How Middle Grade Teachers Think about Algebraic Reasoning

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Algebraic reasoning is an essential habit of mind for building conceptual knowledge in K-12 mathematics, yet little is known about how middle school mathematics teachers think about algebraic reasoning. In this article we describe a research project examining how algebraic reasoning was considered by grades 6, 7, or 8 mathematics teachers in a two-week professional development course and over the following two months. We found these 19 teachers initially described algebraic reasoning in a way requiring only procedural knowledge to solve problems with a single solution, solution strategy, or representation. Teachers reported that three activities influenced a shift in their thinking about algebraic reasoning, specifically by requiring conceptual knowledge to solve problems using multiple solutions, solution strategies, or representations. While some teachers also associated aspects of generalisation and functional thinking as part of algebraic reasoning, two months after the professional development no teachers continued to associate these aspects as part of algebraic reasoning. These findings suggest the kinds of activities other teacher educators can use to develop teachers' thinking about algebraic reasoning, and supports the need for additional research and interventions to support middle school teachers' consideration of algebraic reasoning in more advanced ways.

Keywords · algebraic reasoning · in-service teachers · functional thinking · generalisations

Introduction

In the United States (US) and Australia, algebra curricula and instruction are important focal points in mathematics education. In the US, algebra is a content strand in grades K-12 in both the National Council of Teachers of Mathematics standards (NCTM, 2000) and the Common Core State Standards for Mathematics (National Governors Association, 2010). Major goals of the CCSSM include strengthening students' conceptual understanding of mathematics, engaging eight key mathematical practices during teaching and learning, and improving the coherence in learning expectations. Widely adopted in the US, these standards provide clear expectations for what students will learn in each grade level. While the old standards placed a greater emphasis on memorising facts or procedures, the new standards call for students to solve problems that require deeper knowledge of mathematical concepts and to explain their reasoning (CCSSM, 2010). Similarly, the Australian Curriculum (Australian Curriculum Board, 2011) contains a content strand Number and Algebra that spans all years of education, with an emphasis on algebra near the end of the compulsory years. Both sets of academic standards include mathematical standards that were developed as a result of research studies of mathematics education in high-performing countries, which suggested the need for the mathematics curriculum to become more focused, rigorous, and coherent in order to improve mathematics achievement (Atweh & Goos, 2011).

The shift in what students need to know and understand mathematically requires teachers to change how they teach mathematics concepts. Teachers must shift from focusing on computation and memorisation to focusing on mathematical sense making, problem solving, and reasoning. Students are now expected to develop their own understanding through meaningful activities and discourse that will allow them to deeply understand the concepts and use them as building blocks for learning in subsequent grades. Teacher quality has been consistently identified as the most important school-based factor in how and what students learn (Cai & Knuth, 2011), with the success of teacher efforts to develop students' algebraic reasoning being related to the ability of teachers to facilitate tasks in their classrooms.

Algebraic reasoning is an essential habit of mind for building conceptual knowledge in K-12 mathematics (Kaput, 2008), yet little is known about how K-12 mathematics teachers think about algebraic reasoning in the context of their classroom (Ellis, 2011; Kaput & Blanton, 2005). In this project, we aimed to address this research need by examining how middle school in-service mathematics teachers, who taught grades 6, 7, or 8 in the US, considered algebraic reasoning. Our research question was: how do teachers develop their understanding of algebraic reasoning in the context of their classroom through a two-week professional development session? This question focused our efforts on characterising how teachers communicated their understanding of algebraic reasoning throughout the professional development and during the following months, after teachers returned to their classrooms.

Literature Review

Mathematics education research and national standards recommend teaching algebra and arithmetic as integrated topics in elementary and middle school, which has been termed as algebraic reasoning (CCSSM, 2010; NCTM, 2001). Blanton and Kaput (2004) defined algebraic reasoning as "a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413). Researchers widely agree that for a majority of students to succeed in formal algebra coursework encountered in high school, they need to be exposed to algebraic reasoning in middle and early grades and to demonstrate a fundamental understanding of algebra skills (Bottoms, 2003; NCTM, 2000, 2006; National Mathematics Advisory Panel [NAMP], 2008). Middle school students' prerequisite algebra skills progress from learning about patterns through diagrams and number sequences in elementary school to learning about patterns that represent functions, exploring proportional relationships, and making connections between properties of arithmetic and algebra (Blanton, 2008; Blanton & Kaput 2011; Bush & Karp, 2013). For future success in algebra, middle school students need opportunities to engage in learning that fosters their understanding of generalised arithmetic, functional thinking, and equality (Blanton & Kaput, 2004; Carraher, Schliemann, Schwartz, 2008).

However, developing algebraic reasoning at the middle school level requires more than simply moving the traditional algebra curriculum down into the elementary school level (Kieren, 2004). Instead, integrating algebraic reasoning into primary grades provides an alternative that builds the conceptual understanding, procedural fluency, and problem solving skills into students' early experiences (Kaput & Blanton, 2005; Bush & Karp, 2013). In grades 3 through 5 in the US and Australia, algebra is embedded with number and operations. An early introduction to algebra provides students with more opportunities in higher-level mathematics, and it also serves to support and connect the learning of arithmetic in elementary grades to higher-level mathematics in secondary grades (Kaput, 2008; Lee, 2001).

Integrating algebraic reasoning into arithmetic adds breadth and depth to elementary students' conceptual and procedural understanding and provides them with powerful ways of thinking about mathematics. Generalising mathematical ideas, using symbolic representations, and representing functional relationships are ideas that represent mathematics thinking in the elementary and middle grades (Blanton and Kaput, 2011; Carpenter, Franke, and Levi, 2003). Specifically the development of algebraic reasoning at these levels requires student considering and analysing relationships between quantities, conjecturing, justifying, and proving (Kieran, 2004). Transitioning from arithmetic and computational fluency to thinking more deeply about the structure of mathematics and relationships among quantities represents a shift towards developing ideas fundamental to the study of middle school algebraic concepts. Instruction that supports this kind of thinking is important in developing students' algebraic reasoning during middle school.

Because teachers have great impact on creating change in the classroom and developing students' understanding of mathematics needed for success in algebra, they too need a deep understanding of mathematics and instructional strategies that are useful in fostering and developing students' algebraic reasoning. Many middle school teachers have little experience with aspects of algebraic reasoning because often they have not had an opportunity to cultivate their understanding of algebraic thinking, reasoning, and teaching (Borko, 2004; Kieran, 2007). Teachers need assistance to learn how to provide rich and explicit instruction to develop their students' algebraic reasoning. They need opportunities to problem solve, justify reasoning, use multiple representations, pose questions, and identify and use tasks which promotes students' conceptual and procedural understanding and builds their ability to reason algebraically (Bair & Rich, 2011; Borko, 2004). "If we are to build classrooms that promote algebraic reasoning, we must provide the appropriate forms of professional support that will effect change in instructional and curricular practices" (Blanton & Kaput, 2005, p. 414). Teachers' algebraic knowledge and the teaching of algebra is an important element in the effort to support students' algebraic reasoning and functional thinking (Blanton & Kaput, 2005).

Methodology

Setting

To address the need of providing teachers opportunities to consider algebraic reasoning, we[1] led a professional development experience focused on helping middle school teachers enhance their knowledge about algebraic reasoning and their instructional practice by engaging in mathematical tasks and guided activities that built generality in patterns and relationships. The participants of this study were 19 middle school teachers from the Southern United States who participated in this two-week professional development session and a follow-up meeting two months later. While these teachers worked in school districts following a national curriculum, the allocation of these standards to specific grades had recently been modified prior to the professional development. Thus the professional development in part attended to covering

[1] The two authors of this study were part of a six-member team leading the professional development.

content in grades 6, 7, and 8. Seventeen of the teachers taught mathematics in these grades, one was a special education support teacher, and one teacher taught middle grade science but was interested in transitioning to mathematics. The local school districts encouraged these teachers to participate in the professional development, which was aimed to support teachers in making productive changes to their classroom. The teachers reported little to no experience with the term "algebraic reasoning" despite this term being used in national standard documents.

During the professional development we had teachers engage in several mathematical tasks requiring algebraic reasoning, such as the one given in Figure 1. These tasks required teachers to engage in algebraic reasoning through abstracting quantities from a problem, generalising the relationships between these quantities, building rules and representing these rules through functions, and reasoning and acting on these generalisations to achieve a desired outcome. These actions correspond to the definitions of algebraic reasoning provided in the previous section and align with the work by Driscoll (1999) and Kaput (2008). A major goal of the professional development was to promote teachers' thinking about algebraic thinking in ways that translated to changes in their classroom practice. Specifically, we wanted teachers to recognise that algebraic reasoning goes beyond a particular task and instead develop "a habit of mind that transcends the particular resource being used and allows elementary teachers to see opportunities for algebraic thinking, and functional thinking in particular, in the mathematics they already teach, using the curriculum they have in place." (Blanton & Kaput, 2011, p. 18). We envisioned these changes in the classroom in the types of tasks and general practices the teachers modeled and promoted with their students.

Teachers completed these tasks in groups before we led a whole-group discussion about the task focusing on how algebraic reasoning related to the task. In addition to completing and discussing algebraic reasoning tasks, teachers were asked to read selected pages in Driscoll (1999) and an article about the importance of equal signs (Knuth, Alibali, Hattikudur, NcNeil, & Stephens, 2008). Teachers were asked to write a reflection about how they thought about algebraic reasoning, what algebraic thinking might look like in their classroom, and any changes in their thinking. The first four reflections took place during the consecutive days on the first week of the course. The fifth reflection took place at the end of the second week. A follow-up session took place after teachers had started the school year, during which we had teachers complete similar activities and discuss how previous activities impacted their classroom practices, during which the teachers completed their sixth reflection.

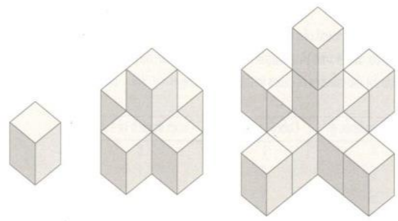

Figure 1. The Block Task: Describe this pattern and use it to write at least one expression for the number of cubes in the nth building.

Data Collection and Analysis

Our research question focused our efforts on characterising how teachers communicated their understanding of algebraic reasoning throughout the professional development and during the following months, after teachers returned to their classrooms.

We conducted observations and collected documents from teachers during and after a two-week professional development session. Our observation protocol focused our attention on the comments teachers made about algebraic reasoning in small and whole group settings. Data from documents consisted of five reflections that teachers completed during the professional development and one reflection two months following the professional development. The reflections asked teachers to individually write down (1) how they interpreted the phrase "algebraic reasoning," (2) what algebraic reasoning looks like in their classroom, and (3) how their view of algebraic reasoning changed due to that day's activities.

We used content analysis (Merriam, 1998) to analyse teachers' reflections and our observation notes and created several codes evident in the data. We clarified and refined our codes based on existing literature. For example, we use Blanton and Kaput's (2011) definition of algebraic reasoning to clarify the patterns of responses attending to generalisation and functional thinking (see Table 1 for full coding scheme). We found 10 codes described the ways teachers talked about algebraic reasoning.

Once we established our coding scheme, we coded the data individually to establish inner-rater reliability. Both researchers then compared the codes and discussed any discrepancies, and the percentage of agreement of a code being both evident and correctly coded was above 90%. To answer the research question we found themes in how teachers made statements about algebraic reasoning at the beginning of the professional development, the end of the professional development, and two months into the school year. We also coded each statement a teacher made about why they changed their thinking about algebraic reasoning by associating the change in thinking with an activity they mentioned. For example, teachers said the Block Task or the similar Counting Cubes task caused them to think about algebraic reasoning as finding patterns and using these patterns to determine the number of blocks or cubes in subsequent buildings. These statements were coded as the Block Task influencing them to think about algebraic reasoning as attending to generalisation. After coding completion and developing of themes, we member-checked these themes with one of the participants to support the validity of the findings.

Table 1: Coding Scheme

Code	Teachers' statements about algebraic reasoning…
Single Solution	indicated only one acceptable solution when solving a problem
Single Solution Strategy	indicated only one acceptable solution strategy when solving a problem
Single Representation	indicated only one acceptable representation when solving a problem
Multiple Solutions	communicated more than one acceptable answer to given problem (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005)
Multiple Solution Strategies	communicated more than one acceptable approach a person could take to solve a given problem (Stein, Grover, & Henningsen, 1996)
Multiple Representations	communicated more than one representation a person could use to solve a given problem (Stein, Grover, & Henningsen, 1996)
Procedural knowledge	exclusively mentioned symbolic procedures, often closely connected with a particular problem type or applied memorised procedures that have little meaning beyond the immediate context (Smith & Thompson, 2007)
Conceptual knowledge	attended to the principles that govern the interrelations between pieces of knowledge (Lesh, Landau, & Hamilton, 1983)
Expressing Generalisation	attended to what is always true in a relation, expression, equation, representation, or function (Blanton & Kaput, 2011) or extended the mathematical directions that an algebraic result may suggest, beyond the result itself (Driscoll, 1999)
Functional Thinking	focused on a relationship between two quantities and used functions to represent this relationship (Blanton & Kaput, 2011)

Findings

At the start of the professional development, teachers made statements about algebraic reasoning that were coded as attending to procedural knowledge with a single solution, solution strategy, or representation. At the end of a two-week professional development session, most teachers described algebraic reasoning by attending to conceptual knowledge and multiple solutions, solution strategies, or representations. While some teachers also associated aspects of generalisation and functional thinking as part of algebraic reasoning, two months after the professional development no teachers continued to make statements about this association. In this section we detail the patterns in the statements teachers gave, beginning with their initial description of algebraic reasoning, how these descriptions changed during the workshop, factors teachers identified as influencing their thinking, and teachers' descriptions of algebraic reasoning two months after the workshop.

Teachers' Initial Descriptions of AR

Initially 11 of the 19 teachers gave responses on reflection 1 that were coded as attending to single solutions, solution strategies and representations. For example, Debra's first reflection said:

> Algebraic reasoning, in my classroom, is the balancing of equations to show the solution for a given variable. It is the expression of what to do with algebraic equations and inequalities. Students will write an equation for each of the equations within the problem…solve for one variable…substitute for the variable found and solve for the remaining variable…check their work using both variables (for both equations).

Debra's statements about "the solution" and only mentioning substitution for solving systems of equations was evidence she thought about algebraic reasoning as having one possible solution and solution strategy to the problem. The exclusive mention of algebraic equations also indicated Debra attended to algebraic reasoning through only a single representation.

Twelve of the 19 teachers who initially made statements about algebraic reasoning only included symbolic procedures, which we coded as attending to procedural knowledge. For example, Jamie said "To me [algebraic reasoning] means you are using numbers and variables to represent a problem in order to solve the equation and make it true. 2n=4 n=2 ; 5+n=12 n=7." The solving of equations Jamie refers to did not indicate students had to go beyond using numerical procedures and thus was coded as attending to procedural knowledge.

The seven teachers who did not attend to procedural knowledge in their reflection 1 instead all attended to conceptual knowledge by interrelating symbolic procedures to other pieces of knowledge. For example, Patricia said:

> Algebraic reasoning in my classroom can be how the students explain their reasoning of how they solved an algebraic problem or the work that is shown. This is what I would like to see algebraic reasoning to look like all the time but I am lucky if the students tell me how they solved the problem (example: ratio, equation, etc.) along with giving the answer with the correct unit or representation (example: five bunnies, could afford four shirts, etc.). (reflection 1)

Patricia's description of algebraic reasoning attended to a numerical answer, the explanation/justification behind the answer, an associated unit, and corresponding representation. This description provided evidence she attended to the principles that govern the interrelations between pieces of knowledge, and was thus coded as conceptual knowledge.

Eight teachers gave reflection 1 responses that were coded as attending to multiple solution strategies, multiple representations, or generalisation (see table 2 for details). For example, Jewel attended to multiple representations and generalisation by saying algebraic reasoning is "forming a pattern using symbols when given numbers and computations, finding equations that describe a situation…[and] looking at a graph and finding the equation of a line." Jewel's attention to algebraic and graphical representations provided evidence she considered algebraic reasoning as attending to multiple representations. Her statement about patterns and generating equations to describe problems attended to what is always true in an equation and thus was coded as generalisation. Only one other teacher initially provided a statement coded as generalisation, while two other teachers attended to multiple representations in their reflection 1.

Table 2: Counts of teacher (n=19) statements coded using our coding dictionary before (reflection 1), during (reflections 2-5) and after (reflection 6) the professional development. Reflections with total counts fewer than 19 resulted from responses unrelated to codes or missing data.

Code	Reflection 1 (before any activities)	Reflection 2 (after Split Time Activity)	Reflection 3 (after Knuth's Article and Block Task	Reflection 4 (after 3-Act Tasks)	Reflection 5 (at end of 2 week PD)	Reflection 6 (2 months after PD)
Single Solution	11 (58%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Single Solution Strategy	6 (32%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Single Representation	2 (11%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Multiple Solutions	1 (5%)	1 (5%)	0 (0%)	0 (0%)	3 (16%)	1 (5%)
Multiple Solution Strategies	5 (26%)	3 (16%)	7 (37%)	7 (37%)	10 (53%)	6 (32%)
Multiple Representations	3 (16%)	2 (11%)	4 (21%)	2 (11%)	4 (21%)	3 (16%)
Procedural knowledge	12 (63%)	0 (0%)	0 (0%)	0 (0%)	1 (5%)	0 (0%)
Conceptual knowledge	7 (37%)	3 (16%)	7 (37%)	7 (37%)	11 (58%)	11 (58%)
Expressing Generalisation	2 (11%)	3 (16%)	4 (21%)	9 (47%)	9 (47%)	5 (26%)
Functional Thinking	0 (0%)	3 (16%)	4 (21%)	1 (5%)	5 (26%)	0 (0%)

Changes in Teachers' Descriptions of Algebraic Reasoning during the Professional Development

As teachers engaged in the professional development, our observations and reflection 2, 3, 4, and 5 documents provided data showing four themes in how teachers considered algebraic reasoning. The first theme was that most teachers shifted from attending to single solutions, solution strategies and representations, to making statements about algebraic reasoning as attending to multiple solutions, multiple solution strategies, and multiple representations. In reflections 2, 3, 4, and 5, no teacher made a statement about algebraic reasoning that attended to single solutions, solution strategies, or representations. The thirteen teachers who initially stated algebraic reasoning as attending to a single solution strategy all later described algebraic reasoning as attending to multiple solution strategies. The seven teachers who did initially address only a single representation all began describing algebraic reasoning as attending to multiple representations.

An example of this first theme is seen in Gabriela's responses. In reflection 1 Gabriela described algebraic reasoning as "how I think about mathematical problems to get to a solution...what do I need to find and what are the steps I take to get there." This description was coded as attending to single solution, single solution strategy, and requiring procedural knowledge. In subsequent reflections Gabriela said "I think my view of algebraic reasoning changed a bit" (Reflection 2) and began describing algebraic reasoning as "being able to solve a problem (or for an unknown value), explain how you solved it and look for ways to solve differently" (reflection 3) and even included an example of using "algebra tiles to show combining like terms" (reflection 2). These statements were coded as attending to multiple solutions strategies and multiple representations.

The second theme in how teachers' statements about algebraic reasoning changed during the professional development was the type of knowledge teachers attended to in their statements about algebraic reasoning. During the professional development eight teachers shifted from attending to procedural knowledge to attending to conceptual knowledge. This shift is also evidenced by Gabriela's responses, this time in her reflection 5: "algebraic reasoning in my classroom will be answering open-ended questions, finding more than one solution, or more than one way to reach a solution, explaining how an answer was reached (justifying an answer with more than just the computational steps)." This response shows her thinking about algebraic reasoning is no longer tied to only using procedures, but instead requires students to relate the solution back to the problem using justification and sense making; therefore Gabriela was coded as attending to conceptual knowledge in her description of algebraic reasoning.

The third theme of how teachers' statements about algebraic reasoning changed was a shift towards attending to generalisation as part of algebraic reasoning. In reflections 2, 3, 4, and 5 there were 14 teachers who gave responses coded as generalisation, with 12 of the 14 teachers having not mentioned generalisation in reflection 1. Susan was one of these 12 teachers who first attended to generalisation in reflection 4 by saying, "algebraic reasoning is taking prior knowledge, looking for patterns and similarities, to develop a more generalised form for any amount you would need to know." In her next reflection she elaborated on a task for her students that would promote algebraic reasoning: "I would give them the…Cubes task. It would require them to look for a pattern with a 3-D shape, generalise the pattern, and apply it to larger shapes to see if the pattern holds true." Both of these statements indicated Susan was attending to what is always true in a relation or representation and extending beyond the result itself, and thus were coded as generalisation.

The fourth theme of how teachers' statements about algebraic reasoning changed was a shift towards attending to functional thinking. In reflections 2, 3, 4, and 5 there were 8 teachers who gave responses coded as functional thinking, all of whom had not attended to functional thinking in reflection 1. For example, Rachel's reflection 2 said "My view of algebraic reasoning has been expanded somewhat from today's activities to include a greater emphasis on functions as central to algebraic reasoning…what functions are, how to determine functions to fit patterns, and how to illustrate functions."

Factors Documented as Influencing Teacher Change

When teachers provided statements about algebraic reasoning we asked them what factors influenced their change in thinking. Of the 25 activities the teachers completed during the professional development, three activities were repeatedly mentioned as being influential to how teachers considered algebraic reasoning: 3 Act Tasks, the Block Task, and the equal sign article. This section details these three activities and how they influenced the shifts in teachers' descriptions seen in the previous section.

The 3 Act Tasks, implemented on day 4, were the most frequently mentioned activity when teachers described what impacted their thinking about algebraic reasoning. 3 Act Tasks are a type of activity for students coined by Dan Meyer (http://blog.mrmeyer.com/2013/teaching-with-three-act-tasks-act-one). Since their creation these tasks have begun to be used and shared among teachers and have even begun appearing in curricula documents (Georgia Department of Education, 2015). During the fourth day of professional development teachers were asked to explore some existing 3 Act Tasks.

Twelve teachers reported being influenced by the 3 Act Tasks in how they thought about algebraic reasoning. These 12 teachers reported these tasks caused them to consider algebraic reasoning as attending to conceptual knowledge (8 teachers), multiple representations (3 teachers), multiple solution strategies (5 teachers), generalisation (8 teachers), and functional thinking (2 teachers). For example, Susan's response was common among teachers: "I have definitely considered algebraic reasoning into my class. My goal is to implement most of these activities into all of my grade levels. The 3-act tasks definitely incorporate algebraic reasoning by having them problem solve to look for patterns." This response indicates Susan now considers pattern solving and thus generalisation as a component of algebraic reasoning because of the 3 Act Tasks, and that she views these tasks as something she can implement in her classroom.

Another example of the impact of the 3 Act Tasks comes from Cathleen, who defined algebraic reasoning as:

> Having the ability to represent a concept/answer multiple ways. The 3-Act Task was impactful because it provided another way of teaching and thinking about teaching. We also had time to use and explore the websites associated with the 3 Act Tasks. Discussing Algebraic Reasoning will be an ongoing discussion in my class and using the 3 Act-Tasks will be incorporated in my class. (reflection 4)

Cathleen's response shows she is now thinking about algebraic reasoning as attending to multiple solution strategies and multiple representations because of the 3 Act Task. Her response is representative of the other teachers citing these tasks as influential on their thinking about algebraic reasoning.

The Block Task, given in Figure 1 and implemented on day 3, was reported by 11 teachers as being influential to how they thought about algebraic reasoning. These 11 teachers reported the Block Task caused them to consider algebraic reasoning as attending to conceptual knowledge (5 teachers), multiple representations (5 teachers), multiple solution strategies (6 teachers), generalisation (5 teachers), functional thinking (4 teachers), and multiple solutions (1 teacher). For example, Debra stated:

> Algebraic reasoning is the multiple representation of a problem and the ability to explain why it works. The Counting Cubes Task [Block Task] is a great example of an assessment that would assess your students' algebraic reasoning. To take cubes and be able to show the different representations of the cubes (as they increase) would show that. (reflection 5)

This response suggests she was attending to conceptual knowledge by having students go beyond just answering the question of how many cubes were needed as well as attend to multiple representations when engaging in algebraic reasoning. Terrance's response shows how this task influenced him to consider generalisation and function thinking as part of algebraic reasoning by stating algebraic reasoning is the ability to

> move from abstracts to concrete and be able to derive patterns, relationships and expressions and even equations. The…activity was especially very good and I will use it…[I will use] the blocks to generate patterns and write a relationship if possible, but it is very capable of generating curiosity, interest and interest in the class. (reflection 3)

These statements indicate the Block Task was influential on teachers thinking about algebraic reasoning.

The equal sign article (Knuth at al., 2008), assigned on day 2 and discussed on day 3, was the third factor teachers attributed to changing how they thought about algebraic reasoning. Eight teachers referenced this article as impacting their thinking about algebraic reasoning, particularly by influencing them to attend to conceptual knowledge (6 teachers), multiple solution strategies (1 teacher), and multiple representations (1 teacher). For example, Debra stated:

> My view of algebraic reasoning is changing every day. At first, my goal (as an educator) was to make certain my students could use steps and procedures to arrive at the correct answer. Now, my view is to make certain the students understand and gain a greater knowledge about what they are doing. The article, "Equal Sign" helped me in seeing the sign as equivalence (relational) and not operational. (reflection 3)

This statement, along with comments from five other teachers, indicated the equal sign article was impactful on teachers' thinking about algebraic reasoning as attending to conceptual knowledge rather than procedural knowledge.

Lasting Changes in Teachers' Descriptions of Algebraic Reasoning following the Professional Development

Two months after the professional development, 11 of the 19 teachers[2] described algebraic reasoning as requiring conceptual knowledge rather than procedural skills. Eight teachers described algebraic reasoning by attending to multiple solutions, solution strategies, or representations. No teachers gave a description of algebraic reasoning that was coded as attending to only one solution, solution strategy, or representation. These responses were similar to those given during the professional development in reflections 2-5.

One dissimilarity between teachers' earlier responses was that teachers did not continue to associate generalisation or functional thinking as part of algebraic reasoning. Only five teachers attended to generalisation in comparison to 14 teachers having attended to generalisation in reflections 2-5. Furthermore, no teachers attended to functional thinking in comparison to nine teachers having done so in reflections 2-5.

Conclusions

The middle school teachers in this study undertook a number of activities designed to alter the way they viewed algebraic reasoning. Almost immediately they appeared to change their views, perhaps because they had not encountered these types of activities prior to the professional development. Our findings suggest these middle school teachers came into the professional development with a view of algebraic reasoning that consisted of attending to a problem with a single solution, using a single solution strategy or representation. Their thinking of algebraic reasoning required only procedural knowledge and did not include generalisation or functional thinking. As the teachers engaged in the professional development, the 3 Act Tasks, Block Task, and the Knuth at al. (2008) reading influenced all teachers to reconsider algebraic reasoning as attending to multiple solution strategies and representations that require conceptual knowledge. Additionally, many teachers began including ideas of generalisation and functional thinking as part of their description of algebraic reasoning. After the professional

[2] Four of these 11 teachers initially described algebraic reasoning as requiring conceptual knowledge, indicating consistent responses across the reflections.

development most teachers still considered algebraic reasoning in terms of multiple solution strategies and representations that require conceptual knowledge but did not continue associating algebraic reasoning as attending to generalisation and functional thinking.

This study addressed issues of the learning which took place in a professional learning program designed to enhance middle school teachers understanding of algebraic reasoning and its teaching. Investigation of these benefits for teachers' learning has potential not only to improve the design and preparation of professional learning programs for teachers but also provides opportunities to influence children's learning in mathematics. These findings are thus significant for three reasons. First, we found teachers did not initially identify algebraic reasoning in a way that aligns with the practices we expect in middle school. Specifically, Blanton and Kaput (2008) and other researchers (Lee, 2001; Carpenter et al., 2003) advocate the importance of attending to multiple solutions, solution strategies, and representations, generalisation, and functional thinking within the middle grades. These patterns suggest either these teachers were not incorporating these practices in their classrooms prior to the professional development or were not associating these practices with algebraic reasoning. The identification of this pattern is important because this suggests teachers were not attending to algebraic reasoning in way that upholds national standards (CCSSM, 2010; National Curriculum Board, 2011) for middle grades students. This finding aligns with other researchers' findings that many middle school teachers have little experience with aspects of algebraic reasoning (Borko, 2004; Kieran, 2007).

A second significance of these findings is that we identified three activities that were impactful for teachers' thinking about algebraic reasoning. 3 Act Tasks are a relatively recent addition to the wealth of information available for teachers online. In reflection 4, after engaging in the 3-Act task, more than twice as many teachers identified "expressing generalisations" as an important aspect of algebraic reasoning when compared to responses in reflection 3, supporting our claim that the 3-act tasks were impactful. Introducing the teachers to the many 3 Act Tasks on Dan Meyer's website and providing teachers opportunities to work through these tasks gave teachers a new perspective on what algebraic reasoning could look like in their classrooms. Additionally, having teachers complete the Block Task and tasks like this from Driscoll (1999) and read the Knuth at al. (2008) article provided teachers opportunities to advance their thinking about algebraic reasoning. Identifying these impactful activities can help other teacher educators focus the limited time they have with teachers to advance thinking about algebraic reasoning.

The third significance of the findings is identifying the limited impact these activities had on teachers' long term view of algebraic reasoning. Generalisation and functional thinking are essential for students to succeed in formal algebra coursework, and thus students need to be engaged in this kind of algebraic thinking in the middle school grades (Bottoms, 2003; NCTM, 2006; NAMP, 2008). The findings indicated many teachers' view of algebraic reasoning changed to include conceptual knowledge and multiple solution strategies after the professional development ended. This suggests these teachers' classroom practices may have begun to reflect their evolving views on algebraic reasoning, in part by incorporating the tasks and instructor moves encountered in the professional development. This type of professional development can lead to teachers promoting algebraic reasoning beyond these tasks in the form of a habit of mind spanning mathematical content, lessons, and grades.

Unfortunately this professional development did not appear to have a lasting impact on how these teachers considered algebraic reasoning as attending to generalisation and functional thinking two months following the workshop. The professional development focused on the type of tasks that foster students' algebraic reasoning. Smith, Hughes, Engle, and Stein (2009) noted that teachers often face challenges beyond identifying tasks that potentially support algebraic reasoning, such as balancing the support of generalisation, functional thinking, and classroom teaching practices. This suggests more research and effort is needed to provide middle school teachers opportunities to develop their thinking about generalisation and functional thinking in ways connected to their classroom practice. For example, more extended professional development could support teachers' thinking about algebraic reasoning, particularly if this effort is integrated into their classrooms. This requires professional developers to identify and support the pedagogy of algebraic reasoning when teachers have selected rich tasks. Future research can focus on identifying ways to promote generalisation and functional thinking with middle school teachers by providing opportunities to engage themselves and their students with new ideas and productive teaching practices.

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