Understanding magnitudes to understand fractions

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Fractions are known to be difficult to learn and difficult to teach, yet they are vital for students to have access to further mathematical concepts. This article uses evidence to support teachers employing teaching methods that focus on the conceptual understanding of the magnitude of fractions.

Introduction

As the joke goes, “three out of two people have trouble with fractions”. Fractions have been used for centuries, not just in mathematics but in a great variety of everyday life situations. Yet, they are very difficult for students to grasp and master. In this article, I discuss principles that teachers can use to directly address the core difficulties that hold back children’s understanding of fractions.

Fractions represent a stumbling block for many primary school children, and there are a number of reasons why we should focus on improving this particular topic. Understanding what fractions are and how to use them are fundamental stepping stones for learning higher concepts in mathematics. Indeed, competency with fractions predicts children’s performance in algebra and their general mathematical achievement in later years (Siegler et al., 2012). Conversely, difficulties in learning fractions can lead to mathematics anxiety and affect opportunities for further engagement in mathematics and science. Nicolas Rouche, a Belgian mathematician who devoted a large part of his career to mathematics education, once wrote, “Fractions are like harmful bugs attacking school kids, and their bites result in unending intellectual and moral after-effects” (Rouche, 1998, page 1). Therefore, it is crucial to children’s mathematical development that teaching instils a proper understanding of fractions.

As shown by evidence from research in cognitive neuroscience and education, being able to understand the magnitude of fractions is an essential and unavoidable stage in the general understanding of fractions (Siegler, Fazio, Bailey, & Zhou 2013). The level of a student’s understanding of fraction magnitude is a strong predictor of competence in algebra (Booth, Newton, & Twiss-Garrity, 2014), and has been positively related to overall mathematics achievement in countries with cultural and educational practices as diverse as the USA, Belgium and China (Torbeyns, Schneider, Xin, & Siegler, 2015). The appreciation of magnitude is one of the hardest concepts to learn, and can be seen as a consequence of children’s failure to understand that natural numbers and fractions have different properties and characteristics.

Procedural and conceptual knowledge

Being able to use fractions requires both procedural and conceptual knowledge. Procedural knowledge can be defined as sequences of actions, as algorithms or as procedures useful to solve problems, while conceptual knowledge (also called conceptual understanding) can be defined as the understanding of the principles ruling a domain, and the interrelations between the different parts of knowledge within a domain (Rittle-Johnson & Alibali, 1999). In other words, you have to know the rules for performing actions on fractions (procedural knowledge), and you have to know why those rules apply (conceptual knowledge). The *Australian Curriculum: Mathematics* includes procedural and conceptual knowledge (as fluency and understanding, respectively) among the four key proficiency strands that children require to be able to successfully learn maths.

Procedural and conceptual knowledge are built up in a process whereby they influence each other's development in an interactive and iterative fashion (Rittle-Johnson & Alibali, 1999). There is a large body of evidence suggesting that conceptual knowledge has a greater influence on the development of procedural knowledge than vice versa (e.g. Byrnes & Wasik, 1991).

Our analysis of current teaching practice revealed a great variety of ways to teach fractions, often with more focus on procedures than concepts, and with fractions isolated from general mathematics lessons (Gabriel et al., 2013a). This analysis showed that procedural knowledge was not sufficient in the absence of conceptual understanding; for example, despite intensive procedural training to find the lowest common denominator, children were generally still unable to add or subtract fractions with different denominators. Other studies have similarly found that there is generally not enough time devoted to ensuring conceptual understanding (Charalambous & Pitta-Pantazi, 2007; Hiebert, 2003). Additionally, it has been reported that the teaching of concepts was too narrowly focused on part-whole interpretation rather than emphasising that fractions are numbers with their own magnitudes (Fuchs et al., 2013).

One way of addressing the problem that children have when learning fractions is to ensure that they have a solid conceptual understanding. This would give children a firm grasp of what fractions actually mean, a good feel for how procedures on fractions should work, and so a better ability to catch their own mistakes and to avoid making procedural errors.

Common errors

There are many apparently distinct ways in which problems with fractions can manifest. One common type of error comes from applying procedures without understanding the underlying concepts (Kerslake, 1986). An example is given in the analysis of an interview with a school child conducted by Kerslake (1986, page 21):

[Having calculated] $\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}=1\frac{5}{12}$ … [the pupil] sees her taught method just as a means of obtaining the right answer, and has no idea of why the 'adding numerators and denominators' method, which one feels she really still prefers, is not appropriate.

This example alludes to the most common error for children (and even adults), which is to wrongly process numerators and denominators as two separate whole numbers and to apply procedures that can only be used with whole numbers (Bonato, Fabbri, Umilta, & Zorzi, 2007; Gabriel et al., 2013a; 2013b). Consequently, typical mistakes appear when adding or subtracting fractions (e.g. $\frac{1}{3}+\frac{1}{2}=\frac{2}{5}$), and also when comparing fractions (e.g. $\frac{1}{7}>\frac{1}{3}$, because 7 is larger than 3). Also, many primary school children always consider fractions as being entities smaller than one, many of them do not seem to understand equivalent fractions, and many have difficulties placing fractions on number lines (Gabriel et al., 2013a; Siegler, Thompson, & Schneider, 2011; Stafylidou & Vosniadou, 2004). Each of these errors can ultimately be attributed to a lack of conceptual understanding, and more specifically, a failure to understand the magnitude of fractions.

Interventions

These observations suggest that a greater emphasis on concepts, specifically magnitude, should improve the efficacy of teaching fractions. This idea has been tested experimentally with children of different age groups in different countries. In this section, I will describe two intervention studies, one from our group (Gabriel et al., 2012) and one from Fuchs and colleagues (2013). These studies both introduced adapted teaching regimes that shifted the focus from teaching procedures to teaching the concept of magnitude.

In our study, we designed and conducted an intervention for Belgian primary school children, aged between 10 and 11 (Gabriel, et al. 2012). We tested 292 children from 4 different schools for 12 weeks for one hour per week. Within each school, we tested four whole classes and randomly assigned two of these classes to the intervention while keeping the other two as controls. The control classes followed their traditional lessons which focused on the part-whole concept and on rote learning of procedures, whereas the intervention group instead followed lessons designed by us.

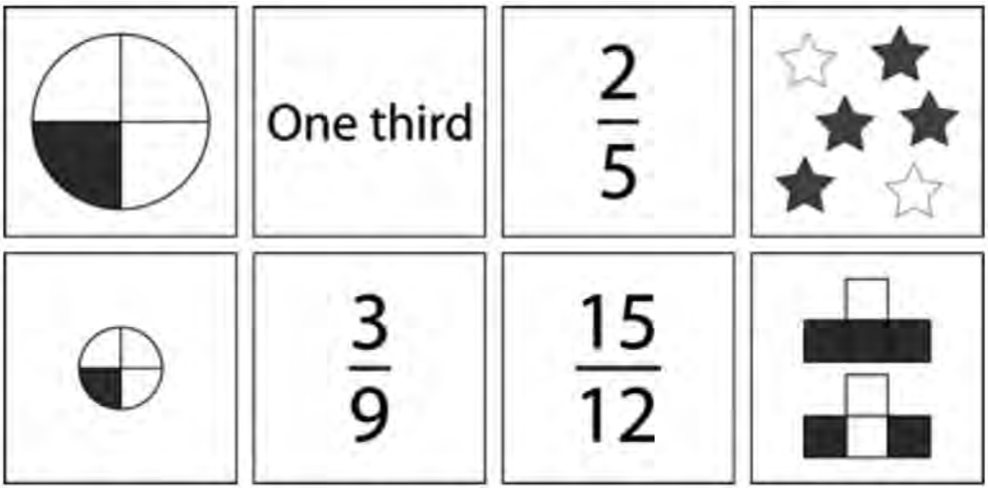

Figure 1: Examples of card faces used in the intervention.

The intervention was based on the use of the concrete-representational-abstract sequence (CRA) and excluded procedural instruction. CRA has been shown to be effective for developing conceptual understanding in several areas of mathematics, including fractions (Butler et al., 2003; Maccini & Hughes, 1997). In this schema, the concrete element of the sequence was a set of wooden disks, cut into a range of sizes from halves to twelfths. Children used the disks as a reference to manipulate quantities and visualise the sizes of fractions. The representational element was a set of cards displaying different representations of fractions (see Figure 1). The abstract element was the step of getting children to consider fractions as numbers in and of themselves.

In the process of designing this intervention, a common impression that we received from the children was that they strongly disliked fractions. Play is known to have positive effects on the learning process (Sawyer, 2006), and so in our design, we used play to try to keep the children motivated and engaged by associating fractions with fun.

The intervention was conducted jointly by researchers and teachers. There were twenty game sessions of 30 minutes each. Within an ordinary size class, children were split into small groups of three to five and played various card games (e.g. memory and blackjack) adapted to use our fraction cards. These games required the children to estimate, compare, and combine fractions represented either symbolically or as figures. Over the course of the intervention, we gradually introduced more complicated fractions, equivalent fractions, improper fractions and a wider breadth of representations of fractions.

We tested all of the children (intervention and control groups) before and after the intervention period. From this we were able to measure the change in conceptual and procedural knowledge of fractions, and to assess the impact of the intervention. The intervention led to a 15–20% improvement in conceptual understanding of fractions (i.e. estimating and comparing fractions, and placing fractions on a number line). The children were able to use their conceptual knowledge to perform simple additions with familiar fractions, but they did not show wide-scale transfer from conceptual to procedural knowledge. Conversely, children in the control group, who received traditional procedure-heavy lessons, improved in procedural skills such as simplification of fractions, but showed no improvement at all in their understanding of concepts.

Fuchs and colleagues (2013) conducted a similar study with 9 and 10 year-old American children. Unlike our study, Fuchs and colleagues (2013) focused their intervention on students with poor mathematics performance. They assigned children to three different groups: an intervention group of poor-performing students (n = 129); a control group of students of similar performance to the intervention group (n = 130); and a target group of normal-to-high performing students to gauge the scale of improvement in the intervention group (n = 282). The intervention and control groups were sampled from children who scored in the bottom 35% of a mathematics achievement test (Wide Range Achievement Test–4 [WRAT-4]; Wilkinson & Robertson, 2006), and the target group was randomly sampled from the rest of the children.

Similarly to our study, the intervention focused on magnitude (i.e. representing, ordering, comparing and placing fractions on number lines), and the control and target groups received instructions which focused on procedures and on the part-whole interpretation of fractions. Their intervention relied on the same theory as ours, but the content and learner activities differed greatly. Children in the intervention group were taken out of their regular classes and put into small groups of three, each group having its own instructor. The intervention was based on CRA, but with far fewer elements of play. The first 22 lessons of the intervention were devoted to teaching concepts with a very heavy emphasis on magnitude, and the remaining five lessons were devoted to procedures. Using a very similar battery of pre- and post-tests to ours, they found that their intervention led to increased improvements in both conceptual and procedural knowledge of fractions over the traditional lessons, and they concluded that this was mediated through the improvement in the understanding of magnitude.

These studies show that gains can be made in improving children's conceptual understanding of fractions, and that magnitude is an important factor in achieving this. They also show that it is necessary to include both conceptual and procedural instruction, and the optimum combination is likely to include a greater emphasis on concepts. Further research will be necessary to clarify when and how such interventions can best improve the understanding of the magnitude of fractions, and how this can impact performance in fraction arithmetic and in mathematics in general. Longitudinal studies tracking the same pupils over time would be ideal for determining if, and to what extent, this improvement of conceptual understanding transfers to other areas of mathematics. We could then see if that type of intervention has long-lasting positive effects and whether such conceptual improvements facilitate children's acquisition and use of procedures.

Teaching implications

From these studies, we can see that by increasing the emphasis on developing conceptual knowledge, it is possible to get children to develop an all-round better understanding of fractions. In current mathematics education, when fractions are taught, children usually learn procedures mechanically and very little time is devoted to the teaching of concepts (Gabriel et al., 2013a; Garet et al., 2011). However, children cannot gain a conceptual understanding of fractions through procedural knowledge alone, and it is clear that teaching of concepts and procedures needs to be rebalanced. This fits neatly with the shift in the strategic intent of the *Australian Curriculum* to emphasise learning for understanding rather than simply learning for knowledge.

There are many published suggestions for introducing concepts more intentionally in the teaching of fractions (e.g. Butler et al., 2003; Clarke, Roche & Mitchell, 2008; Gabriel et al., 2012). From my research, perhaps the most promising way of doing this is to use the CRA method with elements of play. Using materials that encourage playful learning, we can easily introduce many different representations of the magnitude of fractions, and encourage children to estimate the magnitudes of fractions when doing arithmetic operations. Playing with physical objects is an efficient means to get them to grasp the properties of fractions and to appreciate magnitude (Gabriel et al., 2012).

Play actively encourages students to interact with each other, in a way that is not possible with more traditional procedural instruction. This gives teachers more opportunities to listen to the children's thoughts, ideas and comments, and so better assess how their understanding develops. Using play as a teaching instrument also allows teachers to incorporate more open-ended questions in their feedback (e.g. "So, what do you think?" "Is it always the right answer?" "What do you think would happen if…?"), thus creating an environment that encourages children to construct their own understanding.

Conclusion

Fractions are difficult to learn. However, evidence is mounting that the problems children have when learning about fractions can be largely overcome by ensuring that they understand that fractions are numbers that have magnitudes. It is seemingly straightforward to simply increase the emphasis on conceptual knowledge in lessons on fractions. The challenge here is to balance the need to teach children the procedural knowledge for using fractions with making sure they are instilled with the conceptual knowledge to truly understand how, when and why to use fractions. Teaching methods that incorporate play appear to be an efficient way to introduce fraction conceptual knowledge to children, effectively pushing them to actively construct knowledge in a way that is meaningful to them.

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