

Year	Australian Curriculum reference	Outcome
Foundation	ACMNA005	Sort and classify familiar objects and explain the basis for these classifications. Copy, continue and create patterns with objects and drawings.
1	ACMNA018	Investigate and describe number patterns formed by skip-counting and patterns with objects.
2	ACMNA026	Investigate number sequences, initially those increasing and decreasing by twos, threes, fives and tens from any starting point, then moving to other sequences.
3	ACMNA060	Describe, continue, and create number patterns resulting from performing addition or subtraction.
4	ACMNA081	Explore and describe number patterns resulting from performing multiplication.
5	ACMNA107	Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction.
6	ACMNA133	Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence.

Children analyse the outputs in relation to the inputs to determine the rule being used by the machine. This activity develops their understanding of linear relationships and could also be used for squaring numbers. However, the rule that produces triangular numbers from an input (the term number) is too complex to be discovered in this way.

With linear relationships, children learn to distinguish between what stays the same (compared with the starting term) and what consistently changes from one term of the sequence to the next (Switzer, 2013). Because there is a common difference between terms, a frequent error when describing the relationship of a term to its position in a linear sequence, is to disregard the starting term and see the common difference between terms as the constant term of the linear relationship (Riviera & Becker, 2005). The sequence of triangular numbers is non-linear, so this technique of looking for a common difference in order to formulate the multiplicative relationship between a number and its position in the sequence cannot be applied.

The National Council of Teachers of Mathematics (NCTM) in the United Sates recommend that middle school students focus on patterns and relationships in linear functions but also have similar experiences with non-linear functions (Switzer, 2013). By including non-linear relationships, it is hoped that students will be better able to distinguish between linear and non-linear functions, become less reliant on rules without reasons, and achieve a deeper understanding of the process of generalisation.

Our question

Triangular numbers are not specifically mentioned in the *Australian Curriculum: Mathematics* until Year 6, when children are required to "Identify and describe properties of prime, composite, square and triangular numbers" (ACMNA122) with the elaboration of "understanding that some numbers have special properties and that these properties can be used to solve problems" (ACARA, 2015). If, as the curriculum states, the learning of number and algebra are to be developed together, could teaching to achieve this outcome be successfully intertwined with teaching to achieve the Year 6 outcome: "Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence" (ACMNA133)?

To answer this question, a sequence of three 30-to 40-minute lessons were developed for a mixed-ability Year 6 class in an outer suburban school in Sydney with a high percentage of students from non-English speaking backgrounds and a below average Index of Community Socio-Educational Advantage (ICSEA). The lessons were presented by the first author (henceforth referred to as 'the teacher') once a week for three consecutive weeks. The effectiveness of these lessons was assessed by the following criteria:

1. Were they able to recognise the recursive pattern in the sequence of triangular numbers?
2. Were they able to articulate the relationship between a triangular number and its position in the sequence?
3. Were they able to make a connection between triangular numbers and square numbers?

The learning experiences

The sequence of three lessons we developed was based on the curriculum outcomes ACMNA122 and ACMNA133 of the *Australian Curriculum: Mathematics* for Year 6 (ACARA, 2016). In planning these lessons, the following principles were adhered to in order to maximise the engagement of all children in the class.

- A problem solving approach was used (Booker & Windsor, 2010; Bridge, Day & Hurrell, 2012).
- The problem was launched through story-telling, using a picture and a model of a familiar real-life situation to clarify the problem and the language involved (Hourigan & Leavy, 2015; Author, 2003; Muir, Bragg & Livy, 2015)
- Concrete materials were made available so children could explore and physically interact with the problems (Bridge, Day & Hurrell, 2012).
- The children were given the opportunity to think of alternative ways of making a generalisation (Rivera & Becker, 2005; Wilkie, 2014).
- Enabling prompts and extending prompts were prepared for each lesson (Sullivan et al., 2015).
- The children were presented with structurally-related problems to assess and support their ability to generalise (Booker & Windsor, 2010; Hill, Lanin & van Garderen, 2015).
- An over-generalisation predicted to arise from superficially seeing two contexts as having the same structure, was explicitly presented to the class and discussed.
- Questioning and class discussion facilitated the articulation of generalisations (Booker & Windsor, 2010).
- Previous learning was utilised to make connections between triangular numbers and square numbers (Richardson, Carter & Berenso, 2010).

Lesson 1

This lesson was based on a problem about a fruit pyramid found in a book of competition problems for children in upper primary school (Lechner, 2005, p. 44). The fruit pyramid problem was modelled (Figure 1) and students used counters to represent the layers of fruit (Figure 2).

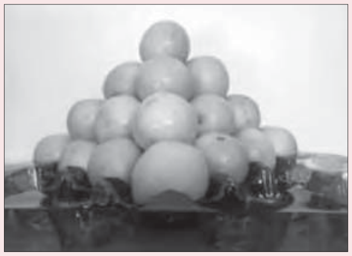


Figure 1. A fruit pyramid with four layers. Each layer consists of a triangular number of fruit.

Figure 2. Children using counters to solve the fruit pyramid problem. This group separated the layers.

The learning targets for this lesson were:

1. Children understand figuratively and numerically, the meaning of a triangular number.
2. Children notice that the position of a triangle in the fruit pyramid (its "layer number") is the same as the number of fruit along one side of the triangle.
3. Children can recognise and generalise the recursive pattern (i.e. that from layer to layer, the increase in the number of fruit increases by one more each time).
4. Children can generalise that the number of fruit in any layer can be found by adding consecutive numbers up to that layer number and can use this generalisation to find the number of fruit in the 100^{th} layer.

Lesson 2

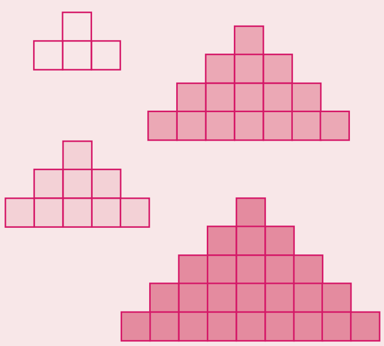


Table 3

Staircase number (number of blocks high)	Number of stairs	Total number of blocks in the staircase
1	1	1
2	3	4
3		
4		
5		
6		
7		
8		
9		
10		

Figure 3. The sheet of up-and-down staircases and table given to the children. They were asked to relate the height of a staircase (the number of blocks high) to the number of stairs it has (i.e. the number of columns of blocks) and the total number of blocks needed to make the staircase.

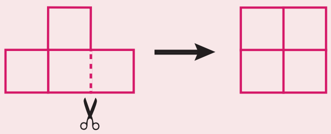
This lesson was based on diagrams of up-and-down staircases made from same-sized blocks (Figure 3), with the staircases numbered according to their height (the number of blocks high).

The learning targets for this lesson were:

1. Children can recognise, generalise and explain the relationship between the staircase number (the number of blocks high) and the number of stairs.
2. Children can recognise and generalise the recursive pattern that the number of blocks needed to build the next staircase increases by two more each time.
3. Children can recognise and generalise that the number of blocks in each staircase is the square of the staircase number and can use this relationship to find the number of stairs in a staircase 100 blocks high.

Lesson 3

This lesson built on Lesson 2 to visually connect triangular numbers with square numbers. The staircases are shown to have a square number of blocks by cutting them into two pieces that form a square (Figure 4).



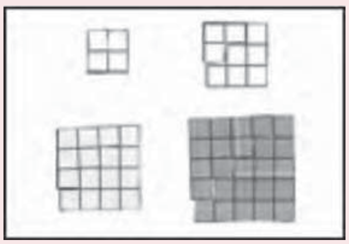


Figure 4. The children cut out the staircases from the sheet (see Figure 3), then cut each staircase into two pieces (each made up of a triangular number of blocks) and placed them together to form squares.

Table 4. A summary table of the numerical sequences.

Staircase number (number of blocks high)	Number of squares in the larger piece	Number of stairs in the smaller piece	Number of squares in the large square
2	3	1	4
3			
4			
5			
6			
7			
8			
9			
10			

A summary table (Table 4) was used to help them make sense of the words “term” and “number sequence”. The term was the number of blocks high. The sequence of triangular numbers is in the second column and the third column, the sequence of square numbers is in the last column.

The learning targets for the lesson were:

1. Children can recognise that a triangular number can be presented figuratively in different forms (as an equilateral triangle of counters or as part of a staircase made from a number of blocks).
2. Children can visually recognise and generalise numerically that two consecutive triangular numbers make a square number.
3. Children can find the value of a term in the sequence of square numbers by squaring the term number.

Results

The Year 6 students showed high levels of interest in solving the problems at hand, with many commenting that they appreciated the tangible and open-ended nature of the tasks. One student stated that the problems were “fun because they were challenging”, and that it was an “exciting way to learn about triangular numbers”. The teacher observed that students were highly focused throughout all three lessons, evident through the on-task discussions amongst small groups over how to best solve the problems. Many students expressed that they were not accustomed to learning mathematical concepts in such an open-ended manner. Initially this caused difficulty for the students, as teacher-scaffolding was withdrawn in favour of the students working with each other to trial different problem solving methods. In the first half of the initial lesson, teacher support was required consistently in order to assist the students to find solutions to the problem. However, by the end of the first lesson, students began turning to each other for support, and this continued throughout the remaining two lessons. The teacher observed that when she initially entered the class and began explaining the problem, there were a number of off-task discussions between students throughout the room, but when the problem was handed to students to solve independently of the teacher, the number of off-task discussions decreased as students focused on the tasks. Indeed, by the final lesson, students were approaching the teacher to discuss the mathematical problems at hand, asking questions that would extend their knowledge beyond what they were learning in the classroom, indicating their engagement.

During the first lesson, students successfully developed their understanding of triangular numbers, reaching all of the learning targets. However, the path to achieving these targets was not smooth-sailing, with many students expressing confusion and frustration at the problem. These feelings were generally due to their uncertainty over how to approach solving the problem without teacher-scaffolding. Indeed, when given the counters to use in solving the problem, most students were confused as to what to actually do with the counters. When prompted to simply "start experimenting", many students began manipulating the counters cautiously, continually asking whether or not they were "doing it right". The teacher indicated to them that there was no right or wrong way to go about exploring the problem, explaining that part of the process was trial and error with a range of methods. This appeared to give the students some confidence in experimenting with the materials, and after several minutes the teacher observed all groups working with the counters, arranging them in a number of different formations. However, in this initial stage, many students placed the counters in a square array or simple piled the counters on top of each other. In an effort to assist them, the teacher reminded them of the pyramid of oranges at the front of the room, which revealed the first four layers of the pattern. At this point, each group nominated a student to view the pyramid of oranges and report back to the group their findings. With a model to observe, many students began realising that the counters were arranged in a triangular formation, and that the number of counters in each layer of the pyramid increased first by 2, then by 3, then 4, etc. Discussing this as a whole class, the students were soon chorusing the pattern, indicating their understanding. It was only at the conclusion of the lesson that the teacher revealed that the numbers they had been working with were triangular numbers. The students were directed to the fact that they created equilateral triangles with the counters, and a collective "ah" sounded around the room. To determine the extent of their understanding, the first author asked students to attempt to explain the pattern to the class. Although some required assistance in wording their response, it was evident that the students were able to generalise the relationship between the triangular numbers and layer numbers, the triangular number being the sum of all the numbers from one up to the layer number.

The students experienced similar success in the second lesson, gaining an understanding of square numbers and reaching all of the specified learning targets. Due to their experience of the investigative approach to solving mathematical problems in the first lesson, students did not experience the same level of confusion at having teacher-scaffolding removed when faced with the staircase problem. Once given the task, all students immediately began examining the problem either individually or with a partner, with few requesting teacher support. The teacher noticed students experimenting with a range of strategies to solve the problem at hand illustrated by diagrams and equations (Figure 6). Their drawings revealed to the teacher the difficulties that the students were having in solving the problem. For example, many students confused the term number (the number of stairs high) with the value of the term (the number of stairs in the staircase). Although the students were having difficulties, they persisted in solving the problem, expressing their thinking and how they were attempting to determine the answer.

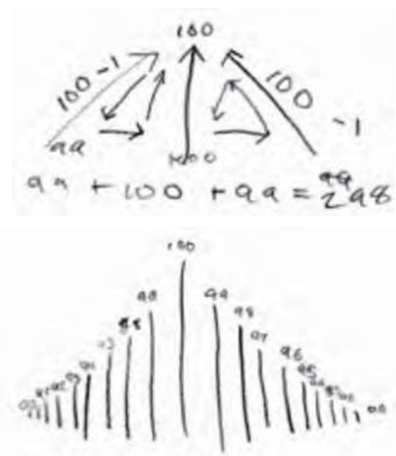


Figure 6. A student's diagrams for determining how many stairs are in a staircase 100 blocks high and how many blocks are needed to make it.

After some time working on the problem, many of the students noticed the square numbers of blocks, that is, that "staircase number 1" is one block high and contains one block, "staircase number 2" is two blocks high and contains four blocks, "staircase number 3" is three blocks high and contains nine blocks and so on. In exploring this concept as a class, students were again able to continue the pattern through chorusing the answers as a class, however in this lesson, students did not relate the square numbers they found to the stair context.

Explaining the relationship between the new number of stairs and the next square number also proved to be difficult for the students, with several expressing confusion. In an attempt to assist, the teacher helped them connect the number of blocks added each time with the number of blocks in the next staircase. Those students who understood began assisting their peers, using the zig-zag pattern modelled by the teacher to show generalisation of this relationship.

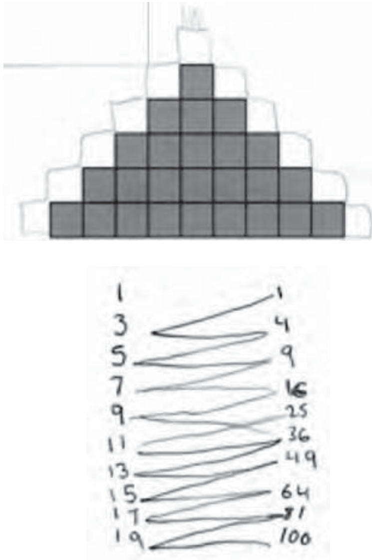


Figure 7. A diagram showing that the number of stairs in the new staircase is the same as the number of blocks added to the previous staircase. Students showed the relationship between this number and the total number of blocks as a 'zig-zag pattern'.

At the conclusion of the lesson, the teacher asked the students if they knew what kind of numbers they had been working with, to which one replied "square numbers". Once again, a collective "ah" sounded around the room, and some students began using the relationships between the numbers to work out how many stairs would exist in a staircase 100 blocks high. Although the teacher was at first disheartened by the fact that not all students had grasped the concept, she was interested to observe the students who understood the number relationships explaining them to their peers. Indeed, these conversations continued as the students went out to recess, indicating that not only did the students enjoy the task, but they also expressed a keenness to actually solve the problem.

In the final lesson, students were asked to explore the relationship between square and triangular numbers. This lesson proved more difficult for students, with the majority unable to reach all of the learning outcomes. When asked to rearrange the number of blocks in each small staircase to form one large square (visually representing the fact that the number of blocks in each staircase is a square number), most students were confused. Despite being told that they could only cut each staircase into two separate pieces, many of the lower-ability students cut their staircases into more than two pieces or into individual squares and many were unable to visually represent the square numbers as a square of small squares. However those students who were successful were pleased to show the others how one larger piece and one smaller piece could be put together to form one large square. In discussing this as a class, students worked together to complete a table that examined the relationship between the number of squares in the larger piece, the number of squares in the smaller piece, and the total number of squares in the large square. Again, a collective "ah" sounded throughout the room when students realised that the number of blocks in each separate piece used to form the larger squares was a triangular number. Students were equally as fascinated when they subsequently discovered that the total number of blocks in each larger square was a square number. They generalised that two consecutive triangular numbers, when added together, form a square number. However, some students expressed total confusion, many becoming frustrated at their lack of understanding. This caused the teacher to reflect on how this lesson could be further scaffolded. Perhaps these students needed to understand that although the two pieces of the staircase did not look like triangles, there was a triangular number of small squares in each piece. This teaching could have been made more explicit.

Despite the frustration of some students with this last lesson, many expressed an interest in continuing to learn the concept of triangular numbers, with several requesting that the teacher (the first author) return in subsequent days to continue lessons.

Conclusion

The sequence of lessons was found to be an engaging and effective introduction to triangular numbers for children in Year 6. Our experience supports the observations of Papic, Mulligan and Mitchelmore (2011) and Billings, Tiedt and Slater (2007) that children with no prior experience of a geometric growing pattern are able to continue it intuitively and generalise from the pattern. It also supports Rivera and Becker's (2005) conclusion that when students are given problems that require them to generalise, figural understandings should be emphasised before numerical understandings, and children should then be encouraged to move flexibly between the two representations to form generalisations and explain them. The use of concrete materials and a systematic tabulation of results enabled students to articulate the important relationship between a triangular number and its position in the sequence of triangular numbers (i.e. that the value of any triangular number in the sequence is the sum of whole numbers from one up to the number of the term). At first, many students found understanding the pattern between the numbers difficult, as previously they had only been

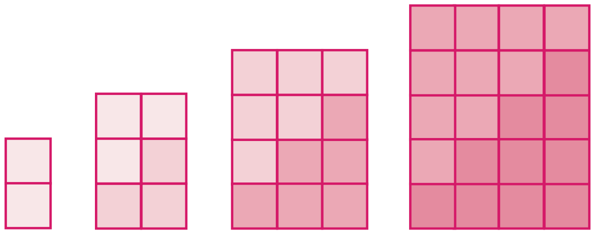


Figure 8. The first four figures in the sequence of rectangular numbers. These can be formed by combining the same two triangular numbers.

exposed to linear relationships (having a consistent difference between terms).However, after realising in the first lesson that non-linear relationships existed, the students seemed more confident in working with number patterns that are non-linear. An extension to the up-and-down stairs lesson that might help them find the value of a triangular number more efficiently from its position in the sequence, would be a task requiring them to form a sequence of 'rectangular numbers'. A rectangular number is the product of a number and one more than the number. A figure with a rectangular number of small squares can be formed by placing together two identical triangular parts of the staircases shown in Figure 3 to produce Figure 8. Each triangular number can then be seen as half of a rectangular number, so in the sequence of triangular numbers, the triangular number in the *n*th position in the sequence is $\frac{n \times (n+1)}{2}$.

Following this discovery, there are many problems with a similar mathematical structure that can be introduced to enable children to apply their understanding. A popular problem with a context enabling it to be acted out is "the handshake problem" (Lechner, 2005, p. 111). There are also some excellent artistic and creative activities that can be used to consolidate the idea of triangular numbers, for example, children might act out the song *The Twelve Days of Christmas* where the number of people on stage are one ("a partridge in a pear tree") then three ("two turtle doves and a partridge in a pear tree") and so on. Wolfe (2014) describes how she choreographed a dance to show that two consecutive triangular numbers make a square number. The teaching of triangular numbers is rife with possibilities!

References

Australian Curriculum and Assessment Reporting Authority [ACARA]. (2015). *Australian Curriculum: Mathematics*, Version 7.4. Sydney. Retrieved (Jan 1, 2016) from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Billings, E. M. H., Tiedt, T. L. & L. H. Slater (2007). Algebraic thinking and pictorial growth patterns. *Teaching Children Mathematics, 14*(5), 302–308.

Booker, G. & Windsor,W. (2010). Developing algebraic thinking: Using problem-solving to build from number and geometry in the primary school to the ideas that underpin algebra in high school and beyond. *Procedia Social and Behavioral Sciences 8*, 411–419.

Bridge, C., Day, L., & Hurrell, D. (2012). From routine to rich: Developing an algebraic reasoning task for a middle/upper primary class. *Australian Primary Mathematics Classroom, 17*(4), 8–12.

Hill, T., Lanin, J. , & van Garderen, D. (2015). Promoting and assessing mathematical generalising. *Australian Primary Mathematics Classroom, 20*(4), 3–8.

Hourigan, M. & Leavy, A. (2015). Geometric growing patterns: What's the rule? *Australian Primary Mathematics Classroom, 20*(4), 31–40.

Lechner, G. (2005). *Creative problem solving in school mathematics.* Australian Edition. New York: MOEMS.

Markworth, K. A. (2012). Growing patterns: Seeing beyond counting. *Teaching Children Mathematics, 19*(4), 254–262.

McMaster, H. (2003). Introducing mathematics topics through storytelling and problem solving. *Reflections, 28* (1), 41–44.

Muir, T., Bragg, L. A., & Livy, S. (2015). Two of everything: Developing functional thinking in the primary grades through children's literature. *Australian Primary Mathematics Classroom, 20*(1), 25–30.

Papic, M., Mulligan, J. & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–269.

Richardson, K., Carter, T., & Berenso, S. (2010). Connected tasks: The building blocks of reasoning and proof. *Australian Primary Mathematics Classroom, 15*(4), 17–23.

Riviera, F.D. & Becker, J.R. (2005). Teacher to teacher: Figural and numerical modes of generalizing in algebra. *Mathematics Teaching in the Middle School, 11*(4), 198–203.

Samson, D. (2014). Visualising and generalising with square arrays. *Australian Mathematics Teacher, 70*(2), 4–12.

Sullivan, P., Askew, M., Cheeseman, J., Clarke, D., Mornane, A., Roche, A., & Walker, N. (2015). Supporting teachers in structuring mathematics lessons involving challenging tasks. *Journal of Mathematics Teacher Education 18*:123–140.

Switzer, J.M. (2013). Linking models: Reasoning from patterns to tables and equations. *Australian Mathematics Teacher, 69*(1), 3–7.

Szetela, W. (1999). Triangular numbers in problem solving. *The Mathematics Teacher, 92*(9), 820–824.

Wilkie, K.J. (2014). Learning to like algebra through looking: Developing upper primary students' functional thinking with visualisations of growing patterns. *Australian Primary Mathematics Classroom, 19*(4), 24–33.

Wolfe, C. (2014). Dancing triangular squares–the process of creating a mathematical dance. *Mathematics Teaching 242*: 11–13.