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Beyond Error Patterns: A Sociocultural View of Fraction Comparison Errors in Students With Mathematical Learning Disabilities

Katherine E. Lewis, PhD[1]

Abstract

Although many students struggle with fractions, students with mathematical learning disabilities (MLDs) experience pervasive difficulties because of neurological differences in how they process numerical information. These students make errors that are qualitatively different than their typically achieving and low-achieving peers. This study builds upon a quantitative study of fraction comparison errors and a qualitative study of students' understandings to explore why students with MLDs make errors on the easiest fraction comparison problems. A detailed analysis of videotaped individual tutoring sessions with two adult students with MLDs revealed that both students understood mathematical representations in atypical ways, which may help explain the unique and persistent error patterns identified in students with MLDs. This study illustrates how building upon both quantitative and qualitative studies can provide a more nuanced understanding of student errors, which in turn can directly connect to implications for instructional interventions.

Keywords

math learning disability, dyscalculia, fractions, tutoring

Researchers estimate that approximately 4% to 8% of students have a mathematical learning disability (MLD) sometimes referred to as dyscalculia (e.g., Shalev, 2007). These students with MLDs have a neurological difference in how they process numerosities, which lead to difficulties processing quantitative information (Butterworth, 2010). Students with MLDs have been found to be slower and make more errors when processing symbolic (e.g., "3") and non-symbolic (e.g., "◆◆◆") representations of quantity (e.g., Piazza et al., 2010). Researchers have found that these students make errors when comparing and estimating quantities (Mazzocco, Feigenson, & Halberda, 2011), performing arithmetic calculations (e.g., Geary, Hoard, Byrd-Craven, & DeSota, 2004), and solving basic number facts (e.g., 4 × 5 = 20; Mazzocco, Devlin, & McKenney, 2008). Longitudinal studies have demonstrated that the difficulties experienced by these students persist across years (e.g., Geary, Hoard, Nugent, & Bailey, 2012; Mazzocco et al., 2008). Despite the increase in research on MLDs over the past several decades, the majority of research has focused on basic number processing or whole number calculation (Lewis & Fisher, 2016). This has left critical content areas, such as fractions, underexplored.

Understanding the difficulties that students with MLDs experience with fractions is critically important because proficiency with fractions is essential for later mathematical development (Bailey, Hoard, Nugent, & Geary, 2012) and future academic and career opportunities (National Mathematics Advisory Panel, 2008). There is some evidence that the nature of the difficulties experienced by students with MLDs in the context of fractions is qualitatively different. For example, Mazzocco, Myers, Lewis, Hanich, and Murphy (2013) conducted a longitudinal study and determined that students with MLDs made errors unlike their typically achieving and low-achieving peers. Students with MLDs made errors on the *easiest* fraction comparison problems—problems in which the fractions have the same denominator and problems in which one of the fractions is one-half. These comparisons are considered to be the easiest for students to master because students can rely upon their whole number reasoning to compare fractions with the same denominators (e.g., 5/8 is greater than 2/8 because 5 is greater than 2; Sophian, 2000) and students have an intuitive understanding of one-half from a young age (Hunting

[1]University of Washington, Seattle, USA

Corresponding Author:
Katherine E. Lewis, College of Education, University of Washington, Box 353600, Seattle, WA 98195, USA.
Email: kelewis2@uw.edu

& Davis, 1991). The errors made by students with MLDs on these easy fraction comparison problems persisted from fourth through eighth grades (Mazzocco et al., 2013).

It is unknown why students with MLDs would experience difficulties with the easiest comparison problems and why these difficulties would persist over the years. This is of critical importance because difficulties with magnitude comparison are thought to be at the heart of difficulties experienced by students with MLDs (e.g., Price & Ansari, 2013). Further, little is known about how to effectively provide remediation for these kinds of difficulties given that most students find these kinds of comparisons unproblematic. Finally, if students with MLDs are unable to accurately compare simple fractions, it is unlikely that they will be able to productively engage with more complex fraction topics. Although these error patterns differentiate the performance of students with MLDs from their peers, it remains unclear what the origin of these error patterns are and why these qualitative differences emerge.

Exploring why these error patterns emerge and persist for students with MLDs necessitates different methodological and theoretical approaches than typically employed in research on MLDs. Research on MLDs tends to draw upon cognitive theoretical perspectives and employ quantitative methods. For example a psycINFO search for "math* learning disab*" or "dyscalculia" yielded 310 empirical articles, 230 of which were classified as employing quantitative methodology, with only five of those studies employing qualitative methodology. The predominant use of quantitative methods limits the nature of the questions that can be asked and answered (Poplin, 2011). Quantitative statistics are appropriate for answering questions about "what." Knowing *what* errors students with MLDs make does not directly connect to implications for intervention. Although quantitative methods have been used to identify that students with MLDs make more errors and that these errors are unlike those made by their peers (e.g., Mazzocco et al., 2013), documenting the existence of these errors does not suggest what should be done to address them. Poplin (2011) referred to this uniformity of methodological approach as the "hegemony of quantitative methodologies" and argued that "when the human sciences use only quantitative data, we end up with a narrow, piecemeal view of reality, and thus, narrowed solutions" (p. 150). The predominant use of quantitative methods has resulted in myriad studies identifying what kinds of errors students make without revealing the reasons those errors are made or why they might persist despite instruction.

To make progress in our understanding of MLDs, particularly in the underexplored domain of fractions, it is necessary to leverage both quantitative and qualitative research in tandem. The current study builds upon a longitudinal quantitative study of fraction comparisons errors in students with MLDs (Mazzocco et al., 2013) and a small-scale qualitative case study of two adult students with MLDs (Lewis, 2014). By drawing upon both, the current study extends each. This study replicates the findings from Mazzocco et al. (2013) in another sample of students—in this case, adults who have met strict MLD criteria. It broadens the implications of Mazzocco et al.'s (2013) findings by going beyond error quantification and provides a more holistic view of the student's learning. The original case study (Lewis, 2014) is strengthened because the case study analyses are connected to difficulties identified in students with MLDs more widely. Central to this qualitative analysis is the theoretical frame, which informs and shapes the analytic endeavor. This study draws upon a Vygotskian notion of disability situated within a sociocultural theory of learning.

Sociocultural View of Mathematical Learning Disabilities

Vygotsky's understanding of disability was situated within his general theory of human development. Vygotsky (1981) argued that human development progresses along two lines: the biological and sociocultural. For typically developing individuals, these two lines of development intersect. For individuals with disabilities, the sociocultural tools that have developed over the course of human history may be incompatible with the individual's biological development (Vygotsky, 1929/1993). For example, spoken language is not accessible to a deaf child and therefore does not serve the same mediational role to support the child's development of language as it would for a hearing child. In the case of students with MLDs, standard mathematical mediational tools (e.g., numerals, drawings, manipulatives) that support the development of typically developing students may be incompatible with how students with MLDs cognitively process numerical information.

Because of the incompatibility of the biological and sociocultural lines of development, a child with a disability "is not simply a child less developed than his peers, but is a child who has developed *differently*" (Vygotsky, 1929/1993, p. 30; emphasis added). Therefore, a quantitative documentation of the student's errors is insufficient. Instead, it is necessary to document the ways in which the student uses and understands standard mediational tools in qualitatively different ways. This theoretical perspective suggests that the differences resulting from the student's MLD may be most readily apparent in the student's interaction with mediational tools like mathematical representations. Therefore, the analysis of MLDs must be sensitive to these qualitative differences in use of mediational tools and must identify the understandings that the students rely upon rather than simply the skills they lack.

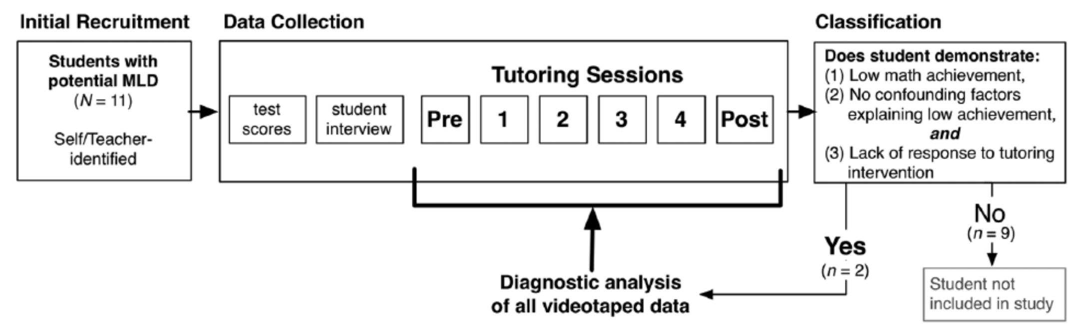

Figure 1. Schematic overview of methods used in this study. Adapted with permission from Journal for Research in Mathematics Education, copyright 2014, by the National Council of Teachers of Mathematics. All rights reserved.

Current Study

This study employed qualitative methods and built upon prior case studies of two adult students with MLDs (Lewis, 2014). The case studies were originally conducted as part of larger study that investigated MLDs in context, during students' attempts to learn basic fraction concepts during one-on-one videotaped tutoring sessions. The two students did not benefit from a series of fraction tutoring sessions that were effective for typically achieving students. Analysis revealed that these students relied on a small set of operationally defined persistent understandings (see Appendix for examples), which were detrimental to the students' attempts to learn. The current study built on these case studies to specifically explore the unique error patterns identified in Mazzocco et al. (2013). The qualitative analysis explores these students' fraction comparison errors and what their process and explanations reveals about why these errors were made and why they might persist. This study brings together a quantitative and a qualitative study of MLDs to address the following research questions:

> **Research Question 1:** Do the case study students demonstrate errors similar to those documented in Mazzocco et al. (2013), specifically, difficulties comparing fractions with the same denominator and comparing fractions with 1/2?
>
> **Research Question 2:** Do any of the persistent understandings identified in the original case studies (Lewis, 2014) occur with these errors and provide a plausible explanatory frame for why these errors might persist?

Method

Multiple sources of data were collected for each student, which were used for determination of the student's MLD status and case study analysis. Although most studies rely exclusively on low math achievement scores to identify students with MLDs (Lewis & Fisher, in press), in this study more stringent identification criteria were employed. To meet the MLDs criteria, students had to demonstrate (a) low math achievement, (b) no social or environmental factors that could explain that low achievement, and (c) a lack of response to instruction (Fletcher, Lyon, Fuchs, & Barnes, 2007). Students were recruited for this study—and all data were collected—before a classification determination was made. Only those students who met the MLD classification criteria were included as case study participants (see Figure 1 for an overview of the design of the study). Therefore, both students who met the MLD requirements demonstrated a history of low achievement, had no confounding factors explaining their low achievement, and did not benefit from a series of tutoring sessions that were effective for typically achieving students. The author served as the interviewer and tutor for all data collected.

Participants

Eleven students with potential MLDs were selectively recruited from a local middle school, high school, and community college based on self-nomination or teacher nomination. In addition to these 11 students with potential MLDs, typically achieving students were recruited to evaluate the effectiveness of the tutoring protocol and to empirically establish expected learning gains from pretest to posttest. To ensure the mathematical content was appropriate for the typically achieving students, fifth-grade students were recruited because, like the students with potential MLDs, these students had prior experience with fractions but had not mastered the topics covered during the tutoring sessions. The parents of all fifth-grade students at a local elementary school received an email inviting their child to participate in a series of mathematics enrichment tutoring sessions. All five students whose parents responded and consented were included as comparison students.

Participant classification. Out of the 11 students, nine were excluded from the MLD classification for failing to meet one or more criteria. Low math achievement (below the

Table 1. Classification and Demographic Information for the Two Students With MLDs.

Name	Criterion	Data
Emily	Low math achievement	<25th percentile on state mandated achievement test
	No confounding factors	Native speaker. Not low SES. No attention issues.
	No response-to-instruction	Pretest: 49%, Posttest: 54%, Change: +5%
Lisa	Low math achievement	Compass placement test placed her in basic arithmetic
	No confounding factors	Native speaker. Not low SES. No attention issues.
	No response-to-instruction[a]	Pretest: 59%, Posttest: 44%, Change: −15%

Note. MLDs = mathematical learning disabilities. SES = socioeconomic status.
[a]Because Lisa was administered a truncated version of the pretest, to calculate response-to-instruction only items that had a corresponding pretest item were included.

25th percentile) was based on the student's score on a norm-referenced standardized math achievement test. Confounding factors (e.g., language fluency, attention, behavior, or other affective issues) were assessed through a student interview and an evaluation of student behavior during the tutoring sessions. The student's response-to-instruction was determined based on a comparison of a videotaped pretest and posttest given in the session immediately preceding and following the sequence of one-on-one tutoring sessions.

Nine students were excluded from the MLD classification for one of several reasons: (a) performance at ceiling on the pretest ($n = 2$), (b) observed or self-reported attention or behavior problems ($n = 3$), (c) failure to complete all data collection sessions ($n = 2$), or (d) response-to-instruction ($n = 2$; e.g., substantial gains from pretest to posttest suggesting that poor prior instruction was a possible cause of their low math achievement). Only two students, "Lisa" (a White, 19-year-old community college student) and "Emily" (a White 18-year-old recent high school graduate), met all the qualifications for having an MLD (see Table 1). Both students scored within one standard deviation on the pretest but not on the posttest. This suggests that they had similar prior understanding to the fifth-grade students, but did not similarly benefit from the tutoring protocol.

Materials and Procedures

Mathematics achievement test. The California state mandated *Standardized Testing and Reporting (STAR) Achievement Test* scores were collected from students to establish the student's low math achievement. Low math achievement was operationally defined as below the 25th percentile. Although researchers are calling for more stringent cutoffs to identify students with MLDs, the 25th percentile remains the most commonly used cutoff for MLD classification (Lewis & Fisher, in press) and MLD classification in this study was not based upon the establishment of low achievement alone. If standardized assessments scores were not available, the student's poor performance on the college Compass Placement Test, resulting in enrollment in a remedial math class, was used to establish the student's low math achievement.

Interview. Students were interviewed about their background and experiences learning math to identify social or environmental factors that could explain a student's low achievement. The questions focused on the following topics: student's background, experience learning math, resources at the student's disposal, student's perceived level of effort, and student's home language (see Lewis, 2011, for interview prompts).

Pretest/posttest. Videotaped semi-structured clinical interview pretests and posttests were administered to all participants. The test (see Lewis, 2011 for the protocol) was designed to cover all fraction concepts targeted in the tutoring sequence. Most problems included multiple iterations involving isomorphic problems but with increasingly difficult fractional values. If a student failed to answer two or more of the iterations correctly, the remaining iterations of that problem type were not administered. The change from pretest to posttest was used to evaluate the student's response-to-instruction. Expected response-to-instruction was empirically defined; the fifth-grade students had an average gain of 15% from pretest to posttest and an average posttest score of 84% (pretest $M = 68.7\%$ $SD = 19.4\%$; posttest $M = 83.6\%$, $SD = 12.6\%$). A lack of response-to-instruction was defined as less than a 10% gain from pretest to posttest and a posttest score at or below 60%.

Tutoring sessions. Four hour-long weekly videotaped tutoring sessions were conducted with each student. These sessions were designed based on research on the teaching and learning of fractions (e.g., Armstrong & Larson, 1995; Empson, 2001; Mack, 1995; Post, Wachsmuth, Lesh, & Behr, 1985). The instructional goals of the tutoring session were (a) to build an understanding of a fraction as a single value, which is determined by the relationship between the numerator and denominator, and (b) to use manipulatives and representations to explore the concepts of fraction equivalence and fraction addition and subtraction.

- Tutoring Session 1 used foam fraction manipulatives and focused on establishing the meaning of the numerator and the denominator.
- Tutoring Session 2 continued to build upon these concepts, exploring the conventions of representing fractions with drawn area models and using area models to compare fractional amounts.
- Tutoring Session 3 used fair sharing in conjunction with area models to explore equivalent fractions.
- Tutoring Session 4 used manipulatives and area models to explore fraction addition and subtraction problems.

Problems were carefully sequenced to ensure that each question built upon previously established mathematical content, and follow-up prompts were created to anticipate the range of student answers (see Lewis, 2011, for complete protocol). Similar to prior tutoring work, each question was conceptualized as an opportunity for the student to learn and a means of assessing the student's understanding (e.g., Mack, 1995).

Although the tutoring instruction was essential for the MLD classification criteria employed in this study, the focus of this research was *not* on the tutoring itself. Instead, the videotaped sessions provided a context in which the difficulties that arose for the students with MLDs could be analyzed. Several fifth-grade students benefited from this tutoring protocol, which suggests that this instructional sequence should be considered a reasonable learning environment.

Analytic Approach

As part of the original case studies (Lewis, 2014), all videotapes of the sessions (pretest, tutoring sequence, posttest) were transcribed and parsed into individual problem instances. Each problem instance began with a question and ended with a student's answer. A microgenetic analysis was conducted that involved iterative passes through the data in an attempt to generate analytic categories that captured the nature of the student's understanding (see Schoenfeld, Smith, & Arcavi, 1993, for an example). The patterns of student reasoning that reoccurred were operationally defined and referred to as "persistent understandings" (see Appendix for operational definitions).

The operational definitions for these persistent understandings were developed and refined through iterative passes through the data (Barron, Pea, & Engle, 2013), which involved identifying candidate persistent understandings, specifying inclusion and exclusion criteria, considering alternative explanations, and attempting to identify counter-examples that would contradict the proposed persistent understandings. This process included considering alternative hypotheses to explain the data and then reviewing the data to determine if each hypothesis was supported or refuted.

A team of four coders used these operational definitions to code the data. Each problem instance was coded for correctness and evidence of any of the persistent understandings identified for that student (a total of six persistent understandings were identified for each student, see Lewis, 2014, for more details). Reliability for this coding was 94.6% for Lisa and 95.4% for Emily. Any discrepancies in coding were discussed in a research meeting with all four coders and were resolved by watching the video and discussing whether there was sufficient evidence in the video to warrant the attribution of the operational definition. All discrepancies were resolved using stringent criteria for coding—if one of the four coders was not convinced that the episode matched the operational definition, it was not coded as such.

After the individual case analyses were completed, a cross-case analysis was conducted to identify commonalities across Lisa's and Emily's persistent understandings. Overarching analytic categories were created for the persistent understandings that resulted in similar kinds of errors. The present analysis focuses on three of these categories of persistent understandings: *fractional complement, single factor*, and *halving* (see Appendix for operational definitions). Note that the specific operational definitions for both Lisa and Emily were somewhat different because these operational definitions emerged from the data. It was only after the case analyses were completed that larger analytic categories were identified. The commonality between operational definitions was determined because they led to similar kinds of errors.

Fraction comparison analysis. Building on the case study analysis, the current analysis focused specifically on fraction comparison problems. A final analytic pass of the data was conducted to facilitate the present analysis. Each problem instance was coded for problem type. Problem types codes were *representation* (e.g., student draws or interprets a fraction representation), *comparison, equivalence, operations*, or *N/A*. All comparison problems were analyzed to determine if they involved same denominator comparisons or comparisons with one-half. Student accuracy on all target comparison problems was calculated. For same denominator and one-half comparison problems, the students' process and explanation were analyzed to determine if the students' errors were associated with any persistent understandings identified in the original case study (Lewis, 2014).

Establishing trustworthiness. Methods to establish the credibility of the analysis were intentionally included in the design. First, during the development of the coding scheme, the candidate operational definitions and video episodes were presented and discussed with members of a research group composed of graduate students and faculty members in math education. Second, in the development of the operational definitions, counter-examples in the video data were deliberately identified and alternative hypotheses were considered. The generation of these specific operational definitions emerged because of the persistence of these behavioral characteristics in the data. Third, the systematic coding of data and evaluation of coding reliability ensured that these operational definitions were sufficiently precise to identify behavioral characteristics in video data. Finally, the video data collected with the typically achieving fifth-grade students were content logged and selectively transcribed to evaluate whether any of the persistent understandings identified for either student with MLDs were evident in any of the fifth-grade students and therefore could be attributed to the tutoring protocol itself.

Results

The analysis was comprised of three parts; each will be discussed in turn. First, the students' performance on comparison problems were considered to evaluate whether Emily and Lisa experienced the same error patterns noted in Mazzocco et al. (2013), specifically, (a) comparisons of fractions with the same denominators and (b) comparisons involving the fraction one-half. Second, the persistent understandings that were associated with the students' comparison errors were identified. Third, the persistent understandings and errors were analyzed together to determine if these persistent understandings provided a productive explanatory frame for the comparison errors made by Lisa and Emily and if they were evident across problem types in the data.

Error Analysis

An evaluation of comparison problems was used to establish that Emily and Lisa demonstrated a similar pattern of errors to those documented in the Mazzocco et al. (2013) study. Throughout all tutoring sessions, Emily answered 63% of all same denominator problems incorrectly and 37% of all problems involving one-half incorrectly. Similarly, Lisa answered 14% of comparison problems with the same denominator incorrectly and 76% of problems involving one-half incorrectly. When errors occurred during the tutoring sessions, they were immediately addressed, but the errors persisted throughout the sessions. For example, on the posttest both students were asked to compare the fractions 2/5 and 3/5 represented as area models. Although both students were able to correctly interpret the area models as 2/5 and 3/5, they both incorrectly determined that 2/5 was the larger fraction. Additionally, on the posttest both students were asked to compare 2/3 and 2/4. Although both students correctly identified 2/4 as 1/2 (not a required part of the problem), after doing so, they both incorrectly answered the question. Emily determined that 1/2 equaled 2/3, and Lisa determined that 1/2 was larger than 2/3. Given the high percentage of errors made on these problems and the continued evidence of errors during the posttest, Emily's and Lisa's performance were both judged to be consistent with the findings of Mazzocco et al. (2013).

Associated Persistent Understandings

The second phase of analysis involved determining which persistent understandings, if any, were associated with the same denominator and one-half comparison problems. Two persistent understandings occurred with these comparison errors: *fractional complement* and *single factor*. The fractional complement understanding was associated with same denominator comparison errors. This persistent understanding involved the ways in which Lisa and Emily interpreted fraction representations. Instead of interpreting a given representation (e.g., area model of 3/4) as the fractional value (3/4), both students sometimes interpreted the representation as the fractional complement (i.e., 1/4; see Figure 2a). The second persistent understanding, single factor, was associated with comparison problems involving the fraction one-half. The single factor understanding involved judging the magnitude of the fraction based on only the numerator or denominator value (see Figure 2b). For example, Lisa might attend to the number of total pieces in both 1/2 and 4/5 and argue that halves are larger than fifths and incorrectly determine that 1/2 is larger than 4/5. Similarly, Emily might attend only to the number of pieces and incorrectly determine that 1/3 was larger than 1/2 because 1/3 was comprised of more pieces.

In addition to these two persistent understandings, a third persistent understanding, *halving*, warrants discussion. Although not associated directly with fraction comparison problems, the halving understanding provides additional insight into how the students understood the quantity 1/2 (see Figure 2c). Both students understood the fraction 1/2 as the act of splitting something into two parts, rather than the quantity (i.e., one of two parts). This understanding of 1/2 may suggest that these students did not have the intuitive understanding that most children have about the quantity 1/2 (Hunting & Davis, 1991) and comparisons of quantities to 1/2 (Spinillo & Bryant, 1991, 1999).

Three persistent understandings were implicated in the fraction comparison errors; fractional complement and single factor were directly associated with these errors, and halving was indirectly associated. In the next section, exemplars of Lisa's and Emily's solutions to same denominator and one-half comparison problems will be presented.

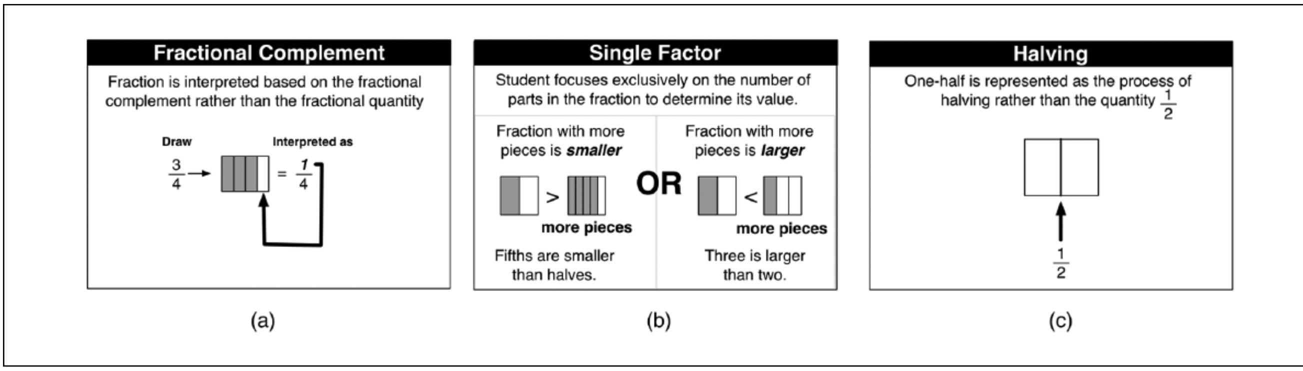

Figure 2. Illustration of "fractional complement," "single factor," and "halving" persistent understandings.

These examples highlight the ways in which the students relied upon the persistent understandings in the context of comparison problems, and the ways in which these persistent understandings led to errors.

Same denominator comparison problems. The fractional complement understanding emerged during the students' attempts to solve same denominator comparison problems. For example, during the pretest, Emily was asked to compare the fractions 2/8 and 5/8. After drawing a correct representation for both fractions (see Figure 3), she began interpreting 5/8 as 3/8 and attending to the non-shaded pieces (relevant transcript in bold).

Tutor: What if we had the problem two-eighths and five-eighths, which one would be bigger there?

Emily: [*writes 2/8 and 5/8. Draws eight rectangles, shades in two. Draws eight rectangles, shades in five*] So, this is . . . [*points to each of the 5 shaded pieces in drawing of 5/8.* ***Points to each of the 3 non-shaded pieces of drawing of*** 5/8] Um. [***Writes*** 3/8 ***Points to each of the 6 non-shaded pieces in the drawing of*** 2/8] I don't know.

Although Emily correctly represented both fractions, these drawings did not support her comparison of the fractional amounts. Once drawn, Emily shifted from attending to the five shaded pieces to attending to the three non-shaded pieces for 5/8 and interpreted her drawn representation as 3/8, the fractional complement. In addition, Emily also attended to the six non-shaded pieces of 2/8 by pointing to each of the pieces in turn before determining she did not know how to answer the question. What this excerpt reveals is not only did Emily have difficulty comparing 2/8 and 5/8, but that when presented with the fractions 2/8 and 5/8, she was unsure whether she should be comparing the shaded pieces (representing the fractional quantity) or the non-shaded pieces (representing the fractional complement).

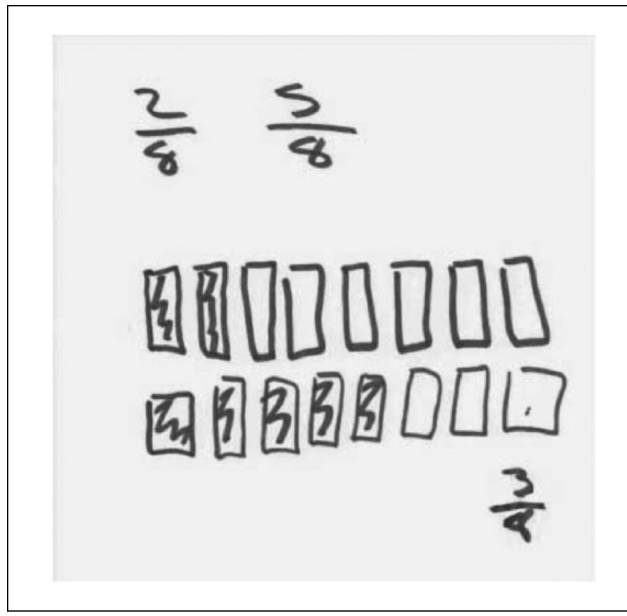

Figure 3. Emily's written work for the comparison of 2/8 and 5/8. Reprinted with permission from Journal for Research in Mathematics Education, copyright 2014, by the National Council of Teachers of Mathematics. All rights reserved.

One-half comparison problems. The single factor understanding emerged during the students' attempts to solve one-half comparison problems. For example, during the first tutoring session, Lisa was asked to compare the fractions 1/2 and 3/4. She incorrectly determined that 1/2 was larger than 3/4 and justified her answer by explaining that 1/2 had larger pieces than 3/4 (see Figure 4).

Tutor: So if we were comparing three-fourths and one-half, which one of those is going to be bigger?

Lisa: The half.

Tutor: Ok, and how do you know?

Lisa: Because it's larger, because it's closer to a whole. You have to split this [*pointing to* 3/4] up four ways to

Figure 4. Lisa's written work for the comparison of 1/2 and 3/4.

> create a whole. Whereas the half only needs to get split into two.

Lisa justified her answer that 1/2 was larger than 3/4, focusing exclusively on the number of partitions (i.e., denominator) for each fraction and determined that the halved pieces would be larger than those split into four pieces. In comparing these fractions, she did not attend to or refer to the numerator of the fractions. Although there was not sufficient evidence to meet the operational definition of halving in this instance, Lisa did refer to the fraction 1/2 as the act of splitting something in two, rather than the quantity 1/2. As in this instance it is possible that Lisa's and Emily's tendency to understand the fraction 1/2 as the act of splitting increased the likelihood that the single factor understanding was more likely to be invoked.

Persistent Understandings as a Productive Explanatory Frame

The persistent understandings provide a plausible explanatory frame for why Lisa and Emily—like students with MLDs—experienced persistent difficulties with these easy comparison problems (see Figure 5). Not only do these persistent understandings provide insight into the fraction comparison errors made by Lisa and Emily, they also characterize the difficulties that the students had across other problem types as well (see Table 2). Therefore, the utility of these operationally defined persistent understandings is their ability to contextualize the difficulties that Lisa and Emily experienced with fraction comparison problems and relate it to their difficulties learning fraction concepts more generally.

Fractional complement understanding. The tendency to attend to the fractional complement was evident throughout both Emily's and Lisa's sessions and caused difficulties on non-comparison problems as well. For example, the fractional complement understanding was evident when Lisa attempted to use equivalent fractions to solve the problem 2/3 + 1/4 = __. Although she was able to create equivalent fraction area models for both fractions (8/12 and 3/12) and she correctly interpreted the area model of 3/12, she incorrectly interpreted 8/12 as 4/12, attending to the fractional complement (i.e., unshaded pieces).

This fractional complement understanding reoccurred in almost all sessions and was often associated with an incorrect answer, in 81% of cases for Emily and 87% of cases for Lisa. In contrast, the fifth-grade students did not similarly orient to the fractional complement and it was not problematic for any of them across the tutoring sessions or any problem types. Therefore, not only does the persistence of the fractional complement understanding for both Emily and Lisa provide a potential explanation for why both students would make errors on fraction comparison problems involving the same denominator, it also suggests that the reliance on a fractional complement understanding led to difficulties more generally.

Single factor understanding. Unlike the other two persistent understandings, the single factor understanding was almost exclusively used in conjunction with comparison problems. This may be because the operational definition of this understanding involves the student's judgment of fractional magnitude, which was most evident on comparison problems. However, there was some indication that Lisa did not treat the numerator and denominator as a single value in other contexts as well. For example, to write the fraction 5/10 she sometimes wrote it as "5 $\frac{1}{10}$ " (i.e., five one-tenths). This suggests that for her the values comprising the fraction were not coordinated and may have contributed to her understanding fractional magnitude in terms of only one of the two values rather than a single multiplicative relationship. Although this understanding occurred primarily in the context of comparison problems, it was often associated with an incorrect answer, in 93% of cases for Emily and 71% of cases for Lisa.

Halving understanding. Although the halving persistent understanding was not evident on any comparison problems for either student, it was evident on other problem types for both students. The halving understanding was most evident on problems when the students were directly asked to represent the fraction 1/2. For example, on the posttest the students were asked to draw 1/2. They each

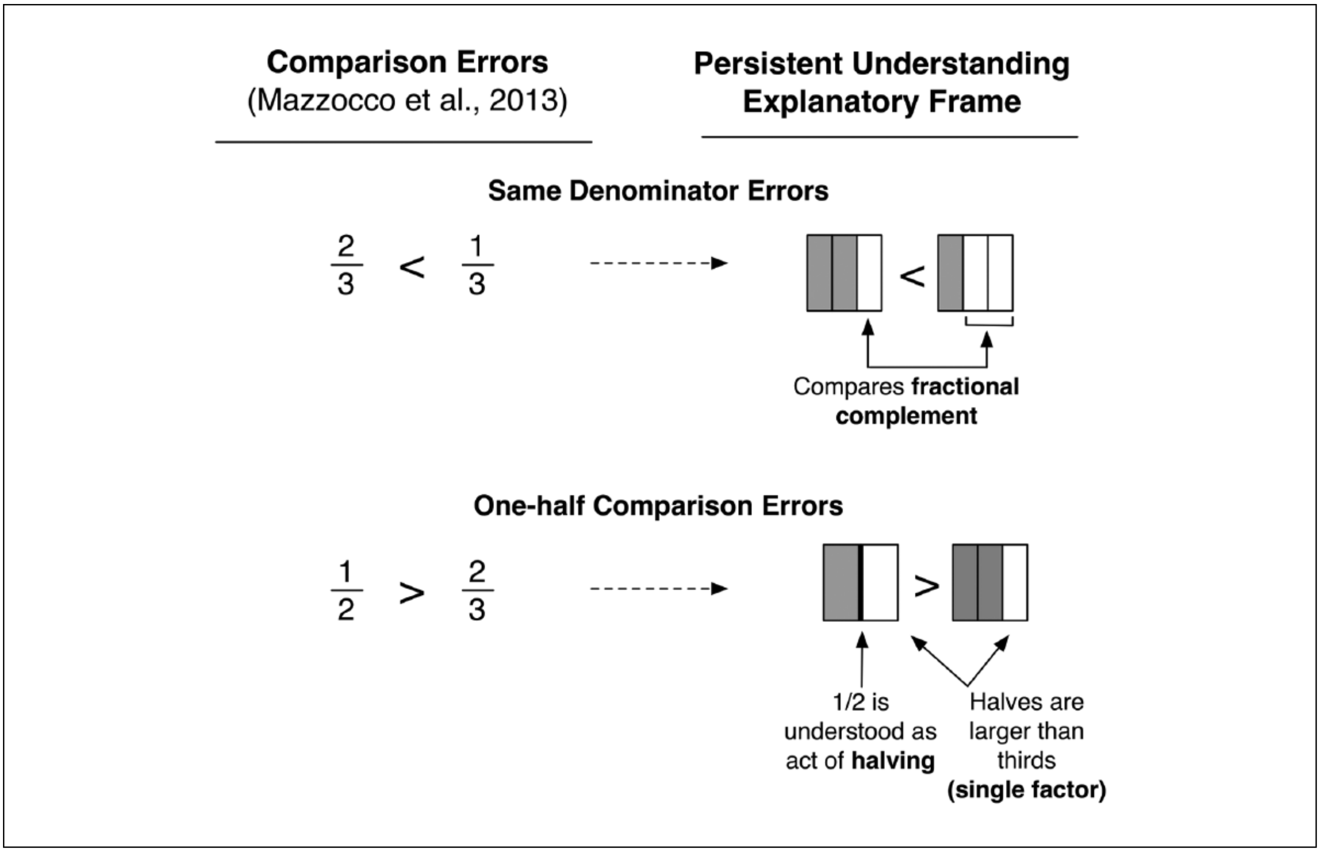

Figure 5. Illustration of the relationship between the comparison errors identified in Mazzocco, Myers, Lewis, Hanich, and Murphy (2013) and the persistent understandings identified for Lisa and Emily.

Table 2. Summary of Qualitative Data Analysis.

			Persistent understanding evident		
Problem type	Total questions[a]	Correct	Fractional complement	Single factor	Halving
Lisa					
Comparison	89	57%	7.87%	25.84%	0.00%
Representation	203	68%	5.42%	2.96%	4.93%
Equivalent fractions	99	79%	3.03%	5.05%	2.02%
Operations	40	50%	22.50%	2.50%	5.00%
Total	431	67%	6.96%	8.12%	3.25%
Emily					
Comparison	72	71%	5.56%	13.89%	0.00%
Representation	215	73%	4.19%	1.86%	3.72%
Equivalent fractions	114	82%	7.02%	0.00%	0.00%
Operations	44	55%	13.64%	0.00%	11.36%
Total	431	73%	6.07%	3.15%	2.92%

[a]Frequency calculations exclude problems that were classified as "tutor guided" or "not applicable" (e.g., student's response about which problems felt hardest).

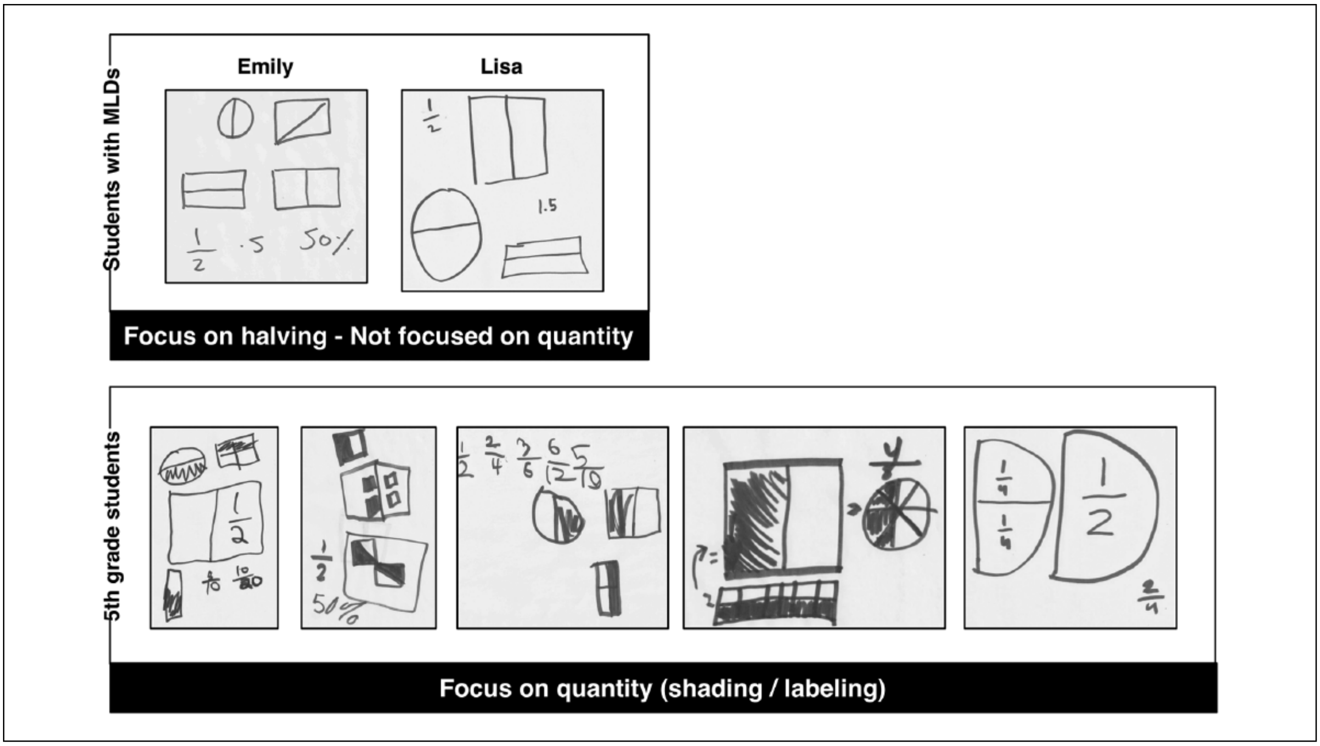

Figure 6. Written work of representations of 1/2 drawn by Emily, Lisa, and the fifth-grade students on the posttest. Adapted with permission from Journal for Research in Mathematics Education, copyright 2014, by the National Council of Teachers of Mathematics. All rights reserved.

drew several different shapes (see Figure 6) that were partitioned in half and omitted the standard shading of one of the two pieces. Lisa's and Emily's responses can be contrasted with those of the fifth-grade comparison students, which all highlighted the fractional quantity using canonical shading or labeling (see Figure 6). When Emily was asked to explain why the non-shaded halved shape represented 1/2, she said,

> Well, this is one [*traces with pen around the circle*]. And it's cut in half, and there is two [*points with pen back and forth between the two pieces*] of the exact same—I mean, they are not the *exact* same, but they are supposed to be the exact same size.

This excerpt highlights that her understanding of 1/2 involved the partitioning of the shape and the balance between the pieces rather than the fractional quantity—one out of the two total pieces. The halving understanding was often associated with an incorrect answer, in 92% of cases for Emily and 71% of cases for Lisa.

The persistence of the halving and single factor understandings for both Emily and Lisa contributes to a plausible explanation for why both students would make errors on fraction comparison problems involving 1/2 and suggests that their reliance on these understandings led to difficulties more generally.

Discussion

The purpose of this study was to leverage both extant quantitative and qualitative studies to investigate the unique errors made by students with MLDs on fraction comparison problems. In particular, this research sought to (a) evaluate whether two adult students with MLDs demonstrated the characteristic error patterns documented in younger students with MLDs (Mazzocco et al., 2013), (b) determine what persistent understandings were associated with these errors, and (c) determine if the persistent understandings provided a plausible explanatory frame for the comparison errors in addition to errors on non-comparison problems.

An analysis of Lisa and Emily's cases revealed that they both demonstrated difficulties on comparison problems with the same denominator and comparisons with the fraction 1/2. These errors persisted throughout the tutoring sessions and were evident at the time of the posttest. These findings replicate and extend those of Mazzocco et al. (2013) for two adult students with MLDs and suggest that students with MLDs may continue to struggle with these kinds of comparison problems into adulthood.

The detailed diagnostic analysis of the cases revealed that both Lisa and Emily's comparison errors were associated with two persistent understandings, which reoccurred and were in conflict with the canonical mathematical understandings. The fractional complement understanding—a tendency to attend to the fractional complement rather than the fractional quantity—was evident in Lisa and Emily's attempts to solve same denominator problems. Unlike most students who draw upon their whole number reasoning (Sophian, 2000), Lisa and Emily attended to the fractional complement (e.g., comparing three pieces for 5/8 to six pieces for 2/8), which often resulted in them incorrectly determining that the smaller fraction was larger. Similarly, the students' single factor understanding appeared to cause difficulties for comparisons involving the fraction 1/2. Both students judged comparison based on a single value (often the denominator) rather than the coordination of the numerator and denominator. Although most students develop an intuitive understanding of the fraction 1/2 from a young age (Hunting & Davis, 1991), Lisa and Emily's representation and interpretation of the fraction 1/2 suggested that they understood this fraction as a halving *action* rather than a quantity. Understanding the fraction 1/2 as a halving action may suggest why they were not able to reason about comparisons to the quantity 1/2 in productive ways. These persistent understandings provide a plausible explanation for why both students struggled with same denominator and one-half comparisons. Therefore, these operational definitions provide a reasonable explanatory frame to understand why these two students with MLDs would continue to make these kinds of comparison errors on the easiest comparison problems.

By building upon both quantitative and qualitative research, this study provides a nuanced understanding of MLDs for two students and illustrates the utility of this approach for designing interventions. This study suggests that the fraction comparison error patterns are but one marker that reflects an atypical orientation to representations of quantity. Placing the comparison errors within the context of the larger case, we see that although the fractional complement, single factor, and halving understandings can provide an explanation for these error patterns, they also result in errors for other problem types. The case studies of Lisa and Emily provide an elaborated view of MLDs in which a variety of error patterns can be understood to be subsumed within a larger category related directly to the student's understanding of fractional quantity.

Situating these findings within the Vygotskian theory of learning, which framed the analysis, raises issues to consider in our understanding of MLDs. Representations of fractional quantity (i.e., standard mediational forms) did not serve the same purpose for Lisa and Emily as they do for most students. Both students' attention to the fractional complement and their representation of 1/2 as a halving action indicate that these representations are not serving as a stable representation of fractional quantity. Consequently, these representations should be thought of as at least partially inaccessible. Furthermore, it is not simply that Lisa and Emily had less skill in fraction comparison; instead, they relied upon qualitatively different resources (fractional complement, single factor, and halving understandings). Therefore, it is of utmost importance that studies of MLDs go beyond documenting what students lack and begin to document what resources students have. Identifying the resources that students draw upon when solving problems can explain why error patterns emerge and suggest avenues for intervention. Situating this work within a Vygotskian theoretical framework is beneficial because it enables an analysis of the qualitative differences displayed by the students rather than focusing exclusively on documenting deficits. Additionally, this theoretical approach suggests that mediational tools, like mathematical representations, are central to understanding the ways in which a disability manifests. It frames difficulties experienced by the students as issues of access and therefore directly connects to implications for instruction.

Implications for Practice

Two related implications for practice can be derived from this study. First, fraction comparison problems can be used as targeted evaluation of student's understanding of fractional quantity. Studies of MLDs have routinely used comparison measurements of whole and approximate quantities to evaluate students' numerical number processing (Price & Ansari, 2013). This study, along with Mazzocco et al. (2013), suggests the diagnostic utility of same denominator and one-half comparison problems to assess students' understanding of fractional quantity. Second, interventions for students who demonstrate these errors should focus on building a foundational understanding of fractional quantity. Particular attention should be given to how students use and explain various mathematical representations in an intervention. For Lisa and Emily, standard representations did not hold the same meaning. Interventions may need to rely upon more experiential ways of representing fractional quantities (e.g., weight or length) to help students make sense of fractions.

Limitations and Future Directions

Several limitations of the present study should be acknowledged. First, this study focused on two students with MLD. Because of this, the particular findings cannot be generalized as one would in a large-scale quantitative study. Case study research enables investigation of how or why particular error patterns emerged for the case study students. It demonstrates that the errors made on same denominator and one-half comparison problems were not isolated but rather related to larger patterns of student thinking. That said, it is unknown if the patterns of understanding evident in Lisa's and Emily's cases are unique. Future research should investigate whether these persistent understandings are evident in larger samples of students with MLDs, particularly given the similarity between Lisa's and Emily's persistent understandings. Second, because the tutoring sessions did not exclusively focus on comparison problems, fewer same denominator and one-half comparison problems were assessed than in Mazzocco et al. (2013). Finally, the persistent understandings that emerged from this analysis were tied to the specific instructional sequence used in this study. It remains an open question whether different explanations for the fraction comparison errors would emerge in a different instructional environment (with different tools and representations). Future research should explore these and other characteristic patterns in a variety of instructional contexts.

Conclusion

This study explored how two students' difficulties with fraction comparison problems were related to how they understood and made sense of fractional quantity more generally. An analysis of the students' persistent understandings provided a more nuanced perspective illuminating why these kinds of unique error patterns may occur and persist for students with MLDs (Mazzocco et al., 2013). As in these cases, it may be that an atypical understanding of fractional quantity underlies these errors more generally and suggests that interventions for students with MLDs should focus on addressing students' underlying conception of quantity. By building upon quantitative and qualitative studies, we can ask different kinds of questions and gain different insight into MLDs.

Appendix

Operational Definitions of Persistent Understandings for Lisa and Emily.

Lisa	Emily
Fractional complement	
Problems were coded as indicative of a fractional complement understanding if Lisa (1) used the words "take," "gone," or "missing" (or any derivation) to refer to the numerator quantity, (2) used "left" to refer to the fractional complement, (3) was gesturing or referring to the fractional complement (represented by the non-shaded area model region or missing fraction pieces) as the focal fractional quantity, (4) used shading to represent the removal of pieces, or (5) interpreted the fractional quantity as the fractional complement.	Problems were coded as indicative of a fractional complement understanding if Emily (1) interpreted a fractional representation based on the part composed of fewer pieces. (2) interpreted the fraction as the fractional complement (corresponding to the non-shaded or missing pieces), irrespective of whether it was the smaller of the two quantities.
Single factor	
Problems were coded as indicative of a single factor understanding if Lisa (1) judged the size of the fraction based solely on the denominator using an inverse relationship (e.g., 1/3 is bigger than 4/5 because thirds are bigger than fifths), (2) judged the size of the fraction solely on the numerator using an inverse relationship (e.g., 3/5 is smaller than 2/5), (3) asserted that 1/2 was the largest fraction, or (4) represented the unit fraction separate from the numerator value (e.g., 5/10 represented as $5\frac{1}{10}$).	Problems were coded as indicative of a single factor understanding if Emily (1) focused on the number of pieces in the whole, rather than on the size of the pieces, particularly when size was the relevant dimension, (2) asserted that the larger denominator had the larger size pieces, (3) asserted that the fraction with the larger denominator was the larger fractional value, or (4) focused on the number of pieces without referencing the size of the pieces in the context of equivalent fractions. In addition, problems in which Emily asserted that the fraction with the larger *numerator* was the smaller fraction (e.g. 3/5 is smaller than 2/5 because 3 is larger than 2) were also included.
Halving	
Problems were coded as indicative of a halving understanding if Lisa (1) justified her answer by focusing on the balance and similarity between the two quantities (part–part understanding) rather than focusing on the one part out of the total number of parts (part–whole understanding), (2) represented 1/2 by drawing a shape, partitioning it in two, and omitting the shading, or (3) used gestures and gave explanations consistent with 1/2 as a splitting action rather than a quantity.	Problems were coded as indicative of a halving understanding if Emily (1) understood the partitioning of a shape to represent that unit fraction quantity (e.g., partitioning into 2 is a representation of 1/2, partitioning into 3 is a representation of 1/3, etc.) or (2) used gestures and gave explanations consistent with 1/2 as a splitting action rather than a quantity.

Note: Adapted with permission from Journal for Research in Mathematics Education, copyright 2014, by the National Council of Teachers of Mathematics.

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