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The Didactical Contract Surrounding CAS when Changing Teachers in the Classroom

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Abstract

The article discusses three empirical examples of Computer Algebra System (CAS) use in a Danish upper secondary school mathematics class that had experienced a recent change of teacher. All examples lead to didactical problems surrounding the situation and unclear expectations between teacher and students, involving loss of students' mathematical skills and confidence, loss of global mathematical perspective, and the students losing sight of the mathematical objects in question. The article is the result of collaboration between two mathematics education researchers and an upper secondary school mathematics teacher, who experienced severe difficulties when taking over a class from another teacher. CAS was experienced as a crucial part of and reason for these difficulties. As a means for investigating the potential reasons behind the difficulties, a selection of constructs from the Theory of Didactical Situations (TDS) is applied. In particular, it is observed that unclear contractual relations about the role of CAS bring with them misguided winning strategies and metacognitive shifts, eventually causing the students to 'lose the game'.

Keywords: CAS, ICT, technology, didactical contract, winning strategies, the derivative

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El Contrato Didáctico Alrededor de las CAS Cuando se Cambia el Profesorado en el Aula

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Resumen

Este artículo discute tres casos empíricos en los que se usan *Computer Algebra System* (CAS) en una clase de matemáticas de un instituto danés que ha experimentado un cambio reciente de profesorado. Todos los ejemplos conducen a problemas didácticos que están alrededor de la situación, así como expectativas no claras entre docente y estudiantes, involucrando pérdida de las habilidades matemáticas de los estudiantes y de su confianza, pérdida de la perspectiva matemática global, y pérdida de vista de los objetos matemáticos por parte de los estudiantes. El artículo es resultado de la colaboración entre dos investigadores en didáctica de la matemática y un profesor de instituto que ha experimentado dificultades severas cuando ha retomado la clase que empezó otro docente. El uso de las CAS ha sido una parte crucial de estas dificultades. Como forma de investigar las razones potenciales detrás de dichas dificultades, se ha seleccionado un conjunto de constructos procedentes de la Teoría de Situaciones Didácticas (TDS). En concreto, se observa que relaciones contractuales no claras del uso de las CAS conllevan las estrategias ganadores y los cambios meta cognitivos equivocados, causando que los estudiantes "pierdan el juego" con el tiempo.

Palabras clave: CAS, TIC, tecnología, contrato didáctico, estrategias ganadoras, derivada.

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At a Danish conference on ICT in mathematics education, held in Copenhagen in June 2014, Michèle Artigue gave a keynote presentation on successes and failures in relation to the use of digital technologies in mathematics education, looking broadly at the past three decades (Artigue, 2014). One of the observations made by Artigue was that in 1985, all mathematics educators seemed to agree that ICT would have a crucial role to play in the teaching and learning of mathematics, but not necessarily in the practice of the discipline of mathematics itself. Thirty years later, the picture appears almost reversed. Now, this is not to say that ICT has not had any impact on mathematics education at all, of course it has. But the impact was not as grand scale as originally foreseen; neither has it 'automatically' resolved students' difficulties of learning in mathematics nor given rise to a consistent and pedagogical reform of teaching practices in mathematics. Nevertheless, it has become difficult to imagine a professional in a mathematics-related discipline or carrier today carrying out his or her work without involving some kind of mathematics software, and therefore curricular ideas about 'mathematical competence' often involve developing abilities with such tools.

The fact that this picture is somehow reversed appears to give rise to new didactical problems, which we are experiencing in the mathematics programs of the Danish educational systems at the moment, not least due to a use of strong mathematical tools introduced without consistent pedagogical intentions. At times the implementation of such tools can be more or less straightforward, especially in cases where the mathematical software is developed with the purpose of supporting the learning of mathematics. However, the use of software which has originally been developed for professional use (e.g. *Maple*), or even with the purpose of easing the work of the mathematics student (e.g. *Photomath* or *Cossincalc*), may have unforeseen didactical consequences when applied in a teaching and learning situation, where the mathematics it is to operate on is not yet conceived by the students. For example, Jankvist and Misfeldt (2015) argue that we may sometimes even talk about "CAS-induced difficulties in learning mathematics", and illustrate this by means of the 'desolve' command and students' conception of differential equations.

In this article, we discuss three examples from a Danish upper secondary school mathematics class, all illustrating didactical problems related to change of teacher in a CAS heavy environment. Before we describe and analyze the examples, we will account for the educational setting in which the situations took place, i.e. the role of CAS in the Danish upper secondary mathematics program as well as the circumstances around the particular upper secondary class. Also, a brief introduction to the theoretical constructs, which we have chosen to apply in our analysis of the examples, shall be given. We rely on constructs from Brousseau's Theory of Didactical Situations (TDS), in particular those of strategy, fundamental situations, didactical contract, and the didactical milieu.

Educational Setting

With a new reform of 2005, CAS found its way into the everyday mathematics teaching in the Danish upper secondary schools[1]. In the ministerial orders for mathematics, it says:

> The program is organized so that calculators, computing and mathematics programs are essential tools in the students' learning and problem solving. The organization includes training in the use of these devices: to perform calculations; for symbolic manipulation of formal expressions; for handling statistical data used to gain an overview of graphs; for equation solving; and for symbolic differentiation and integration. Further, it includes the use of calculators, computing- and mathematics-programs in the organization of the experimental approach to topics and problem solving. (UVM, 2013, our translation from Danish)

The ministerial orders also include a list of "didactical principles", one entry which relates specifically to CAS: "CAS tools should not only be used to perform the more complex symbolic computations, but also to support learning of skills and mathematical concept formation" (UVM, 2013, our translation from Danish).

Unfortunately, the ambitions use of CAS meets difficulties in the everyday teaching practice in Danish upper secondary school. How CAS is used differs greatly from teacher to teacher, from school to school and from textbook system to textbook system. Of course, the eventual assessment (Trouche et al., 2013), i.e. in this case the written exam, is very influential on the use of CAS in the teaching. Danish upper secondary school is three years, and students may take mathematics at one of three levels (C, B or A), depending on the number of years they take it, e.g. if a student takes mathematics every year for all three years, the student will have A-level. The class of students we shall consider in this article followed mathematics at B-level. In terms of assessment, this involves a four-hour written exam that falls in two parts. During the first hour, no aids besides pencil and paper are allowed. The following three hours all aids are allowed (except use of Internet and any communication with the outside world). During this part, students are assumed to have access to and be familiar with a CAS tool – in fact it is difficult to complete the test within the timeframe without some use of CAS.

As mentioned, teachers and schools often have different policies in regard to the introduction of CAS. For example, in some Danish upper secondary schools CAS is not introduced until after Christmas in the first year, i.e. the first semester follows a more traditional paper-and-pencil approach. The particular upper secondary class, from which the examples in this article are taken, is one that was introduced to CAS from day one of first semester. In their second year of upper secondary school, the class had a change of teacher; their regular teacher went on paternity leave, and the class was assigned a temporary, but experienced, teacher during this period (the third author of this article). The regular teacher had asked the temporary teacher to revise what the students knew about the topic of differential calculus. Hence, prior to the teacher change the students had been introduced to the concept of the derivative, and they were also somewhat familiar with the related commands and operations of CAS, which the temporary teacher took as the starting point of the asked for review. Now, this particular teacher is a so-called ‘maths counsellor’ (cf. Jankvist & Niss, 2015), who next to performing regular teaching also assists students who are found to have specific mathematics-related learning difficulties, misconceptions, impediments, etc., meaning that the teacher possesses extensive didactical and pedagogical mathematics education related insights. Nevertheless, upon encountering this particular class of students, the teacher ‘sounded the alarm’ by asking us as mathematics education researchers (first and second authors of this article) to sit in on a handful of lessons to see what was going on with these particular students, their mathematics understanding, and their use of CAS.

Hence, the analysis presented in this article addresses how CAS use can become both a catalyst and an indicator of the didactical problems related to changing teacher. Of course, changing teachers and the resulting variation in classroom rules and values are broad problems with many potential resulting difficulties. But in the case presented here, CAS is a focal point for these difficulties, which allows us to zoom in directly on the interplay between CAS, teacher-student expectations and norms, and students' mathematical competence and conceptual development.

Approach and Method

The first and second author agreed to discuss the experienced problem with the third author and follow his class over some time in order to help understand and hopefully overcome the problems. In essence, the third author explained that in one way or another his teaching did not work with this particular class, and that he believed the problems he experienced to somehow be related to the use of CAS technology by his students, both now and with the previous teacher. Hence, the first and second author were invited to visit the teacher's class and try to make meaning of the problem experienced by the temporary teacher. We all agreed to seek for answers to the experienced problems in the students' and teacher's use of CAS, but we also agreed to keep an open mind in our possible 'diagnosis'. In total, we observed the class on three different occasions (described below), and had a handful of meetings with the teacher besides this. Also, we collected lesson plans, teacher notes, and student assignments. Finally, we interviewed a few students about their assignments.

Following these meetings and observations, we decided to report our discussions and analyses of the situation(s) in this joint article. Hence, the research reported builds on a tradition of teacher-researcher collaboration (Jaworski, 2005) and collaborative action research (Raymond & Leinenbach, 2000), understood as a collaborative inquiry process, where we simultaneously investigate and discuss the situation with this specific class and aim at developing knowledge about how students work with CAS along the way. In that sense, the joint inquiry can be seen as an answer to the problems experienced by the third author of the article, as well as an investigation of a negotiated research question about students' use of CAS.

Raymond and Leinenbach (2000, p. 284) describe collaborative action research as "a medium for teachers to systematically look at the problems they face in their classrooms in an effort to find practical solutions" often in collaboration with educational researchers or other domain experts. In their joint investigation the teacher takes initiative to contact the researcher for help on certain challenges introduced by a new curriculum program. We similarly depart in the third author's challenges with understanding why his teaching did not work as usual, and investigate the question of the students' use of CAS with constructs from the Theory of Didactical Situations, since this framework offers a broad description of the teaching situation.

Theoretical Constructs from TDS

Brousseau's (1997) *Theory of Didactical Situations* (TDS) is a mathematics education theory that can be seen as an answer to a number of empirically observed problems and phenomena. In the theory of didactical situations students construct knowledge as a result of interaction with the didactical environment (or *milieu*). This environment is set up and governed by the teacher, and in that sense teaching and learning are only indirectly connected following a constructivist conception of learning: "The student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria, rather as human society does. This knowledge, the result of the student's adaptation, manifests itself by new responses which provide evidence for learning" (Brousseau, 1997, p. 30).

Students' interaction with the environment can be guided by the teacher or the teacher's expectations (such situations are described as *didactical*) and they can be a result of the students' genuine interest in the mathematics proposed by the environment (referred to as *adidactical*). The teachers' expectations and classroom norms are described as a result of a *didactical contract* between teacher and students. This contract is often implicit, although necessary and it guides the actors' behavior and mutual expectations:

> The DC is the set of reciprocal obligations and 'sanctions' which [1] each partner in the didactical situation imposes or believes to have imposed with respect to the knowledge in question, explicitly or implicitly, on the other; [2] or are imposed, or believed by each partner to have been imposed on them with respect to the knowledge in question. The DC is the result of an often implicit "negotiation" of the mode of establishing the relationships for a student or group of students, a certain educational environment and an educational system. (Education Committee of EMS, 2012, p. 54)

The didactical contract becomes most apparent when it is broken, and the teacher-student expectations are no longer at the center of the educational activities. Some ways of breaking the contract are necessary for good teaching to occur; mainly students should once in a while break the contract and start investigating the mathematics they work with, without considering the teacher's expectations.

The didactical environment is connected to mathematical knowledge and to students' learning through *fundamental situations.* It is an assumption in TDS that for every piece of important mathematical knowledge there exists a *fundamental situation* resembling this knowledge in a student activity or task. Brousseau uses the metaphor of a *game* to describe these activities. The 'game' should be designed in a way so that the winning strategy implies that you have constructed the intended knowledge.

Brousseau departs in a number of unintended but typical problems in teaching situations, namely the Topaze effect, the Jourdain effect, the metacognitive shift, the improper use of analogy, and the changing of teaching situations. In relation to our analysis the most relevant effect is the metacognitive shift, where the topic of the mathematics teaching changes away from considering the mathematical objects towards something else (e.g. specific procedures). Brousseau describes metacognitive shift as when a teacher (or educational system) "take her own formulations and heuristic means as object of study rather than genuine mathematical knowledge" (Brousseau, 1997, p. 26). This phenomenon is almost unavoidable "as long as the teacher is unable to withdraw herself from the obligation to teach at all costs" (Brousseau, 1997, p. 27). Revealing and correcting metacognitive shifts require reflection and conscious action by the teacher.

We use the framework of TDS to describe how different values, practices and assumptions around the use of CAS among other change the winning strategies in didactical situations, which results in the need to rethink the relation between environment and winning strategy and hence also affect the constructed knowledge.

Example 1: The Product Rule for Differentiation

The first example is taken from a lesson in which the temporary teacher had planned first to go through a collection of exercises from an assignment, which the students had handed in previously, and then carry on with the day's homework. The topic of the hand-in exercises was the derivative and the differentiation rules. In particular one exercise involving the product rule appeared to have caused the students some difficulties. In this exercise, which came from the textbook, the students were asked to find $f'(x_0)$ for six different functions, numbered $f_1(x)$, $f_2(x)$,…, $f_6(x)$. The fifth of these, $f_5\,(x) = (\sqrt{x}+3)(\sqrt{x}-3)$ for $x \in [0;\infty[$, had appeared particularly troublesome, so the teacher had the students do it again in class. A point of the teacher was to tell the students how a problem may be solved in several different ways, and that doing so may also be a way to check one's result. More precisely, the teacher had in mind to first use the product rule to find $f_5'(x_0)$, and then next have the students observe that in this case it is easier to multiply the two factors in the original expression to get $f_5(x) = x - 9$, and use this to find $f_5'(x_0) = 1$.

At the time of the lesson, the students had been introduced to defining a function in CAS and having CAS differentiate it, e.g. defining $f_5(x)$ and let CAS differentiate this in one go. However, none of the students had done so in their hand-in assignment. The students' main use of CAS consisted in typing up their solutions, and handing them in electronically. The students, who had applied CAS, had generally used it to differentiate the individual functions (factors), say f and g and then applied the product rule itself, i.e. $f' \cdot g + f \cdot g'$. Most students had left this expression, i.e. $1/\left(2\sqrt{x}\right) \cdot \left(\sqrt{x}+3\right) + \left(\sqrt{x}-3\right) \cdot 1/\left(2\sqrt{x}\right)$ as a result, and hence not found $f_5'(x_0) = 1$. Only one student had realized the possibility of multiplying the expressions of the two functions, and then differentiated the resulting expression. No students had commented on the fact that a use of the product rule requires an argument for the involved functions being differentiable in x_0.

The teacher began the lesson by writing up $f_5(x)$ and defining the first factor $(\sqrt{x}+3)$ as f , and the second $(\sqrt{x}-3)$ as g. Already at this time hands were raised in the classroom, and questions were asked as to from where the '*5*' in $f_5(x)$ came, and how there could be two f's in the teacher's expression. In fact, some of the students found it difficult to apply the product rule, since the function they had to apply in on was also named f, i.e. f_5. The teacher referred to the formulation of the exercise in the book, explained that the '*5*' was just a numbering, and then rewrote the product rule to $f' = h' \cdot g + h \cdot g'$, and renamed the first factor of $f_5(x)$, i.e. $(\sqrt{x}+3)$, to h. The students were then asked to solve the problem using the product rule, and to do so without applying CAS.

Most of the students were able to arrive at the expression $f_5'(x_0) = 1/(2\sqrt{x_0}) \cdot (\sqrt{x_0} - 3) + (\sqrt{x_0} + 3) \cdot 1/(2\sqrt{x_0})$, but when having to reduce this it became more difficult. Several students were unsure whether expressions had to go over or under the fraction line when multiplying, and when doing the actual reductions some students, for example, did not realize that $\sqrt{x_0}/\sqrt{x_0} = 1$. However, after some 10 minutes, most students had arrived at the answer $f_5'(x_0) = 1$.

Next, the teacher asked the students to reduce the expression first, i.e. multiply the parentheses, and then differentiate the resulting expression. During this part of the exercise, our attention was drawn to a student who had actually done okay during the first part of the exercise. When having to multiply $\sqrt{x}$ with $\sqrt{x}$, she wrote down $(\sqrt{x})^2$, she realized that the terms $3\sqrt{x}$ and $-3\sqrt{x}$ cancelled out, and she ended up with the expression $(\sqrt{x})^2 - 9$. Now, having to differentiate this, she regarded $(\sqrt{x})^2$ as a composite function, thus needing to apply the chain rule for differentiation. Of course, the student eventually came to the correct result. But to say the least we were quite astonished at what we witnessed, and slowly began to grasp the claim of the teacher that "something else was going on here".

The original purpose of the teacher with the particular session, i.e. to have the students realize that there were different paths to solving the problem, and that taking different paths could be a way of checking one's result, somehow drowned in the algebraic difficulties which the students had while having to do the problem with 'paper and pencil'. The discussion of the hand-in assignment ended up taking most of the lesson, and hence did not leave enough time for going through the actual homework for the lesson. Eventually, the teacher asked the students to begin looking at the next hand-in assignment. Here the students here were to use CAS to differentiate a function given in the variable *t*. One interesting, although probably common, observation was that several students could not get CAS to work properly due to the simple fact that they were trying to differentiate the function with respect to *x*.

Yet a comment should be made in regard to the observation of this particular lesson. Even though the intention of the teacher was to have the students find the derivative of $f_5(x)$ by paper and pencil, practically none of the students actually used paper and pencil. Everything was written in their computer's CAS program (*TI Nspire*) – even though the students were asked to do all reductions without using the CAS commands. That is to say, in this respect CAS merely played the role of text processing program for the students (Iversen, 2014).

Subanalysis 1: Loss of Mathematical Skills and Confidence

We understand the above situation as a matter of losing a clear conception of mathematical, and in particular algebraic, skills (Jankvist & Misfeldt, 2015). A weakening of classical algebraic skills is a relative simple consequence of using CAS, since the automatization of algebraic skills of course leads to weakening skills in a paper and pencil domain. But the students might also experience a lack of acknowledgement of their CAS-based skills and work practices. The consequences of such weakened skills are in this case that the students lose the ability, not only to come up with different solution strategies, but also to follow the teacher's suggestion of different paths to the solution. Furthermore, the lack of basic skills in some cases makes the students' mathematical work very complicated, as the student applying the chain rule to the function $(\sqrt{x})^2$ illustrates. From one perspective, the strong dependence on CAS for algebraic tasks deprives the students of relevant strategies and approaches towards the tasks. Hence, the students end up applying overwhelmingly complicated strategies towards the tasks provided in the didactical environment. From another perspective, the teacher frames the situation in a way where the students' lack of algebraic skills becomes crucial, and without acknowledging and building on the students' abilities with CAS. The result is a tendency among the students to apply rules without reasoning, in the sense that the students apply the product rule and the chain rule without first considering the algebraic context, which they are situated in. This is understandable because the didactical contract can give rise to a mutual belief that recently taught 'rules' should be used in the solution of a following task – and thus it becomes a matter of considering the didactical contract over insights from fundamental algebraic skills such as $(\sqrt{x})^2 = x$.

Hence, the loss of algebraic skills does not only influence learning of mathematics (as described in Jankvist and Misfeldt, 2015), but it also changes the strategy which students apply towards the tasks in the environment, and may lead to stronger contractual dependency. Of course, it is an interesting, and somehow 'unnatural', aspect of the situation in this first example that the teacher asks the students not to use CAS when differentiating the function. This demand from the teacher might explain that the students are focused on contractual relations and weakened in their mathematical confidence. This can at least partly explain that the students seem to apply rules without reason. We shall come back to such contractual relations later in our analysis of the second example.

Example 2: Optimization by Differentiation

In the next example, which took place three weeks after the first, the students were to work on optimization problems in relation to finding maximum and minimum of functions using the concept of the derivative. More precisely, the students were given a worksheet with four problems of varying difficulty. The first of these was an almost exact copy of an example from the textbook, and made up the basis for the teacher's exposition to the class. The second problem was also an example from the textbook. The third and fourth problems were more difficult ones, but since none of the students ever got to these, we shall not discuss them here.

The point of departure for the teacher's exposition was the often-used example of how differential calculus may be applied to dimension a cylinder-shaped barrel or can with a given volume minimizing the surface area. The teacher went over the example in the textbook, which read: "In industry you might be interested in producing a cylinder-shaped aluminum can with a bottom and a top, which should contain 1 liter of fluid (oil, soup, etc.). The can should be produced from the least amount of material, i.e. its total surface area should be as small as possible." (Carstensen et al., 2013, p. 130, our translation from Danish).

The teacher began by drawing a cylinder with radius r and height h on the electronic whiteboard. After some discussion with the students, an expression for the total surface area was agreed upon: $A = 2\pi r^2 + 2\pi rh$. The requirement for the volume to be 1 liter, translated to 1000 cm^3, gave that $V = 1000 = \pi r^2 h$, i.e. $h = 1000/(\pi r^2)$. By means of discussion, the students and teacher arrived at the following expression for the surface area as a function of the radius: $A(r) = 2\pi r^2 + (2000/r)$, (see figure 1 for a screenshot of the whiteboard). Next, $A'(r) = 4\pi r - (2000/r^2)$ was found, $A'(r) = 0$ was solved and resulted in $r^3 = 2000/4\pi$, which again led to

$r = 5.42$. The last thing needed was to argue that this value of r was in fact a minimum (again, see figure 1).

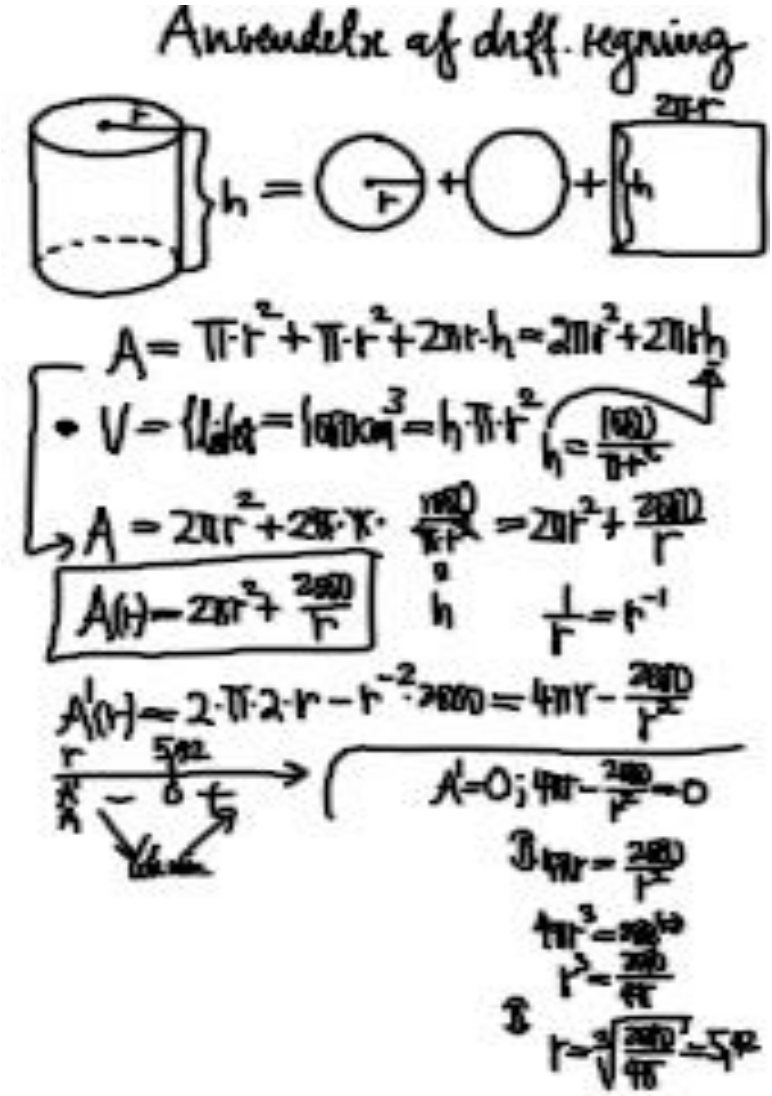

Figure 1. Screenshot of teacher's presentation on the whiteboard. The heading reads "Application of differential calculus"

Next, the agenda was on the first task of the worksheet. This task was an almost exact copy of the example above; only V was now set to 400 cm^3. The students were asked to do this task by paper and pencil, not using CAS. Several students asked for the screenshot of the whiteboard, which the teacher then gave them via the intranet. The students' reason for this, however, was a bit surprising. While observing in the classroom, we saw students who opened the screenshot in *MS Paint*, erased the values that had to do with the 1000 cm^3, and drew in new values based on the 400 cm^3. One common mistake, which this resulted in, was replacing the value 2000 in the expression for r^3 by 400, instead of the correct 800.

The second task of the worksheet concerned the making of an open box. More precisely, given a rectangular sheet of metal, 50 cm times 80 cm, equal sized squares, x^2, had to be removed in the corners (see students' drawing on figure 2). Now, the size of the squares, i.e. x, had to be decided so that the volume of the box be maximized. Since the height of the box will be x, the expression which one should arrive at is $V(x) = x(50 - 2x)(80 - 2x) = 4(x^3 - 65x^2 + 1000)$, which then has to be differentiated and solved for $V'(x) = 0$, leading to two potential solutions, one which has to be rejected as a maximum (see figure 2), eventually leading to a conclusion of $x = 10$.

Now, for this task, the students were told that they could use CAS all they wanted to. For instance, it could then make sense to use CAS to draw up the graph for $V(x)$, and then by mere inspection find the maximum value of x, or by using the min-max facility of CAS. However, none of the students did this. What the students typically did was to write up the expression for $V(x)$, then either attempt to multiply things by hand, or use CAS, to find the expression $V(x) = 4x^3 - 260x^2 + 4000x$. Next, they used CAS to find $V'(x)$. Realizing that setting this equal to 0 would lead to a second-degree equation, some attempted to solve this by hand, while others used CAS. And then they did 'paper and pencil' reasoning to decide which of the solutions led to a maximum (a la figure 2). In a sense, the students' use of CAS was limited to that of a sophisticated calculator. It is remarkable to notice that the students' primary use of CAS, in the differentiation of $V(x)$, could have been carried out much quicker by hand, while their paper and pencil reasoning could have been supported and carried out much quicker by means of CAS. In a sense, the picture is reversed as to what one would ideally expect. (Yet an observation concerned the fact no students relied on their textbook while working on these tasks, which would have made sense, since both tasks actually were examples in the textbook.)

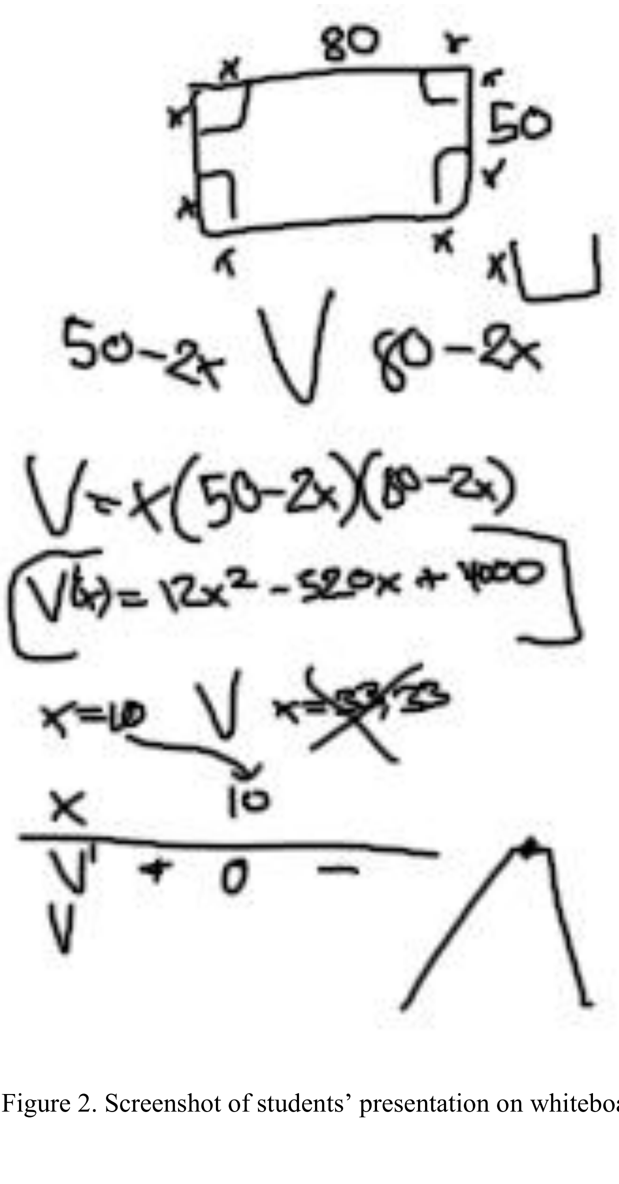

Figure 2. Screenshot of students' presentation on whiteboard

Subanalysis 2: Loss of Global Mathematical Perspective

The first part of the example shows us that several of the students in this case did adopt a repetitive approach to learning. Bringing in the picture from the electronic whiteboard and manipulating the teacher’s worked example might be a good approach to learning in some situations, but here it illustrates that some of the students are rather dependent on the teacher’s approach in their own work with the example. In that sense, the students’ work with this example is didactical – as opposed to adidactical – and under strong influence of the didactical contract. This insight also helps us to make meaning of the students’ approach to CAS in the case.

The students’ solution presented on the whiteboard was heavily dependent on CAS, but in a rather rudimentary way. It seems like a natural choice for the students to address this problem by investigating a graph of the resulting function. If we seek reasons for not doing that, it is obvious that the didactical contract and the didactical nature of the students’ working situation provide some explanation. The teacher had shown a procedure-based paper-and-pencil approach to working with the optimization task. This approach is taken very directly by the students as a way of addressing the task. However, the students at the whiteboard (figure 2), well-aware of their weak paper-and-pencil skills, choose to address all the algebraic manipulations by means of CAS, and then perform the same procedures as the teacher did, distributing all algebra to CAS, not taking into consideration if the tool can do this quicker, easier, or more elegantly.

The first task offers the students a didactically controlled situation, where the winning strategy was clearly devolved by the teacher, i.e. the students had to solve a task similar to that just reviewed on the whiteboard, and they were not to use CAS while doing so. However, in the second task the situation is didactically uncontrolled, although it appears that the students are given a clear instruction, i.e. that they may use CAS in any way they find appropriate. The reason is that it is unclear to the students what the winning strategy then becomes. For instance, is it an accepted winning strategy to just plot the function and find the minimum by mere inspection? Apparently, the students do not believe so – although they are familiar with the required CAS commands. Instead, they apply the paper-and-pencil strategy of the first task, involving only symbolic calculator functions of CAS. As we shall see in the third example, students regard this to be within the scope of the didactical contract (cf. student quote later). But in reality, the second task offers them an opportunity of breaking the didactical contract – an opportunity greatly missed. In this sense the students' loss of sight of the winning strategy leads to yet a loss; one of global mathematical perspective.

Example 3: Conditions of Monotonicity of a Function

The third example is taken from a lesson two weeks after the lesson of example 2. After the lesson we had the opportunity to interview a few of the students individually about their solution to the weekly assignment. One of the tasks in the given assignment was: "A function f is given by: $f(x) = x^3 - 3x^2 + 4$. Find f', and account for the conditions of monotonicity of f." The students had previously been introduced to such typical tasks – also before the time of the second example. In particular, the temporary teacher had provided the students with a 7-step 'recipe' for tackling such tasks, which was given to them again as part of the assignment:

1. Define the given function f in *TI Nspire*.
2. Find the derivative $f'(x)$.
3. Solve the equation $f'(x) = 0$, using *TI Nspire*.
4. Draw a 'line of monotonicity'.
5. Plot in the local points of extrema on the 'line of monotonicity'.
6. Find the operational sign for f' on each side of the potentially found local points of extrema.
7. Account for the conditions of monotonicity of the function.

Taking $f(x) = x^3 - 3x^2 + 4$ and walking through the recipe, we would in step 1 define the function in CAS. In step 2, we find the derived function using CAS to $f'(x) = 3x^2 - 6x$. In step 3, also using CAS, we solve $f'(x) = 0$, i.e. $3x^2 - 6x = 0$, providing $x_1 = 0$ or $x_2 = 2$. In step 4 we draw the 'line of monotonicity', which is illustrated in figure 3. Step 5 consists in assigning the found values of x_1 and x_2 to the 'line of monotonicity' (cf. figure 3), i.e. the monotonicity intervals of the function. In step 6, we find the operational sign for the derivative on each side of the local points of extrema, e.g. by calculating values such as $f'(-1) = 9$, $f'(1) = -3$ and $f'(3) = 9$, resolving in operational signs $+$, $-$, $+$ (cf.

figure 3). Step 7 now consists in concluding that f is increasing in the intervals $]-\infty;0]$ and $[2;\infty[$, and decreasing in the interval $[0;2]$, i.e. f has a local maximum in $x_1 = 0$ and a local minimum in $x_2 = 2$.

Using the recipe, the students were to do three tasks on a handout, one of them being that above. However, none of the students had chosen to follow the steps of the recipe while attempting to do the tasks. For example, none of the students had defined the functions in CAS and used CAS to find the derivative. Further, the majority of the students had not concluded anything, i.e. had not done anything resembling step 7. On the handout, the students were also provided with an illustrative example of how to walk through the recipe. In this example, the monotonicity intervals were found for a given function.

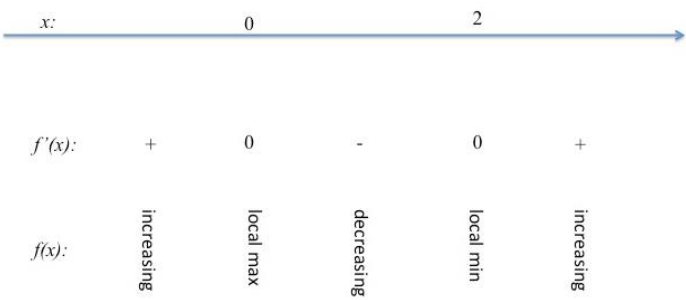

Figure 3. The 'line of monotonicity'. Steps 4 through 7 of the 'recipe' for $f(x) = x^3 - 3x^2 + 4$

The answers in the students' hand-ins were somewhat messy and offered a variety of conceptual misunderstandings. Examples are:

- Having found the two points of extrema to $x = 0$ and $x = 2$ in the above task, two students concluded that they now had found the "zeros of f".
- One student wrote: "We have now differentiated the function, after which we let it approach zero." (Something which teacher did not recall having ever seen before!)
- Several students did not distinguish between $f'(x)$ and $f'(x_0)$. And this despite the fact that a double-lesson had previously been spent on explaining this difference, i.e. that $f'(x)$ is the tangent function,

and $f'(x0)$ is the slope of the tangent of the function f in the point x_0, and that $f'(x_0)$ is a value.

In fact, only one student in the class approached the task following the recipe step by step. All other hand-in answers missed some of the steps and offered comments as the above.

As mentioned, we had the chance to confront a few of the students with the way they used CAS in the above example. During these interviews, we asked the students why they just did not plot the function instead of going through the lengthy process described above. One student answered: "You're not supposed to do that!" This of course refers explicitly to the teacher's expectations – and hence implicitly to the didactical contract – which includes the 7-step recipe. Other students similarly indicated that using plot commands was in a sense to "miss the purpose of the task".

Subanalysis 3: Loss of Mathematical Objects

The intended winning strategy from the teacher's perspective in regard to the task of the third example was that the students, by being forced through the seven steps of the recipe, would come to work with both the algebraic aspects –although here attempted distributed to CAS– and the graphic aspects of the derivative, and see how the different representations relate to each other. The idea was that they thus would come to understand something about the mathematical object of the derivative. To win the 'game' it is crucial that the students go through all seven steps – which they did not. If the students do not go through all steps, the intended winning strategy is lost, and with that the fundamental situation in the activity. But why do the students deviate from the laid out path? The reason for this appears, once again, to be related to contractual issues surrounding the use of CAS, and that these in themselves appear unclear to the students. As seen from the student quote above, the students realize that pure CAS-strategies, as for example plotting a function and finding extrema by inspection are to be considered outside the didactical contract. So, even though the students appear to be clear on the contract related to a pure paper-and-pencil approach, as we saw in the first task of the second example, the mixing of the two approaches, i.e. CAS and paper-and-pencil, somehow blur the contractual bounds for the students.

If we look at the seven steps of the recipe, the first three ask for a use of CAS, while the next four follow a traditional paper-and-pencil approach. The recipe itself is of course an adaption of an older recipe with six steps, where step 2 and 3 were carried out by hand and without step 1. Due to the introduction of CAS in upper secondary school, the old recipe was altered to include CAS in relevant steps, while attempting to still keep the mathematical object of the derivative in focus. For example, asking the students to account for monotonicity by means of the min and max commands in CAS or having them use a plot-and-inspect-strategy would clearly hide away the role of the derivative. Hence, from this perspective the recipe is meaningful. Still, from the students' perspective, the recipe may appear to fall between two stools, i.e. that of paper-and-pencil and that of CAS. As stated, the students neither define f in CAS, nor use their CAS program to find the derivative. Effectively, the students only use CAS in step 3 of the recipe. Having used CAS in step 3, several students then believe themselves done with the task, concluding peculiarities such as having found the "zeros of f" or having let f "approach zero" – illustrating that the role of the derivative in relation to investigations of monotonicity is rather unclear to these students. The non-understanding of other aspects of the derivative such as $f'(x)$ being the tangent function, $f'(x_0)$ the slope of f in x_0, etc. support this further.

In essence, for the students, focus is shifted away from the mathematical object of the derivative. Where the focus is redirected to is not necessarily unequivocal. For some students the focus is probably shifted to the recipe itself or to trying to remember the recipe – or at least some of its steps. For others, the approach appears to be shifted to the intended role of CAS procedures in the recipe. But whatever the shift, the effect is the same; focus is removed from the mathematical object of the derivative, i.e. a metacognitive shift has taken place. CAS, and more specifically the change in the role that CAS has played for the students in the classroom, has contributed to this metacognitive shift by blurring the overall picture of what the winning strategy is, and eventually removing the fundamental situation for the derivative from the activity of investigating the function.

Discussion: CAS and Varying Contracts

In this article we have described a number of problems, all related to the use of CAS in a specific upper secondary school class that had been subject to a change of teacher. The theory of didactical situations has allowed us to describe thoroughly how the situation with unclear and very changing expectations to the students' use of CAS gave rise to a loss of mathematical skills and confidence, a loss of global perspective and loss of existing fundamental situations, due to the introduction of new unintended winning strategies in the didactical environment. In order to cope with these problems, teaching with CAS in this specific class was extremely regulated by the didactical contract. Such strong regulation is not unproblematic, since it deprives students the ability to freely investigate and develop their mathematical skills and competences.

Despite the strong regulation, new solution strategies do emerge. However, these strategies –which in our case often are not very clear to the students– may also be characterized by losing their learning potential, because they cease to resemble the same fundamental situation as before. One example would be students who plot the graph of a function instead of using the 7-step recipe. Even though plotting is a well-functioning winning strategy to meet the environment, it is not a fundamental situation for the derivative anymore.

In our analysis, we see that this strong regulation of the didactical environment leads to metacognitive shifts away from investigating mathematics and towards investigating the rules and regulations that govern the classroom in general and the students' use of CAS in particular. As seen, one example of strong contractual control was revealed when we asked a student to describe to us why she did not use CAS to plot a specific function in her homework assignment instead of or prior to following the 7-step recipe for function investigation. She provided the obvious explanation: “You're not supposed to do that!” Hence, in this case the combination of a teacher promoting a relatively classical –not very CAS-dependent– approach to the solution of this type task, and students that are strongly dependent on CAS for algebraic manipulations, leads to a situation where the use of CAS is necessary for the students, but controlled by the didactical contract. Such a situation makes it difficult for the students to recede the didactical contract into the background and limits the development of adidactical situations. The example shows very clearly that the didactical contract and the teacher's expectations are constantly present when the students use CAS in the classroom.

The problems that we describe in the three cases are of course particular to this specific classroom and we suggest them to be strongly related to the recent change of teacher. However, the cases do, in our view, point to problems that transcend this specific classroom. Contractual regulations of tools and methods are always present in classrooms, and CAS use is one of the many parameters that does need regulations, and obviously change of teacher means that these regulations are re-negotiated. Nevertheless, in the Danish case there seems to be a systematic layer to the problem, because even though pre-service education has done very little in terms of preparing Danish upper secondary teachers to regulate CAS, neither textbooks nor written curriculum guide teachers in this work. Even though the ministerial orders explicitly state that computer algebra systems and other software technology should be used not only for solving problems but also for learning mathematics, they do not provide sufficiently detailed guidance about what this means and how it should be realized. In our analysis of problems in the specific class, the previous teacher's regulations and norms are important aspects of the problems experienced by the temporary teacher. The change in teacher and the associated contractual changes are very likely to have confused the students. Also, it is clear that the obvious difference in how the previous and the present teacher use CAS has weakened the authority of the temporary teacher and his norms and ideas, which is clearly exemplified concerning how to use CAS. In a broader perspective, it is interesting that the different approaches to and attitudes towards technology that may come with a change of teacher bring with them such strong problems in terms of learning. That different teachers have different attitudes towards technology is well established (e.g. Lavicza, 2010), but what seems to be the case in Danish upper secondary school is that different teachers' attitudes live side by side in an unnegotiated manner. These unnegotiated attitudes come into play in situations of teacher change – and these attitudes may very well be responsible for some of the problems described in this article.

Conclusion: Didactical Consequences of an Unclear CAS Contract

The three examples or situations which we have described and analyzed by means of TDS constructs in this article all involve confusion and unintended approaches from the students' side. We have attempted to provide an explanation for these observations with outset in the temporary teacher's difficulties with developing a consistent contractual approach to CAS that was aligned with his own approach to CAS and still acknowledged the students' competences and habits from their previous teaching. Hence the role of CAS in the didactical milieu of the cases described above is not entirely clear.

In the first example, the dependence on CAS causes students to 'forget' or lose basic mathematical skills, while the changed role of CAS (from central to peripheral in the students' mathematical skills) makes the students loose confidence in their own skills. In the second example, the unclear role of CAS causes the students to lose sight of global mathematical perspectives. In both examples, students appear to experience unclear and non-negotiated contractual relations, and in the second example the winning strategy of the didactical milieu becomes unclear to the students. The situation of losing sight of the winning strategy is further illustrated by example 3, where the students as a result of both contractual unclarities and a deviation from the milieu's intended winning strategy lose sight of the mathematical object under investigation. This leads to a metacognitive shift, where the students' focus is shifted away from the mathematical object to something else; either the recipe procedure or to a guessing of CAS' role in this. In both cases, the winning strategy is blurred and the fundamental situation of the mathematical object is no longer present in the milieu. In this sense, CAS becomes an unintended 'game' changer bringing with it the unintended mathematical behaviors of the students.

The strong effect of CAS on the didactical situations, as suggested by our analyses, may make it relevant to investigate specific didactical problems related to CAS. In that sense, we should consider CAS-based learning processes not just as a psychological problem related to learning, but also as a didactical problem related to teaching, and to the organization of teaching, teacher collaboration and teacher training. As shown in the analysis, the students' difficulties with the non-negotiated contractual relationships may very well stem from their teachers' equally unnegotiated attitudes towards the use of CAS. The analysis suggests that rather than teachers just including CAS ad hoc in their teaching, we need a joint strategy of 'accommodation'[2], where curriculum and textbook presentations are rethought so that CAS comes to play a natural and clear role in the teaching and learning of mathematics – for students as well as for teachers.

This of course involves a re-design of the milieu, and an identification of possibly new fundamental situations related to the mathematical objects. Further, it would involve making the didactical contract related to CAS use much more clear, which in turn must be expected to make the intended winning strategies more clear to the students, and hence also assist us in avoiding metacognitive shifts as the ones observed and described in this article. To put it bluntly, it is a matter of redefining the 'game' – and maybe not only the 'game', but also the means for playing it, i.e. the CAS tools. That is to say, maybe it is not only a matter of 'accommodating' the teaching strategies for using CAS, but also the CAS tool itself. As mentioned in the beginning of the article, several CAS packages applied in educational settings are not initially developed for education, but rather for professional use. And some of the packages that are targeted education are not necessarily the result of didactical development.[3] A joint development of curriculum, resources (such as textbooks) and tools are needed. Hopefully this could lead to a reflected use of CAS in the CAS heavy environment that both teachers and students are required to function within.

Notes

[1] CAS has been sporadically used in Danish upper secondary school since the 1990s, but then it was restricted in relation to exams and its pedagogical use was optional. With the 2005 reform, the use of CAS in practice became mandatory, and the written exam presupposes CAS use.

[2] The idea of a strategy of accommodation is inspired by Fried (2001), who suggests a similar one in relation using the history of mathematics in mathematics education.

[3] Didactically informed CAS obviously does exist. One example is Cassyopeé. However, most of the CAS used in Danish upper secondary school are professional software (such as *Maple*, *Mathcad* and *Mathematica*), school directed 'clones' of professional software (such as *TI-Nspire*), or software directed specifically at supporting students' work (such as *Microsoft Mathematics* or *Wordmat*).

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