Probability: A matter of life and death

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Introduction

Life tables are mathematical tables that document probabilities of dying and life expectancies at different ages in a society. Thus, the life table contains some essential features of the health of a population. We will examine life tables from a mathematical point of view.

Probability is often regarded as a difficult branch of mathematics. Life tables provide an interesting approach to introducing concepts in probability. Concepts such as complementary events and conditional probability become easy to understand when presented in the context of a life table. Furthermore, in our experience, they can generate useful class discussion as students begin to link the mathematics to life, and death, in society.

Table 1 traces the survivors by age in a synthetic population from birth to age 100, based on the age-specific probabilities of dying of Victorian females in 2011–2013. The variable x represents age and l_x denotes the number of survivors still alive at their x-th birthday.

The initial number of the population of 100 000 at birth (age 0) is arbitrary but commonly used. This is called the radix of the life table. Of these 100 000 persons, 99 727 survive to their first birthday, which means that 100 000 – 99 727 = 273 die before reaching age 1. Of the 99 727 who reach age 1 year, 99 682 survive to age 5 and therefore 99 727 – 99 682 = 45 die between their first and fifth birthdays. Finally, in the last line of the table, we see that 3312 out of the initial population of 100 000 reach their 100th birthday. These people all die at some point after this age. This table has been extracted from Australian Bureau of Statistics (2014). Newer versions of the table will be published by the ABS as they become available.

Table 1 presents one data column of a life table, namely the number of survivors by age. In practical applications, life tables are more extensive, with additional columns for probabilities of dying by age, deaths by age, total years lived, and future life expectancy by age. Table 1 is also shown in an abridged form, with the initial population at age 0, survivors to age 1, survivors to age 5, and every fifth year of age thereafter to age 100. Unabridged or complete life tables show data for every year of age, rather than every fifth year. However the simple life table in Table 1 will allow us to present the key ideas.

Table 1 Abridged life table l_x column, number of survivors to age x.

Age (x)	Survivors (l_x)	Age (x)	Survivors (l_x)
0	100 000	50	97 770
1	99 727	55	96 770
5	99 682	60	95 314
10	99 652	65	93 140
15	99 613	70	89 837
20	99 506	75	84 459
25	99 395	80	75 566
30	99 282	85	60 381
35	99 096	90	37 807
40	98 825	95	15 460
45	98 412	100	3312

The problems below illustrate how life tables can be used to explain ideas in probability theory. And one of the best ways to learn mathematics is by solving problems! We proceed by posing some problems. Then we provide detailed solutions and comments on links to mathematics in the *Australian Curriculum v. 8.1* and other classroom matters.

Problems

Use the data in Table 1 to answer the following questions.

1. What is the probability that a person survives to their 40th birthday?
2. What is the probability that a person dies before age 80?
3. What is the probability that a person who has reached age 20 will survive until age 80?
4. What is the probability that a person aged 80 survives for at least the next 20 years?
5. What is the probability that a person will die before age 5?
6. What is the probability that a person who is aged 5 will die before age 10?
7. Compare the answers to the last two questions. Why is it more likely that a person will die in the first five years of life than the second five years of life?
8. Consider 5 people with independent probabilities of dying/survival. What is the probability that they all survive to age 80?

9. Consider 5 people with independent probabilities of dying/survival. What is the probability that they all die before age 80?
10. Consider 5 people all aged 30 with independent probabilities of dying/survival. What is the probability that at least 3 survive to age 80?
11. A woman gives birth on her 25th, 30th and 35th birthdays. What is the probability that the woman and her three children all survive to celebrate her 100th birthday, assuming independent probabilities?
12. The life expectancy of the population is the average number of years per person lived from birth to death. Estimate life expectancy of the population using the life table.

Solutions and comments

1. Answer: 0.98825

The probability that a person survives to their 40th birthday is calculated by dividing the number of survivors in the life table at age 40 by the original number of the population at age 0:

$$\frac{l_{40}}{l_0} = \frac{98\ 825}{100\ 000} = 0.98825$$

This problem illustrates the calculation of the probability of a simple event.

2. Answer: 0.24434

The probability that a person dies before age 80 is 1 minus the probability that a person survives to age 80:

$$1 - \left(\frac{l_{80}}{l_0}\right) = 1 - \left(\frac{75\ 566}{100\ 000}\right) = 0.24434$$

This problem illustrates the calculation of the probability of a complementary event, which arises in Year 8 of the *Australian Curriculum*.

3. Answer: 0.75941

The probability that a person who has reached age 20 survives to age 80 is calculating by dividing the number of survivors to age 80 by the number of survivors at age 20:

$$\frac{l_{80}}{l_{20}} = \frac{75\ 556}{99\ 506} = 0.75941$$

This problem illustrates the calculation of conditional probability, which arises in Year 10 of the *Australian Curriculum*. The solution comes naturally even before one formally introduces the concept of conditional probability.

4. Answer: 0.04383

The probability that a person aged 80 survives for at least the next 20 years is the probability that a person aged 80 survives to age 100:

$$\frac{l_{100}}{l_{80}} = \frac{3312}{75\ 566} = 0.04383$$

This problem illustrates again the calculation of conditional probability.

5. Answer: 0.00318

The probability that a person will die before age 5 is 1 minus the probability that a person survives from birth to age 5:

$$1 - \left(\frac{l_5}{l_0}\right) = 1 - \left(\frac{99\ 682}{100\ 000}\right) = 0.00318$$

This problem is another illustration of a complementary event.

6. Answer: 0.00030

The probability that a person who is aged 5 will die before age 10 is

$$1 - \left(\frac{l_{10}}{l_5}\right) = 1 - \left(\frac{99\ 652}{99\ 682}\right) = 0.0030$$

This problem combines calculation of the probability of a complementary event and calculation of conditional probability.

7. Problems 5 and 6 remind us that some children die very young. This makes a connection between our calculations and a sad fact of life. The probability of dying in the first five years of life is around ten times the probability of dying in the next five years in this life table. This is due to infant mortality—deaths before the first birthday—which are commonly much higher than for any other age in childhood. In the life table in Table 1, there are 273 deaths between birth and age 1 (100 000 – 99 727). In contrast, there are only 45 deaths in the next four years between age 1 and age 5 (99 727 – 99 682). This is why, in general, even abridged life tables split the first five years at age 1, to account for higher levels of infant mortality. This problem would be suitable for class discussion, although one should be sensitive to the fact that some students may have witnessed infant mortality in their own families.

8. Answer: 0.24640

The probability that five people all survive to age 80, assuming independence, is:

$$\left(\frac{l_{80}}{l_0}\right)^5 = \left(\frac{75\ 556}{100\ 000}\right)^5 = 0.24640$$

This exercise provides an opportunity to discuss independent events, which arises in Year 10 of the *Australian Curriculum*. The assumption that the life spans of the five people are independent of each other is a reasonable assumption.

9. Answer: 0.00087

The probability that five people all die before age 80, assuming independence, is

$$\left(1-\left(\frac{l_{80}}{l_0}\right)\right)^5=\left(1-\left(\frac{75\ 556}{100\ 000}\right)\right)^5=0.00087$$

Here we combine the notions of complementary events and independence.

10. Answer: 0.90787

This problem is an application of the binomial distribution which arises in Mathematical Methods. The probability that at least 3 of 5 people all aged 30 survive to age 80 is:

$$\binom{5}{3}\left(\frac{l_{80}}{l_{30}}\right)^3\left(1-\left(\frac{l_{80}}{l_{30}}\right)\right)^2+\binom{5}{4}\left(\frac{l_{80}}{l_{30}}\right)^4\left(1-\left(\frac{l_{80}}{l_{30}}\right)\right)^1+\binom{5}{5}\left(\frac{l_{80}}{l_{30}}\right)^5\left(1-\left(\frac{l_{80}}{l_{30}}\right)\right)^0$$

$$=(10)\left(\frac{75\ 566}{99\ 282}\right)^3\left(1-\left(\frac{75\ 566}{99\ 282}\right)\right)^2+(5)\left(\frac{75\ 566}{99\ 282}\right)^4\left(1-\left(\frac{75\ 566}{99\ 282}\right)\right)+\left(\frac{75\ 566}{99\ 282}\right)^5$$

$$=0.90787$$

This example shows that the binomial distribution can arise in the context of life tables constructed from Australian data.

11. Answer: 0.02362

This is the joint probability that the woman survives from 35 to 100, that her first child survives from birth to age 75, her second from birth to age 70, and her third from birth to age 65:

$$\left(\frac{l_{100}}{l_{35}}\right)\left(\frac{l_{75}}{l_0}\right)\left(\frac{l_{70}}{l_0}\right)\left(\frac{l_{65}}{l_0}\right)$$

$$=\left(\frac{3312}{99\ 096}\right)\left(\frac{84\ 459}{100\ 00}\right)\left(\frac{89\ 837}{100\ 00}\right)\left(\frac{93\ 140}{100\ 00}\right)$$

$$=0.02362$$

In essence, the solution does not involve any complicated mathematical concepts. The complexity arises in putting together several simple ideas.

12. Answer: 84.66 years

This problem is more advanced than the others, but it gets to an important application of life tables. It could be suitable for students in their first statistics course at university.

The calculations required are shown in Table 2.

Of the initial population of 100 000, 273 died before their first birthday. Let us assume that these 273 people died, on average, halfway through their first year. Then they lived a total of (273)(0.5) = 136.5 years.

Forty-five people died between age 1 and age 5. If they died, on average, midway between their 1st and 5th birthdays, their total years lived are

$$(45)(3.0) = 135.0 \text{ years.}$$

Continue down the table similarly.

Finally, 3312 people survived to age 100 and died at some point after that. If we assume that they died on average at age 102.5, then their years lived are

$$(3312)(102.5) = 339\,480.0.$$

Now find the total number of years lived by the 100 000 people (8 465 871.5) and then find the average numbers of years lived per person

$$\frac{8\,465\,871.5}{100\,000} = 84.66 \text{ years.}$$

Using one-year age groups instead of five-year age groups, and more refined calculations of average age would lead to a better estimate, but not by much; actual life expectancy for this population was 84.68 years as opposed to 84.66 years calculated here (see Table 2).

Table 2. Calculating life expectancy.

Age x	Survivors l_x	Deaths	Average age (years)	Number of years lived by those who have died = (Deaths) × (Average age)
0	100 000	273	0.5	136.5
1	99 727	45	3.0	135.0
5	99 682	30	7.5	225.0
10	99 652	39	12.5	487.5
15	99 613	107	17.5	1872.5
20	99 506	111	22.5	2497.5
25	99 395	113	27.5	3107.5
30	99 282	186	32.5	6045.0
35	99 096	271	37.5	10 162.5
40	98 825	413	42.5	17 552.5
45	98 412	642	47.5	30 495.0
50	97 770	1000	52.5	52 500.0
55	96 770	1456	57.5	83 720.0
60	95 314	2174	62.5	135 875.0
65	93 140	3303	67.5	222 952.5
70	89 837	5378	72.5	389 905.0
75	84 459	8893	77.5	689 207.5
80	75 566	15 185	82.5	1 252 762.5
85	60 381	22 574	87.5	1 975 225.0
90	37 807	22 347	92.5	2 067 097.5
95	15 460	12 148	97.5	1 184 430.0
100	3312	3312	102.5	339 480.0
			Total	8 465 871.5
			Average life expectancy = $\frac{\text{Total}}{l_0}$	84.66

Conclusions

Life tables are interesting mathematical tools that have many applications in health sciences and demography. They can be used to estimate life expectancy of a population which is a fundamental summary measure of the health of a population. Indeed, closing the gap between life expectancy of Indigenous Australians and that of the rest of the nation is one of the stated targets of the Council of Australian Governments (COAG). By the way, the very readable book by Angus Deaton (2013), who was awarded the Nobel Prize for Economic Sciences in 2015, makes considerable use of life expectancy in discussing inequalities in health throughout history, and across the globe. Life tables also are important in calculating relative survival statistics for cancer; see Thursfield et al. (2012, Appendix VI).

However, the main aim of this paper has been to show how life tables can be used to illustrate basic ideas in probability in the classroom. They lead to interesting discussions in the classroom on topics such as infant mortality and life expectancy—and the broader social contexts of these issues. Life tables can contribute to realising the aims of the *Australian Curriculum* at many levels.

There is much more to study on this topic and demography in general. For example, Keyfitz and Beekman (1984) provide an introduction to demography through problem solving.

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