Proof and rhetoric: The structure and origin of proof—from Ancient Greece to Abraham Lincoln's speech in defence of the Union and Paul Keating's Mabo speech

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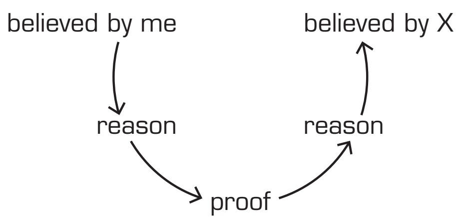
Introduction

According to the latest news about declining standards in mathematics learning in Australia, boys, and girls, in particular, need to be more engaged in mathematics learning. Only 30% of mathematics students at university level in Australia are female (ABC News, 2014) although paradoxically it would seem, the majority of lawyers in Victoria are female, a profession which requires a good grasp of language, rhetoric and logic (Merritt, 2014). So why not engage girls—with one of their strengths in early childhood (and later)—language and its acquisition (Padula & Stacey, 1990) and at the same time assist boys in their teenage years when their language development has (usually) caught up with girls (Goodman, 2012; Gurian, Henley & Trueman, 2001)? One of the ways to do this would be to teach pure mathematics—in the form of proofs.

Hardy (1941) states: "Pure mathematics is on the whole distinctly more useful than applied. For what is useful above all is technique, and mathematical technique is taught mainly through pure mathematics." Bertrand Russell has stated that pure mathematics consists of "entirely of such asseverations as that, if such and such a proposition is true of anything then such and such another proposition is true of that thing" (Bogomolny, 2013). In other words: proofs. Proof is the essence of mathematics but what can we learn from its origins and structure to assist students who want to learn about it?

Cheng (2015), writing about category theory, gives us a new slant on proof. She states that in mathematics knowledge comes from proof. She details how mathematicians like her move from belief, to reason, to proof. Once she believes something, how does she convince someone else? She shows them her proof.



So the procedure is:

- I start with a truth I believe that I wish to communicate to X.
- I find a reason for it to be true.
- I turn that reason into a rigorous proof.
- I send the proof to X.
- X reads the proof and turns it into a convincing reason.
- X then accepts the truth into his realm of believed truth.

(Cheng, 2015, p. 281)

Cheng argues that: "the key characteristic about proof is not the infallibility, but its sturdiness in transit. Proof is the best medium for communicating my argument to X in a way that will not be in danger of ambiguity, misunderstanding or distortion. Proof is the bridge for getting from one person to another, but some translation is needed on both sides" (ibid). For this translation to occur a knowledge of the language of proofs is a necessary condition. However, both for the writer and reader of proofs sending and receiving that message or argument is not easy since: "Words themselves are abstract representations that emerge like islands from a sea of associated meanings" (Siegel, 2011).

Language of proofs

Proofs are made up of words and mathematical symbols. One can assume the words would assist understanding but this is not how mathematics students think necessarily (Padula, 2006, 2011), particularly perhaps if English is not their first language. Or, for that matter, this is not what mathematicians think—Gödel, for example, thought it was astounding that we ever understand each other (in natural language) since it is so imprecise and different from mathematical language (Goldstein, 2005). But Arianrhod (2003) states that when you think in terms of mathematical symbols as well as words: thought itself is economised because "the symbolism enables you to see at a glance patterns and generalities, similarities and differences, which may not be obvious if you think only in words" (ibid). And Stewart (1995) writes: "a mathematical proof is a story about mathematics that works" and "both a proof and a novel must tell an interesting story… a good storyline is the most important feature of all".

According to Ernest (1999), the underlying language of mathematics as a whole is a mathematical sub-language, such as English or German, supplemented with mathematical symbols. It has an extensive range of linguistic objects: mathematical symbols, notations, diagrams, terms, definitions, axioms, statements, analogies, problems, explanations, method applications, proofs, theories, texts, genres and rhetorical norms for presenting mathematics. Mathematics could not exist at all without knowledge of this language, he argues, and although elements of this language are explicit, knowing how to use them is tacit. This being the case Ernest writes that these “linguistic objects” should be taught.

Mathematician Steven G. Krantz opines: “A (mathematical) proof is a rhetorical device for convincing someone else that a mathematical statement is true or valid” (Tammet, 2012, p. 56). So let us consider just one element of this language: rhetoric. If we agree that rhetoric is the art of persuasive or impressive speaking or writing, language designed to persuade or impress and that an axiom is a statement that we accept as being self-evidently true (or indeed previously established in other proofs) we can present students with very simple examples of the structure of proofs.

In the book and film *To Kill a Mockingbird* starring Gregory Peck (Pakula, 1962), a defense lawyer in court, the young heroine Scout’s father, Atticus Finch (Peck) uses rhetoric in defense of his black client. There are many other examples in literature, film and television. But where do they originate in western culture?

Origins of proof

Daniel Tammet (2012) argues in his book that proof and logic have their origin in Ancient Greece with their law courts. His argument can be stated in the following manner: Ancient Greece with its rowdy assemblies, passionate debates and litigious citizens was where rhetoric (or persuasive argument) led to the evolution of proof—with rhetoric and logic. It was “the law courts with their public trials, that the building blocks of our system of thought were honed” (ibid, p. 56); and, “With their axioms… the Greek plaintiff was able to methodically construct his case, the Greek mathematician, his theorem” (p. 59); see Figure 1.

Law courts of Ancient Greece

↓

Rhetoric

↓

Logic

↓

Proof

Figure 1. The evolution of proof in western culture, according to Tammet (2012).

It is interesting to note that there were always an odd number of jurors so that no tied judgments were possible and the jury members had to be males over the age of 30.

Structure of proof

Here is one example of a plaintiff's case provided by Tammet (2012) with a comment in brackets added (by the author),

PROPOSITION: A thief (claims the defendant) killed the victim.
But thieves steal the victim's cloak. [And it was not stolen.]
Thus a thief did not kill the victim.

(Tammet, 2012, p. 58)

Euclid's Elements

There are more proofs in Euclid's *Elements*, Tammet (p. 58) continues. Here are two of them—things which are equal to the same thing are equal to each other and the sum of any quantity of even numbers is even. The first:

PROPOSITION: *CA* and *CB* are each equal to *AB*.
But things equal to the same thing are equal to each other.
Thus *CA* is also equal to *CB*.

And the second, Proposition 21, Book IX of the *Elements*:

> If as many even numbers as we please be added together, the whole is even … For, since each of the numbers … is even, it has a half part; so that the whole [number] also has a half part. But an even number is that which is divisible into two equal parts; therefore [the whole number] is even" (pp. 58–59).

Tammet (p. 59) analyses Euclid's Proposition in the following manner:

PROPOSITION: Even numbers (of any quantity) added together make an even number.
CLARIFICATION: Since even numbers have half parts, their sum will also have a half part.
AXIOM: An even number is that which is divisible into two equal parts.
CONCLUSION: Therefore the sum of any quantity of even number is even.

Examples can also be found in history. Consider Abraham Lincoln's State of the Union Address:

'The Union of these States is perpetual'

Lincoln's view of the American Union when it was threatened with the defection of the Southern states was as follows:

> A disruption of the Federal Union heretofore only menaced is now formidably attempted.
> I hold, that in contemplation of universal law and of the Constitution, the Union of these States is perpetual. Perpetuity is implied, if not expressed, in the fundamental law of all national governments. It is safe to assert that no government proper, ever had a provision in its organic law for its own termination. Continue to execute all the express provisions of our national Constitution, and the Union will endure forever—it being impossible to destroy it, except by some action not provided for in the instrument itself (Abraham Lincoln, First Inaugural Address, 4 March 1861).

Which, as Tammet (2012, p. 62) has shown, can be deconstructed as follows:

PROPOSITION:	The Union of these States is perpetual.
CLARIFICATION:	Perpetuity is implied in the fundamental law of all national governments.
AXIOM:	No government ever had a legal provision for its own termination.
CONCLUSION:	Therefore continue to execute the Constitution and the Union will endure forever.

There are speeches by Australian politicians which can be similarly analysed; for example, Paul Keating's Mabo speech of reconciliation (Keating, 1992).

PROPOSITION: We [Australians] cannot sweep justice aside… or the first Australians—the people to whom the injustice has been done.

CLARIFICATION: [R]ecognise that the problem starts with us non-Aboriginal Australians. It begins… with that act of recognition:

- A. Recognition that it was we who did the dispossessing.
- B. We took the traditional lands and smashed the traditional way of life.
- C. We brought the diseases. The alcohol.
- D. We committed the murders.
- E. We took the children from their mothers.
- F. We practiced discrimination and exclusion.

AXIOM: MABO… the bizarre conceit that this continent had no owners prior to the settlement of Europeans, Mabo establishes a fundamental truth and lays the basis for justice.

CONCLUSION: We cannot imagine that the descendants of people whose genius and resilience maintained a culture here through fifty thousand years or more, through cataclysmic changes to the climate and environment, and who then survived two centuries of dispossession and abuse, will be denied their place in the modern Australian nation.

(Paul Keating, Redfern, 10 December 1992)

Applications

One can structure a proof that $\sqrt{2}$ is irrational in a similar way (as Tammet). Take G. H. Hardy's (1941, pp. 34–36) version of the proof where he reasons with the reductio ad absurdum argument of classic logic: if a given proposition is not true then its denial must be true.

DEFINITIONS: Integers are whole numbers. 'Integral' is its adjectival form. A rational number is a fraction $\frac{a}{b}$ where a and b are integers. They have no common factors.

Hardy argues as follows:

PROPOSITION: $\sqrt{2}$ is irrational.

CLARIFICATION: (A) …To say that '$\sqrt{2}$ is irrational' is merely another way of saying that 2 cannot be expressed in the form

$$\left(\frac{a}{b}\right)^2$$

And this is the same way as saying

(B) $a^2 = 2b^2$ cannot be satisfied by integral values of a and b which have no common factors.

… suppose that B is true. It follows that a^2 is even (since $2b^2$ is divisible by 2), and therefore that a is even (since the square of an odd number is odd). If a is even then

(C) $a = 2c$ for some integral value of c; and therefore $2b^2 = a^2 = (2c)^2 = 4c^2$ or

(D) $b^2 = 2c^2$

AXIOMS: Things which are equal to the same thing are equal to each other and the sum of any quantity of even numbers is even.

CONCLUSION: Hence b^2 is even, and therefore (for the same reason as before) b is even. That is to say, a and b are both even, and so have the common factor 2. This contradicts our hypothesis [that $\sqrt{2}$ is rational] and therefore the hypothesis is false (p. 36).

Arianhrod (2003) is more fulsome with her explanation. She uses more equations between Hardy's parts (A) and (B), as follows.

If $\sqrt{2}$ were rational you would have:

$$\sqrt{2} = \left(\frac{a}{b}\right)$$

Then if you square both sides of the equation you have

$$\left(\sqrt{2}\right)^2 = \left(\frac{a}{b}\right)^2$$

$$2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

$$\text{and } a^2 = 2b^2$$

These equations can never be taken for granted; as Stewart (1995) writes: "readers are expected to fill in routine steps for themselves". Obviously, proofs should not be introduced until students have mastered the basics of algebra.

As an exercise in structure students could be asked to read, or watch, Paul Keating's Mabo speech (Keating, 1992a) and to summarise it under the headings: Proposition, Clarification, Axiom and Conclusion. Lecturers may also like to consider Julia Gillard's misogyny speech in this regard (Gillard, 2012a). Both of these speeches are available on YouTube (Keating, 1992b; Gillard, 2012b).

As an introduction to proofs, teachers can illuminate the structure of a proof with examples of rhetoric and deduction from film and literature. For example, after showing the court scene from the film *To Kill a Mockingbird* (Pakula, 1962) or reading a suitable excerpt, such as the summing up for the jury by Atticus Finch from the Pulitzer-prize-winning novel by Harper Lee (1960), examples of rhetorical language using deduction and logic can be discussed. Australian, English and American television courtroom dramas abound with examples of such rhetoric. The British courtroom drama *Silk* (Salmon, 2014) is one which has been enjoyed by many young women and is available in video form on the Internet. Perhaps teachers can co-operate with English teachers in this regard and this inter-disciplinary cooperation would be very much in the spirit of ACARA (2009).

Bogomolny (2013) contains examples of proofs in five groups: simple, charming, surprising, fallacious and invalid. He includes 27 proofs of the proof of the irrationality of $\sqrt{2}$ and these are all in the 'simple' class. (Please note, some of the activities in the simple and other classes require JavaScript to be activated online.) Students could complete some of the activities included in the 'simple' class and tell the teacher or lecturer and/or fellow students about their favourite proof and/or their favourite proof of the irrationality of $\sqrt{2}$.

Once students have mastered Hardy's version of the proof (Hardy, 1941; Padula, 2006) together with Arianrhod's (2003) highly explanatory version, and perhaps, the proof of the existence of an infinity of prime numbers

(Padula, 2003), Hardy's (1941) other example of an 'elegant' proof, they can progress to more difficult proofs such as: Gödel's incompleteness theorems (Padula, 2011) and the story of the solving of Fermat's last theorem (Singh, 2005).

Note

Some female students from non-English-speaking backgrounds find refuge in and do better at mathematics than in more English-language-based subjects, perhaps because mathematical symbols (or the mixture of words and symbols) make it easier for them to both understand and express mathematical ideas (Padula, 2006). But perhaps some others (for whom English is their first language) would benefit from teaching which takes their strengths in language into account and impresses upon them the knowledge that mathematics is language, that they can do it as well as the boys and that mathematical ideas can and do have extremely powerful and beneficial effects in their world. It would also help if the work of, for example: Hypatia (Greek Alexandrian, Neoplatonist philosopher), Ada Lovelace (arguably the world's first computer programmer) and Sophie Germain (Germain primes) were brought to their attention and/or they are asked to do an internet search for female mathematicians in history who persevered and succeeded in their work in spite of the strong societal prejudices of their times. Seeing mathematics through the spectrum of its very beginnings: its structure and origins, will also assist their and boys' understanding of this truly wondrous discipline that is about, not just pattern, but patterns of ideas (Hardy, 1941).

Conclusion

Talking about the followers of Pythagoras, Tammet (2012, p. 56) writes: "The Pythagoreans thus became the first to understand the world not via tradition (religion) or observation (empirical data), but through imagination—the prizing of pattern over matter".

They also introduced mathematical thought and logic into Western civilisation through argument in their law courts.

Bogomolny (2013) states that the beauty of mathematics lies in its abstraction and the universality of mathematical concepts. Hardy (1941) wrote: "The mathematician's patterns like the painter's or the poet's must be beautiful, the ideas, like the colours or the words must fit together in a harmonious way". Proof is such a pervasive and important part of our culture, the very basis of mathematical thought and knowledge, and it should be taught.

The study of proofs, their underlying structure and aspects of their language such as rhetoric is a good place to start. There is no reason why more students cannot be taught to enjoy and appreciate mathematical beauty in the form of proof and rhetoric, especially girls with their early and often good command of language—remembering that mathematics is language, a language with great power (Arianrhod, 2003) and many codes (Patel, 2008). This will provide a fine foundation for the study of pure mathematics, logic and law at university for both females and males.

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