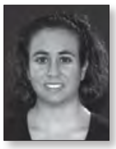
Using Disney's *Frozen* to motivate mathematics: Bringing fractals into the classroom

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The movie *Frozen* took the world by storm and this global popularity of the movie and its music can be harnessed by teachers of mathematics. This article builds on the "frozen fractal" lyric from *Let It Go* to incorporate fractal geometry into primary mathematics classrooms.

Are you interested in using your students' enthusiasm with the movie *Frozen* (Buck, Lee, Morris, 2013) to teach geometry or fractions? Maybe you want to teach mathematics and art together and are looking for ideas? This article will show you some engaging activities you can use to tie together art, primary mathematics, and the movie *Frozen*, by introducing your students to fractal geometry. Through this geometry, we can teach students mathematical concepts like rational numbers, measurement, properties of shapes, similarity, and symmetry.

Frozen: The movie

The animated movie *Frozen* has taken the world by storm, earning more than AU$1.6 billion at the box office worldwide. It is currently the fifth highest grossing film of all time and is the highest grossing animated film (Konnikova, 2014). The song *Let It Go* has been translated into 41 different languages (NPR Music, 2014) and the movie's soundtrack earned the title of the top selling global album for the year 2014, with over 10 million copies sold (IFPI, 2015). Because of the global popularity of the movie, Disney has officially announced their plans to produce *Frozen 2,* a sequel to the original movie (Graser, 2015).

We, as mathematics teachers, can use the continued popularity of the movie and its music to interest our students in the field of mathematics. Most of us have probably noticed the line from the song *Let It Go* that refers to "frozen fractals" swirling around (Anderson-Lopez & Lopez, 2013). We may have been impressed by the clever usage of the mathematical term. But have we stopped to consider "How can I include a discussion about fractals in my mathematics classroom?" or "Can I relate fractals to the mathematics curriculum I need to teach?"

What is a fractal?

A fractal has formally been defined as "a rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a smaller version of the whole" (Thomas, 2002, p. 203). In other words, a fractal is an object that looks the 'same' even as we zoom in. We call this quality 'self-similarity on all scales.' Consider the example in Figure 1. Notice that one branch of the tree, when magnified and rotated, looks like the original tree. So we zoomed in on a branch but the branch is similar to the original tree. If you zoomed in on a branch of this new tree, the same thing would happen again. This is called self-similarity.

Because of self-similarity, fractals are surprisingly easy to create. Consider the tree from Figure 1. To generate this tree, start with a line that is a fixed length, say 32 units long (e.g. 32 cm). Next, create branches that are half this length, 16 units long, and that grow at 120° angles from the original branch. This is Iteration 1 of the fractal. Now keep repeating this process. So, for both of the 16-unit long branches, create 8-unit long branches at 120° angles. This creates Iteration 2. Figure 2 shows this process at work. Because you keep repeating the same steps to smaller and smaller branches, you

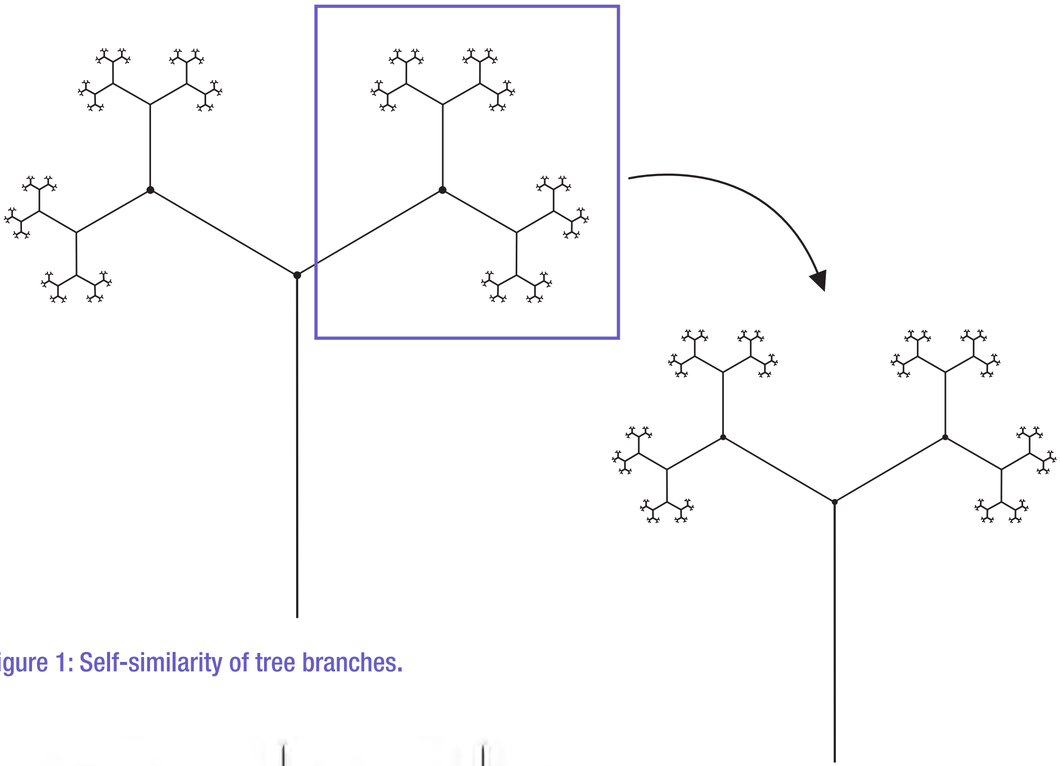


Figure 1: Self-similarity of tree branches.

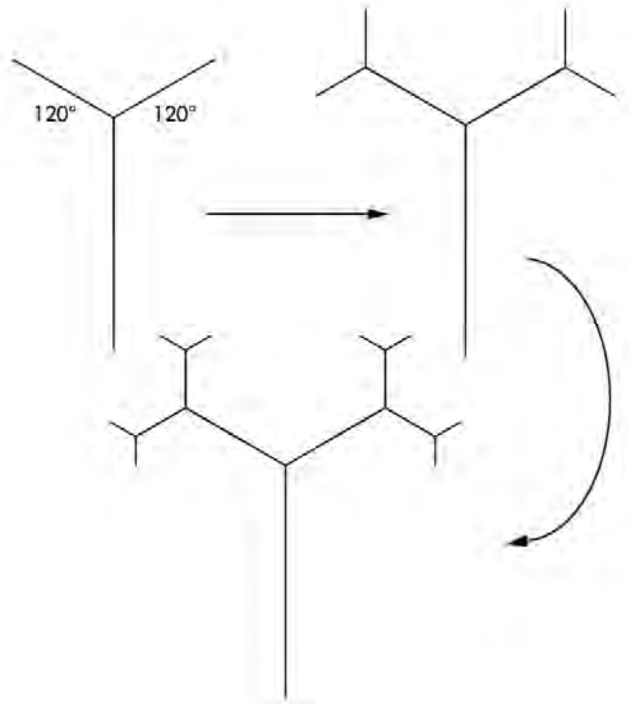


Figure 2: Iterations 1, 2, and 3 of this fractal.

will end up with a figure that has self-similarity. If you did this forever, you would have self-similarity on all scales. Numerous computer programs and applications ('apps') also exist for the purpose of generating fractals.

A brief history of fractal geometry

The concepts behind fractal geometry were first considered by mathematicians during the late 19th and early 20th centuries but at that time were considered too complicated to investigate deeply. It was the later advancement of computers that eventually made the mathematical exploration of fractals possible (Berlinghoff & Gouvêa, 2004).

Benoit Mandelbrot (1924-2010), often known as the "father of fractal geometry" (Gomory, 2010), revived the study of fractal geometry in the late 1960's, eventually coining the term 'fractal' (stemming from the Latin word for 'broken') in the year 1975.

Mandelbrot's interest in this field stemmed from his observation that many objects found in nature are not modelled nicely by lines, circles, cones, or spheres, but rather have a natural, complicated roughness to them. Mandelbrot's most famous and influential work was his 1985 book titled *The Fractal Geometry of Nature* (Gomory, 2010).

Fractals found in nature

Even though the mathematical study of fractals is relatively new, fractals (or fractal-like objects) have always appeared in nature. The concepts behind fractal geometry generally fall under one of two categories: geometric fractals or random fractals. One example of a geometric fractal found in nature is a fern leaf. Consider the fern leaf in Figure 3.

First, look at the entire leaf. Then look at only one branch of the leaf. That one branch looks like a shrunken version of the whole, showing self-similarity. Of course, as with all objects in nature, as we continue to 'zoom in', eventually this self-similarity feature begins to fail. Despite this, many objects in nature are good models of fractals.

Another example of a geometric fractal found in nature is cauliflower. As can be seen in Figure 4, the sub-branches of the cauliflower are self-similar to the cauliflower as a whole.

Random fractals, as opposed to geometric fractals, do not have true self-similarity in that we do not see the exact same pattern repeated over and over. Instead, random fractals have the quality that as one zooms in, we see an approximation of the original pattern. A few objects found in nature that model random fractals are tree branches, lightning, and the Grand Canyon, as seen in Figures 5, 6, and 7.

For more photographs of fractal objects found in nature to share with your students, see McGuire (1991) or conduct an online image search. For example, Australian mathematician Paul Bourke has carried out work using *Google Earth* to capture aerial photographs of the planet that demonstrate fractals, and awe-inspiring images of his work can be found online (Bourke, 2012).

Computer generated fractals

Snowflakes are another example of a fractal, which may be where the lyric in the song *Let It Go* comes from. Although snowflakes are found in nature, they are quite difficult to photograph. Therefore, computer generated snowflakes, such as the one seen in Figure 8, are more easily accessible. To see that the snow-flake shown in Figure 8 is a fractal, first examine the top of the snowflake. We see the top is made up of four parts, each which is like the whole (see Figure 9). As we continue to zoom in on the snowflake, we see the self-similarity of the crystals. This type of fractal is called the Koch Snowflake, and is an example of a "frozen fractal", as referenced in the lyrics of *Let It Go.*

Figure 3: The fractal nature of ferns (above top). Photography by Christine Phelps. Figure 4: The fractal nature of cauliflower (above 2). Figure 5: The fractal nature of tree branches (above 3). Figure 6: The fractal nature of lightning (above 4). Figure 7: The fractal nature of the Grand Canyon (above 5). Figures 4–7 photography courtesy of Mike Piatek-Jimenez.

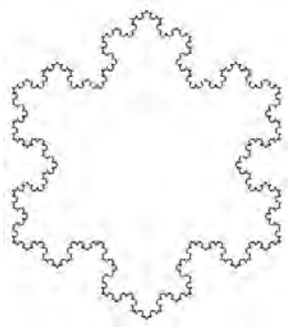


Figure 8: Iteration 4 of the Koch Snowflake.

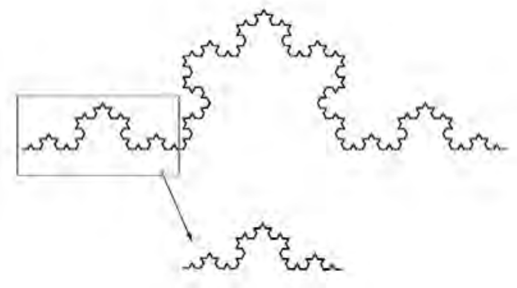


Figure 9: Self-similarity of the Koch Snowflake.

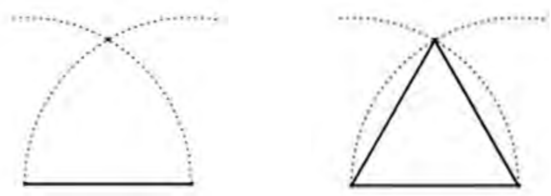
Fractal activities for the primary school classroom

While fractals may look very complex, and actually do in fact include complex mathematics, because they can be created by repeating the same process over and over again, they are easily within reach of primary school students. Though fractals are not a typical mathematics topic in the primary school curriculum, including them could be a way to achieve the *Australian Curriculum: Mathematics* standards while also building on students' interest in the movie *Frozen* (Australian Curriculum and Assessment Reporting Authority [ACARA], 2014).

Below we suggest several activities that build on the frozen fractal lyric and incorporate primary school level mathematics. Wherever each activity meets one of the *Australian Curriculum: Mathematics* strands for Year 1 through Year 7, we will label it with the strand's reference code.

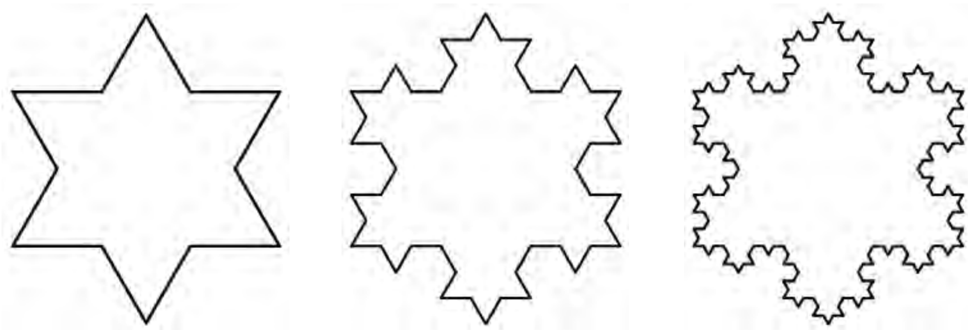
Koch Snowflake and Anti-Snowflake

We already referenced the Koch Snowflake, a fractal that looks like a snowflake. Students can create these snowflakes in class either by hand or with dynamic geometry software. Here is one possible way to have students make this snowflake. First, introduce the activity by talking about the movie *Frozen* and the frozen fractal lyrics from the song *Let It Go*. Tell them that today they are going to make a frozen fractal. Then, ask students to draw an equilateral triangle on a large sheet of paper. You could have one piece of poster paper for each group and students could work together. One way to have students make an equilateral triangle is for them to use a protractor to create the 60° angles and a ruler to make the sides of the triangle the same length. A faster way would be for the students to use a compass and a straight edge. They can begin with one side of the triangle drawn and open theircompass to be the length of that side. Then by putting the sharp point of their compass on each end of the line segment and making arcs above it, where the two arcs intersect is the third vertex of the triangle (see Figure 10(a); the arcs made by the compass are the dotted lines). They can then use their straight edge to draw in the final two sides of the triangle (see Figure 10(b)). Once students erase the compass arcs, they are left with an equilateral triangle.



(a) (b)

Figure 10 (a) and 10 (b): Creating an equilateral triangle with a compass and straight edge.



(a) (b) (c)

Figures 11 (a), (b) and (c): Multiple iterations of the Koch Snowflake.

For the next step, students will need to use rulers to divide each side of the triangle into thirds. On the middle third of each side, the students will need to construct another equilateral triangle, as can be seen in Figure 11(a). For the next iteration, the students will divide each edge (currently there are 12 of them) into thirds, and create a new equilateral triangle in the middle of each edge, as in Figure 11(b). Students can continue this process as much as time allows, to create something similar to Figure 8. If needed, as the teacher, you can provide a final example of what it should look like. You can also discuss what they have learned about "frozen fractals".

Another related fractal that can be used as an extension to the Koch Snowflake is the Koch Anti-Snowflake. To create the Anti-Snowflake, rather than creating equilateral triangles that protrude from the figure each time, students will create equilateral triangles that point inward. A few iterations for the Koch Anti-Snowflake are found in Figure 12.

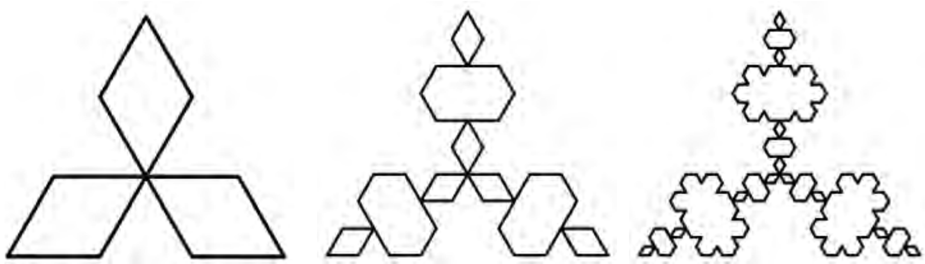
The Koch Snowflake and Anti-Snowflake can be added to multiple places in a mathematics curriculum. Some mathematical topics relating to the construction of the Koch Snowflake and Anti-Snowflake are:

1. **Use of mathematical tools:** Through this activity, students will get practice using rulers, a protractor, and/or a compass. Students can also be introduced to the concept of geometric constructions. [ACMMG112]
2. **Discovering properties of shapes:** For example, if using the compass and straight edge method for creating their equilateral triangles, students can then use a protractor to measure the angles of the triangles to discover that equilateral triangles all have 60^{o} angles. This may lead to rich discussions about categorising triangles by side lengths and by angle measures. [ACMMG022; ACMMG042; ACMMG165]
3. **Symmetry and similarity:** Students can use transformations to explore rotational and reflectional symmetry of figures. Because this figure demonstrates self-similarity when you zoom in on the sides, students can begin discussing concepts of similarity. [ACMMG045; ACMMG046; ACMMG091; ACMMG114; ACMMG142]

We will now introduce other fractals that could be taught after the Koch Snowflake and Koch Anti-Snowflake, to build an entire unit on fractals. This will help students better understand the concepts behind fractals and the relevance of the words used in the song.

Controlled Blue Wash fractal

For this activity, students will begin with a large rectangle that is twice as long as it is tall. Students will need to determine the half-way mark of the longer side of the rectangle and will draw a vertical line to divide the rectangle in half, as shown in Figure 13(a). One option is to print this rectangle on graph paper and students can count squares to determine the halfway mark. Another option is to have students use rulers to find their measurements. Next, the students can draw a horizontal line to divide the square on the right in half, as seen in Figure 13(b). The students can then vertically divide the rectangle on the upper right in half, as in Figure 13(c). As students continue this process they will eventually end up with a figure such as the one in Figure 13(d). The final figure in Figure 13(d) is a geometric fractal. We can see the self-similarity as we zoom in on the top right of the rectangle. A class discussion could focus on the commonalities between this fractal and the Koch Snowflake fractal.



(a) (b) (c)

Figures 12 (a), (b) and (c): Multiple iterations of the Koch Anti-Snowflake.

There are many mathematical topics relating to the construction of the Controlled Blue Wash fractal:

1. **Fractions:** At each stage of the figure, we can talk about unit fractions, equivalent fractions, and operations with rational numbers. We can ask students questions like, "Which piece of this figure represents $\frac{1}{8}$ of the whole? If we take half of that piece, what fraction of the whole is the new piece?" This visual helps students develop the concepts behind multiplying fractions by one-half. Students can also practice adding fractions with different denominators and have visual representations to develop the concepts behind common denominators. [ACMNA016; ACMNA033; ACMNA058; ACMNA126; ACMNA155]
2. **Area and perimeter:** Students can explore how the area and perimeter of the newly created rectangles decrease each time, and can try to find a pattern. They can also observe that the pattern is different for the change in area than it is for the change in perimeter. [ACMMG037; ACMMG061; ACMMG290; ACMMG109]

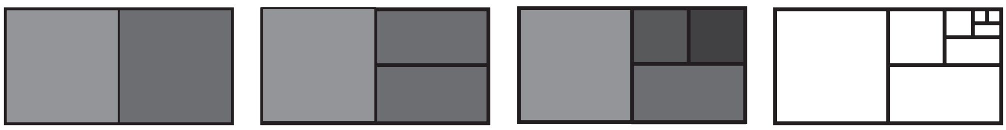
Sierpinski Triangle

For this activity, students will begin by creating a triangle on a large sheet of paper, just as for the Koch Snowflake. In our figures we will use an isosceles triangle, but equilateral or scalene triangles can be used as well. Using a ruler, students will need to find the midpoint of each of the sides of the original triangle, as in Figure 14(a).

The students should then connect the three midpoints to form a new triangle, as in Figure 14(b). For the next iteration, the students will need to find the midpoints of the sides of the three triangles in the corners and connect their midpoints, making Figure 14(c). Repeatedly continuing this process, students will end up with a figure such as the one in Figure 14(d).

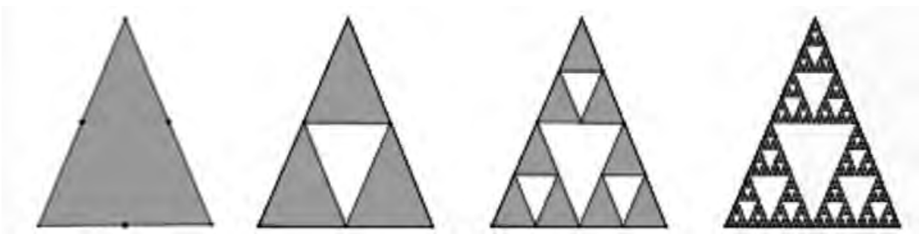
Mathematical topics relating to the construction of the Sierpinski Triangle:

1. **Fractions:** This fractal also leads to rich discussions about operations on fractions. For example, at each iteration, students can be asked, "What fraction of the whole is shaded? If in the next stage we shade in three-fourths of each piece, what fraction of the whole will be shaded?" This provides students with a visual of multiplication by three-fourths. [ACMNA103; ACMNA126; ACMNA154]
2. **Area and perimeter:** Students can confirm that congruent shapes have equal areas and equal perimeters. Students can explore patterns of how the area and perimeter of the smallest triangles change with each iteration. [ACMMG061; ACMMG290; ACMMG137; ACMMG159]
3. **Symmetry:** When constructed from an equilateral triangle, both reflectional



(a) (b) (c) (d)

Figure 13: Multiple iterations of the Controlled Blue Wash fractal.



(a) (b) (c) (d)

Figure 14: Multiple iterations of the Sierpinski Triangle.

symmetry and rotational symmetry exists. For example, lines of symmetry can be drawn from each vertex of the big triangle to the midpoint of the opposite side. Students can explore this concept by drawing the lines of symmetry with each iteration and noting that the lines of symmetry stay the same across all iterations. [ACMMG045; ACMMG046; ACMMG091; ACMMG114; ACMMG142]

The Dragon Curve

This fractal is different from the others described in that it has the qualities of a random fractal rather than a geometric fractal. This might be suitable for a long unit on fractals or a more advanced class. For this activity, the students will each need two pieces of tracing paper and a protractor to make 90º angles with. The students should begin with a short line segment AB on the lower right side of a piece of paper. Using the tracing paper, students should rotate segment AB counter-clockwise by 90º around point B, as in Figure 15(a). Again using the tracing paper, students should rotate the new figure counter-clockwise by 90º around the new endpoint created in the previous iteration, as seen in Figure 15(b). As students continue this pattern, they will make the dragon curve, seen with 6 iterations in Figure 15(c).

Mathematical topics relating to the construction of the Dragon Curve:

1. **Measurement:** Students will practice using a protractor to create 90º angles. This can also spur discussions of parallel and perpendicular lines. [ACMMG112; ACMMG164]
2. **Symmetry:** Students can get hands-on experience working with rotations and can explore how rotational symmetry is different from reflection symmetry. The activity can also be done on a Cartesian plane, which can open discussions about length, distance, and even the Pythagorean Theorem. [ACMMG046; ACMMG114; ACMMG181]

Throughout the unit, you can regularly refer to the movie *Frozen* and post students' fractals around the classroom, creating a "frozen fractal" inspired theme that builds on important mathematics.

Fractals and proficiency strands

Frozen provides a good introduction to motivate and engage students in learning new mathematics. Teaching fractals in the primary school classroom can also be a great way to achieve the *Australian Curriculum: Mathematics* proficiency strands. Fractals are especially relevant to problem solving and reasoning. Consider the Koch Snowflake example. Students can be encouraged to discover the properties of the shape and its similarities and symmetries for themselves, building problem solving skills. Then they can be encouraged to justify their conjectures and explain their thinking, which aids in building their reasoning skills. All of this can occur in an unfamiliar and meaningful situation, an important part of problem solving (ACARA, 2014).

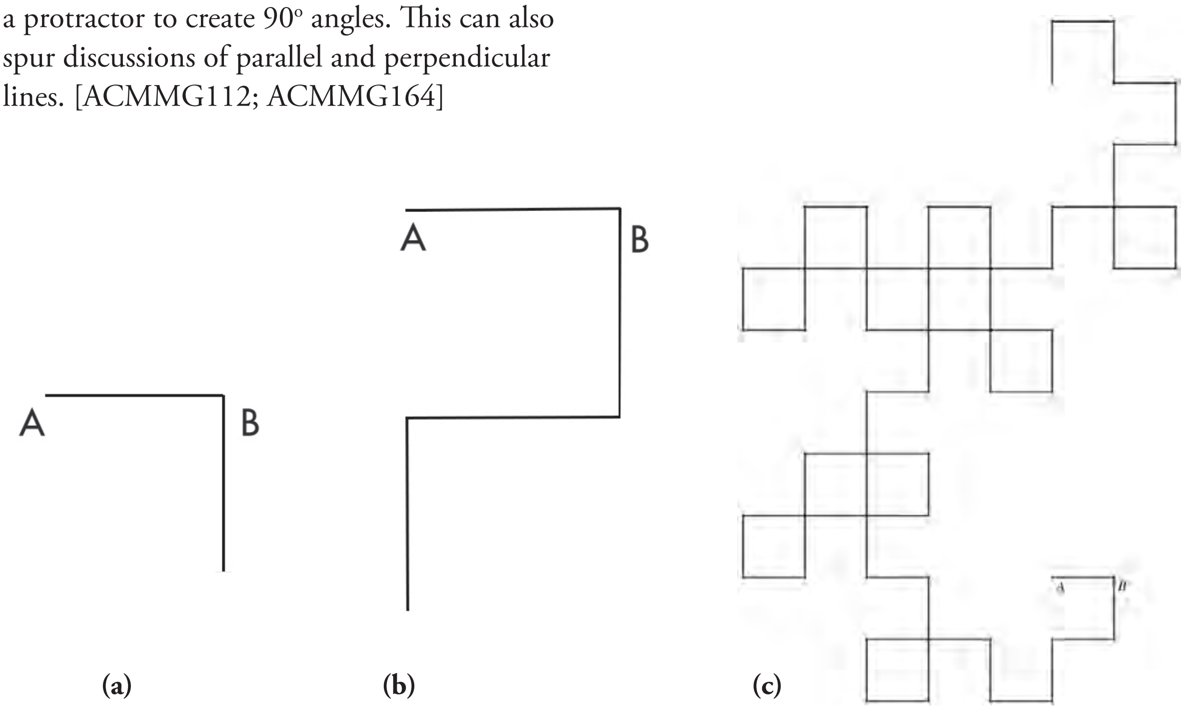


Figure 15: Multiple iterations of the Dragon Curve.

By incorporating fractals in the classroom, you can build on students' interest in the movie *Frozen* to help them develop valuable mathematical knowledge. We have found that teaching fractals is a fun and rewarding experience, and we hope that you do too.

Acknowledgements

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