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Non-mathematics Students' Reasoning in Calculus Tasks

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Abstract

This paper investigates the reasoning of first year non-mathematics students in non-routine calculus tasks. The students in this study were accustomed to imitative reasoning from their primary and secondary education. In order to move from imitative reasoning toward more creative reasoning, non-routine tasks were implemented as an explicit part of the students' calculus course. We examined the reasoning of six students in the middle of the calculus course and at the end of the course. The analyzed data showed that the students' reasoning differed in the middle of the course and after having passed the course, in terms of having more characteristics of creative reasoning. In addition, we found several negative met-befores and met-afters affecting the students' knowledge and interfering with their reasoning.

Key words: Reasoning; Non-mathematics students; Calculus; Non-routine tasks

Introduction

The tasks that are given to students in a mathematics course can be categorized as routine and non-routine tasks. The main difference between a non-routine and a routine task is that in a non-routine task the solver has to, at least partially, construct his/her solution method, while in a routine task, the method is already known by the solver or provided by an external source such as the book or the teacher (Lithner, 2012). We can ask ourselves what competencies students in tertiary education are developing when they are given the same types of tasks in exams as they have met on their mathematics courses. When it comes to exam requirements, students do expect this kind of situation to occur. Their expectations come from the previous exam papers accessible to them, and it is usually part of a didactical contract between students and lecturers (ibid.). The reasoning employed in such situations Lithner (2003) calls imitative reasoning. This kind of reasoning is founded on copying task solutions, for example by looking at a textbook example or remembering an algorithm. Examining final exams from the introductory calculus courses at Swedish universities, Bergqvist (2007) found that most of the tasks can be solved using imitative reasoning. Teachers were concerned with the exam pass rate; therefore, the majority demanded imitative reasoning (Bergqvist, 2012). The situation is no different in Croatian universities. There are no published studies, but browsing through exams and course material accessible on the web pages of various Croatian universities, one can reach a similar conclusion, especially when it comes to calculus courses for non-mathematics students.

Lithner (2008) points out that even after many years of research, students still perform inefficient rote thinking and rely on imitative reasoning. The main problem with such reasoning is that students do not develop the conceptual knowledge necessary for learning different aspects of mathematics (Lithner, 2004). Cox (1994) argued that many first-year university students obtain good grades by concentrating on routine topics, instead of aiming at a deep understanding of fundamental facts. While many teachers are trying to reduce the complexity of mathematical concepts and processes, students are trying to cope with curriculum goals so they often use quicker short-cut strategies to learning and passing exams (Schoenfeld, 1991).

The research conducted by Glasnović Gracin (2011) showed that teachers at primary and secondary educational levels in Croatia rely heavily on textbooks especially for practicing taught subject matter. In an analysis of textbook exercises, Glasnović Gracin (2011) showed a predominance of operation activities on the reproduction or simple-connections level i.e. dominance of routine tasks. These findings showed that only imitative reasoning is necessary for solving mathematics tasks at lower educational levels. In this study, we examined the reasoning of several non-mathematics students who have been exposed to non-routine calculus tasks on a traditional calculus course. A detailed description of the course can be found in the methodology section. The purpose of the study was to examine what characterizes students' reasoning when faced with a problematic situation.

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Theoretical Framework

In this section we will describe and discuss the theoretical frameworks for our study and analysis, and present our research questions.

The Reasoning Framework

In mathematics education literature, many definitions of the term reasoning can be found. Lithner (2008) defined reasoning as the line of thought that is adopted to produce assertions and to reach conclusions when solving tasks. It is not necessarily based on formal logic, nor restricted to proof. It may even be incorrect as long as there are some sensible reasons (to the reasoner) supporting it. Lithner (ibid.) differentiates the thinking process from the product of that process. The product of the thinking processes, the way of understanding, can be observed as behavior, but conclusions made about the cognitive processes involved in that behavior, would still be to some degree, speculative. In this study, mathematical reasoning is a mental act and the purpose is to investigate the students' way of thinking.

There are two types of mathematical reasoning: imitative and creative reasoning (Lithner, ibid.). Everything that includes rote- learning reasoning is, in fact, imitative reasoning, and the opposite reasoning is creative reasoning. The basic idea of creative reasoning or creative mathematically founded reasoning is the creation of new and well-founded task solutions. Creative mathematically founded reasoning fulfills the following criteria: novelty, plausibility, flexibility and mathematical foundation. Novelty includes new reasoning sequence that is created or recreated if forgotten. Plausibility can be described as using arguments to support the strategy choice and/or strategy implementation. The choice of strategy can be supported by predictive arguments i.e. will the strategy solve the difficulty, and strategy implementation can be supported by verificative argumentation i.e. did the strategy solve the difficulty. Flexibility admits different approaches and adaptations to the situation and it does not suffer from fixation that hinders the progress. Mathematical foundation means that the argumentation is founded on the intrinsic mathematical properties of the component involved in the reasoning. Creative reasoning does not have to be a challenge in terms of problem-solving, but conceptual understanding is deeply anchored in it, unlike in imitative reasoning. This means that the construction and anchored argumentation in creative reasoning is impossible to do without considering the meaning of the components involved in the reasoning (Lithner, ibid).

Imitative reasoning is a term that describes several different types of superficial reasoning. In Memorized Reasoning (MR), the strategy choice is founded on recalling an answer and the strategy implementation consists of writing this answer down with no further consideration. Algorithmic Reasoning (AR) is implemented when the strategy choice involves recalling a certain algorithm (set of rules) for solving a given problem. The strategy implementation is trivial, straightforward once the rules are recalled. AR has several variants: familiar algorithmic reasoning, delimiting algorithmic reasoning and guided algorithmic reasoning. The strategy choice in Familiar AR is founded on recognizing the task as being familiar, which can be solved by a corresponding known algorithm. In Delimiting AR, an algorithm is chosen from a set that is delimited through surface relation with the task. Following the algorithm carries out the implementation of the strategy. If this implementation does not produce the desired outcome, the algorithm is abandoned, and a new one is chosen. In Guided AR, the reasoning is mainly guided by two types of sources that are external to the task. In person-guided AR, a teacher pilots the student's solution. Within text-guided AR, in the task to be solved, the strategy choice is founded on identifying similar surface properties to those in the text source (e.g. a textbook).

Met-befores and Met-afters

A met-before is a mental construct that a person uses at a given time based on prior experiences (Tall, 2006). Using met-befores can sometimes be an advantage when the person is learning a new mathematical concept, and sometimes it can be an obstacle that causes severe difficulties. Hence, met-befores affect the learning of new concepts, but new mathematical concepts may also affect older knowledge. Such mental constructs are called met-afters (Lima & Tall, 2008). Met-afters are those experiences met at a later time that affect the retention of old knowledge. Met-afters can also be both positive and negative; the negative effect of a met-after can be an indication of the fragility or inconsistency of previously learned knowledge.
New knowledge that builds on previous knowledge is much better remembered, but concepts that do not fit into earlier experience are learned temporarily and easily forgotten or not learned at all. According to McGowen and Tall (2010), this can be observed when a student, for instance, is interested only in algorithmic reasoning, relying on well-established procedures or algorithms. If there is no conceptual meaning, this kind of knowledge

is stored improperly and is very fragile when the person tries to adapt it to a new situation. This previous knowledge makes it difficult to understand new subject matter. The student is trying to distinguish among accessible rules and is trying to imbed new knowledge into his fragmented knowledge structure.

Research Question

Hiebert (2003) argues that students learn when they are given an opportunity to learn, thus we expected that, in the long run, the non-routine tasks incorporated in the traditional calculus course would have a positive influence on students' reasoning. Therefore, we formed the following research questions: How do average non-mathematics students reason when given non-routine calculus tasks? What characterizes non-mathematics students' reasoning in the middle of the calculus course and after passing the calculus course?

The research reported here should be seen as exploratory, and the conclusions are of relevance to the Croatian context and to other similar university contexts.

Methodology

Participants and Context

This study was conducted at one university in Croatia, and the participants were first-year civil engineering students. In order to move from the "vicious circle", where university calculus courses promote procedural knowledge and imitative reasoning (Tall, 1997), non-routine tasks were implemented in the calculus course for civil engineering students. At this university, calculus courses are designed according to the needs of the specific study program. This means that calculus courses are not shared but each study program has its own calculus course. Selden, Mason & Selden (1998) suggested that the non-routine tasks should be implemented as an explicit part of the curriculum in traditional calculus courses, not at the end, but throughout the course in the exercises sessions or in the homework. This group of civil engineering students was given several non-routine tasks for homework in each exercise session. The students had to hand in the homework to the teaching assistant in the following exercise session as evidence that they actually were solving it, but the solution did not have to be correct. Sometimes the homework tasks were solved in the next exercise session, and sometimes the solution was only commented on. The routine calculus tasks, which were given to the students in the course, required the application of some procedure that was shown to them either by the lecturer or teaching assistant, while the non-routine tasks were more oriented toward conceptual understanding. For instance, the routine tasks that the students solved in exercise sessions required the evaluation of function limit at some point e.g. "Find $\lim_{x\to 1}\frac{1-x^3}{1-x}$", while the corresponding non-routine task given for homework had the following form "Is there an a such that $\lim_{x\to 3}\frac{x^2+x-ax-a+4}{x^2-2x-3}$ exists? Explain your answer."

In order to pass the calculus course, the students in this study program have to pass both a written exam and an oral exam. Moreover, the written exam consists of a mid-term exam and a final exam. In the written exam students have to solve various tasks, the oral exam has an emphasis on mathematical theory. The participants for this study were chosen according to their scores obtained in the calculus mid-term exam. They scored around 70% in the mid-term exam, and we would classify them as average students, i.e. students who possessed some knowledge, were far from failing the exam, but also were not close to excellent scores. Their scores represent the most common results in the calculus exam among civil engineering students.

Method

The empirical data was collected from six task-based interviews. The students were interviewed in pairs. Arksey & Knight (1999) argue that this method is better for establishing an atmosphere of confidence with two students being interviewed at the same time and because the interviewees may 'fill in gaps' for each other. Also, the students' interaction may be of interest. Schoenfeld (1985) gives support for this kind of interview, stating that two-person protocols often provide better insight into and information about students' reasoning and knowledge. The students were interviewed on two occasions: in the middle of the course, just after the mid-term exam, and after passing the course. In the interview, the participants were given specific tasks designed in collaboration with the course's lecturer and teaching assistant. At the start of the interview, the participants were instructed to talk to each other when solving the task, to say out loud what they are thinking at that moment, not to plan what to say, and to behaves though they are alone in the room working together on their homework or an assignment. Similar directions were recommended by Ericsson & Simon (1993) in order to encourage the students to think out loud. During the first minutes of the interview, the interviewer just reminded the participants to keep talking if they were silent for a while. If the students struggled with the given tasks for more than several minutes, the interviewer asked direct questions to try to get the students to explain what they were doing and why they were doing it.

Participation in the study was voluntary and the students had the right to withdraw from the study at any time, we believe therefore that the students invested a significant amount of effort into solving the tasks given. The students in this study will be referred to by their initials in order to assure their privacy. The interviews were video-taped, transcribed and analyzed together with the students' written work.

Tasks

Before the interviews took place, we examined the participants' results and solutions from the mid-term exam and from the final exam in which they had to solve several routine tasks from differential calculus. The tasks in the mid-term contained the function given with a concrete algebraic expression, and the tasks required students to:

- calculate the limit of the function at the point,
- examine whether or not the function is continuous at the point
- determine extreme values for the given function.

The task in the final exam asked students to investigate all properties (domain, zero points, continuity, asymptotes, extrema, intervals of decrease and increase, the point of inflection, intervals of concavity) of a certain function given with algebraic expression (e.g. $f(x) = \frac{x^2}{x^2-1}$) and to draw the graph of the function accordingly.The tasks in these two exams can be characterized as routine tasks since students had met and practiced similar tasks during the calculus course, and they should have been able to employ a familiar algorithm in order to solve them.
In the interviews, the students were given tasks that were designed in conjunction with the course's lecturer and teaching assistant. The guidelines in designing the tasks were that they should differ slightly from the textbook exercises. According to Selden et al. (1998), such tasks can become significant problems, and thus become non-routine tasks. Therefore, the tasks contained concepts that students had met, used and learnt on the calculus course, they were not a challenge in terms of problem-solving, but they did not have a concrete algebraic expression for the given function.

The following tasks were given to the students:

1. It is given function $f: R \to$R. Let $f'(x_0) = 0$for only one pointx_0. Also let$x_0 > 0$and$f(x_0)$<0. If$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty$, how many zero points does function *f*have?
2. Sketch the graph of a function *f* which satisfies the following conditions:
   a. f is discontinuous at $x = 0$ and $f(0) = 1$;
   b. $f''(x) < 0$ for all $x < 0$ and $f''(x) > 0$ for all$x > 0$;
   c. $f'(-1) = 0$ and$f'(x) \neq 0$ for $x \neq -1$.

The first task was given to students after the mid-term exam, and the second task was given to them after they had passed the course. The students were given sheets of papers for use in solving the tasks. They were able to use their notebooks if they felt they needed to look something up. According to the lecturer and teaching assistant, it was unlikely that the students had encountered such tasks in the course, however in order to be able to determine whether a task is routine or not, it is not sufficient to examine the properties of the task alone, but the relation between the task and the solver has to be also considered (Schoenfeld, 1985).

Data Analysis

The students' work was videotaped because their gestures, tone of voice and the interaction between the students play an important part in their behavior and can tell us something about their mathematical reasoning. However, in this paper we will focus mainly on the verbal expressions, thus the main data for analysis consisted of the transcribed interviews, supplemented with the written work that the students produced during the interview. The interviews were transcribed by the author. As the research questions in this study deal with characterizing students' mathematical reasoning, the transcriptions were primarily focused on verbal and written mathematical communication. Therefore, we examined the students' arguments, guesses, and conjectures related to mathematics, produced in written or oral form during the interview.

The analysis for each interview was conducted in the following steps. First we provide a description of the data from the interviews. Then follows an interpretation of the data with the aim of understanding the central parts of the reasoning. The reasoning was characterized with the help of the reasoning framework. First, we classified the reasoning sequence as either creative reasoning or imitative reasoning. If the reasoning sequence was classified as imitative reasoning, we tried to determine whether the reasoning sequence was memorized or algorithmic. In addition, we examined the students' erroneous hypotheses and conclusions to detect mental structures that affected their knowledge base.

Lithner (2008) proposes that a reasoning structure is carried out in four parts: a task is met, a strategy choice is made, the strategy is implemented and a conclusion is obtained. The strategy choice and conclusion of the reasoning sequence were identified in the transcripts, which enabled us to identify separate reasoning sequences. Strategy choice includes recalling a procedure, selecting from accessible procedures, constructing a new procedure or simply guessing. After the strategy has been implemented, a conclusion is reached as the product of strategy implementation. It can be both incorrect and incomplete.

Validity and Reliability

The study in this paper is a qualitative one, so it is impossible to generalize findings. Here the generalizability can be replaced by the notion of 'fittingness' i.e. "the degree to which the situation matches other situations in which we are interested" (Schofield 1990, p. 207). On the other hand, Goetz and Le Compte (1984) use the notion 'translatability' and the notion of 'comparability'. Translatability refers to a clear description of one's theoretical stance and research techniques, i.e. whether the theoretical frames and research techniques are understood by other researchers in the same field. Comparability is the degree to which the parts of a study are sufficiently well described and defined so that other researchers can use the results of the study as a basis for comparison (Goetz & Le Compte 1984, p. 228). 'Thick descriptions' are, therefore, vital for others to be able to determine if the attributes compared are relevant (Kvale 1996). Therefore, we have aimed at making the process transparent by providing a good amount of detail about our study.

Results

Reasoning in Task 1

The work of the students will be presented, each followed by a summary of the reasoning characteristics. The interview was translated from Croatian, '…' denotes a pause, and [] some action, implicitly or explicitly shown.

Students D and P

Description:
Task 1 was given to students D and P when they were in the middle of the calculus course. When presented with the task, the students sat in silence for a while and then student P said:

> "It is equal with zero so… When we search for zero point of the function, it is in fact the first derivative. Here it says $f'(x_0) = 0$, therefore it means that we have only one zero point. And it's 0. This point is in the origin of coordinate system."

Student D remained silent while student P was explaining his solution; however, his facial expression implied that he was not sure that this solution was right. The students then were silent for some time, so the interviewer began asking some questions. The interviewer asked the students if they had ever met derivatives and where derivatives were used in the calculus course. Student P identified $f'(x_0) = 0$ again as the method for calculating a zero point of the function, and when the interviewer explained that in fact x_0 represents critical point of the function, student P stuck with his interpretation. He remembered that they used second derivative "for calculating minimum and maximum". However, student D then intervened:

D: I think $f'(x_0) = 0$ says x_0 is the point of inflection.
[silence for several minutes]
I: Hm…Let's try this way...Where is this point? What else is given?
D: It's in the origin.
I: Do you have any other information?
P: I don't understand. I've never met a task like this before.

D: It’s too difficult. I know how to calculate zero point for the function but this way no… We are not mathematics students.
I: But you’ve met tasks without calculation in the course, haven’t you?
P: Yes.
[silence]

The students gave up solving the tasks, with no further attempt to examine other given conditions.

Interpretation:
The students’ reasoning in Task 1 had an erroneous base from the start which was an obstacle preventing them from moving beyond the first condition. At the beginning, student P used familiar AR when he concluded that the function in fact has one zero point. Here, he relied on superficial property, namely the expression where the function f was equal with the zero and concluded that there is only one zero point (x_0) without investigating other information given in the task. When the interviewer intervened, it made no difference. It seemed that student P had a somewhat vague understanding of the derivatives and their properties. He recalled that the second derivative was used when they calculated if the critical point was minimum or maximum, but since this was not a task with calculation he was unable to use his knowledge flexibly. Student D used imitative reasoning when he concluded that x_0 is the point of inflection. He relied on recalling something he had seen before, not giving any arguments as to why he reached this conclusion. The fact that the students experienced significant frustration because they could not interpret the given condition in the right way was reflected in their reluctance to examine other conditions. Students D and P claimed that the task was too difficult pointing out the fact they had never met a task phrased this way before, and that the task was created for mathematics students since there were no concrete numbers to work with. They pointed out that they knew how to calculate derivatives and zero points. The students’ emphasis on “calculation” and their comment that they “have never met tasks like this before” indicate dependence on algorithmic reasoning i.e. mimicking the same procedure.

Students N and M

Description:
Students N and M read the task silently and afterwards they commented on the amount of written text in the task. Their whole reasoning process was very short, but they also experienced some difficulties in the process:

N: Too much text [laughs].
M: Hm, well…
N: How many zero points does function f have?
M: So for only one x we have $f'(x_0) = 0$… Aha, it means that function has only one extreme.
N: Zero point.
M: Zero point and extreme are not the same.
N: Oh, yes, yes… It’s about extreme values.
M: This extreme value is here [shows first quadrant] …
N: No... x_0 is here [point at positive part of the x axis]… and $f(x_0)$ is negative…. so the point is in fourth quadrant.

The students had the idea to separate function limit in two parts and to see what each part represents. Student M said that “function values go to the positive infinity for all positive x-values”, and student N concluded the same for negative x-values. Both students gave an answer to the question. The interviewer asked them how they came up with the solution.

N: I made a figure of the graph in my head connecting conditions.
M: Yes, yes, I imagined that. It all fits.

Interpretation:
Students M and N solved the given task without any assistance or prompting from the interviewer. The students experienced some problematic situations in the beginning, regarding the position and type of the given point: student N identified $x_0$as the zero point, and student M situated the point in the wrong part of the coordinate system, namely the first quadrant. However, solving the task together was valuable for the students’ reasoning sequence: they helped each other; one member of the pair steered the reasoning sequence on the right track. The students together reached a meaningful conclusion after separating the function limit in two parts and examined functions values of the given function. At the end, the students verified their solution. Their reasoning had the characteristic of being creative, regardless of initial problems. They were able to adapt to the problematic situations. Apparently the reasoning sequence was new to them; their strategy choice and implementation were supported by predictive and verificative argumentation. The arguments that were used to reach the conclusion were based on sound mathematical properties. After the interview the students said that the task looked difficult at first, but when they look at it now it seems easy.

Students B and J

Description:
Student B read Task 1 out loud, and both students had the same reaction to it, that they were intimidated by the amount of data in the task. When they began to interpret the given data, student B concluded that $f'(x_0) = 0$ is a derivative of the constant, but student J corrected him:

J: No, that's critical point.
B: Hm…
[silence]

The interviewer asked the students to express out loud what they were thinking at that moment, but the students stayed silent for several minutes. In order to stimulate the reasoning process, the interviewer prompted the students:

I: Go on... What seems to be the problem?
B: Don't know what to do with the data.
I: What about drawing what is given in the task?

Even with this prompt, the students were not able to move further in the reasoning sequence. The interviewer started to ask leading questions:

I: Where does the point $(x_0, f(x_0))$ lie?
B: In the second quadrant [pointing at the fourth quadrant] because $x_0$is positive and that is on the right from the origin on x- axis, and $f(x_0)$ is negative and that is down on y-axis.
I: What about next piece of data? What does it say?
B: That everything goes to infinity.

At the end of the task, student B concluded that he can sketch parabola from the given data but he was uncertain how to draw it because there was only one point given in the task, and he did not have the formula of the function. However, student J took the initiative:

J: [takes pencil and draws] You do not need another point. See? If you connect all conditions, you get a graph.
B: Oh… and there are two zero points

Interpretation:
Here in Task 1, the students met several problematic situations, such as the wrong interpretation of symbols presented in the task or the inability to move beyond their inference about the amount of text and data in the task. The students' strategy choices are made or at least influenced by the interviewer, and the reasoning sequence was guided by the interviewer. Besides giving a prompt for drawing data in the coordinate system, the interviewer asked what a certain piece of data represents. Students B and J alternated in data interpretation, but neither of them was independent in the reasoning process. The students drew the point $(x_0, f(x_0))$ in the correct quadrant, and then concluded that the function values go to infinity. Student B could not draw a graph of the function because he did not connect the given conditions. Here he was fixed on one specific strategy choice, namely, drawing a graph of a function with only one point given. Consequently, that hindered his reasoning sequence. On the other hand, student J made an adequate graph of the function according to given data and verified her solution. If we examine parts of the students' reasoning sequence, we can detect components of (local) creative reasoning, but more so in student J's reasoning than in student B's. Using their conceptual understanding, examining intrinsic mathematical properties of the concepts in the tasks, the students obtained the solution. But the final conclusion could not have been reached without the interviewer's initial help and supportive questioning; therefore, it is not certain that the students would have been able to solve the task on their own. After the interview was concluded the students said that they were used to calculation in the tasks, and that they would avoid this type of task if it came up in the exam.

Reasoning in Task 2

The work of students will be presented, each followed by summary of the reasoning characteristics. The interview was translated from Croatian, '…' denotes a pause, and [] some action, implicitly or explicitly shown.

Students D and P

Description:
The initial behavior of students D and P was similar as in Task 1, but this time student D was involved in solving the task from the start. Both students decided to read the task, and to interpret each condition:

P: [draws coordinate system] This means that function in 0 would cross coordinate axes. [draws small circle in the origin of coordinate system]

D: Second condition....We used this when we calculated minimum and maximum.
[silence]

The students stopped and it seemed as though they did not know what to do next, so after several minutes of silence, the interviewer asked them to explain the role of the first and second derivative when examining properties of the function. The students remained silent, so the interviewer tried another approach asking:

I (interviewer): How did you calculate intervals of concavity for a concrete function?
P: Hm …It's concave upward when the second derivative is less than zero.
D: Hm...no... I think is concave downward in that case.
P: [draws graph of function where certain parts are concave upward and concave downward] So let's look at parabola… second derivative… greater than zero here where [it] is concave upward, and concave downward here [pointing to the figure]
[silence]
I: What about the next condition? What does -1 represent?
P: It's zero point.
D: Hmm… no it's maximum. See here at the figure [pointing to upper figure]… the derivative is zero in the critical point and then it follows that it's maximum from other given conditions
[silence]
I: And now, can you incorporate all condition in your sketch?
P: No, I can't make a figure in my head.
D: [silence]

Interpretation:
As in Task 1, students D and P experienced significant difficulties in Task 2. Their initial reasoning can be characterized as familiar AR – based on recalling something they had calculated many times. The students' uncertainty prevented them from proceeding further. After the interviewer's intervention, the students were able to move on, but they showed that they were quite dependent on external guidance in their reasoning. In some situations, the students were able to develop their ideas. And in some situations, the interviewer took the initiative, for instance, when the students had to connect second derivative and intervals of concavity for some concrete function. It was only after this prompt that the students' reasoning sequence continued. From the students' behavior, it seems that this reasoning is new to them, but their reasoning sequence is not entirely flexible in a new and problematic situation, even though in many parts, it is mathematically founded, i.e. arguments given are based on intrinsic mathematical properties of concepts involved in the tasks. Local creative reasoning can be found in part of the reasoning sequence, for instance, when student P uses a concrete function, namely a quadratic function to reconstruct his knowledge on the second derivative and its relation to concavity. Or when student D reached a conclusion about a given point, and verified it using sound mathematical arguments. But in the end, the students were not able to reach a final conclusion. Their strategy choice was dependent on the interviewer's hints and lead, and it is unlikely that the students would solve the task on their own. However, after the interview the students said that all the concepts in the task were indeed familiar to them, but they preferred tasks with calculation.

Students N and M

Description:
After several months, when faced with Task 2, students N and M employed the same strategy as they had in Task 1. They read the task silently and then they started talking to each other and interpreting conditions. First of all, student N decided to draw the coordinate system. Then student M said that the function f had a hole in 0 and student N sketched that hole in the coordinate system as an empty circle at the point (0,0) and drew a full black circle at the point (0,1). Then they interpreted condition b. :

M: Second condition... It could be … [silence]
I (interviewer): Where have you used the second derivative?
M: When something is convergent and divergent….but I don't remember when you use the first thing and when the second.
N: For intervals of decrease or increase.
M: [draws a figure of curves being concave upward and concave downward] That is the first derivative, the second is used when something converges or diverges.
N: Ok, you mean concave upward and downward…Discontinuity is here where the function changes its shape.

M: Well, no… that would mean… when $f''(x) > 0$ [writes on the paper], it is concave upward, when $f''(x) < 0$[writes on the paper], it is concave downward.

N: See, here where $y = 1$, it fits… it goes like this [corrects figure, draws curve looking like parabola, having minimum]

M and N switched to condition *c*. identifying -1as the only extreme value of this function. They changed their figure according to a new condition, and again student M was a bit puzzled as to whether the new figure was the correct solution of the task. Student N explained that the new figure fulfills all the conditions:

M: So it's ok that we have two parts?

N: The function has discontinuity in 0.

M: Hm…I tried to imagine something I have seen before, and now I see it doesn't make sense.

Interpretation:

In Task 2, students M and N chose the following strategy: examine each condition separately and draw them in the figure. The chosen strategy was implemented, and the students adjusted the figure in each step to correspond to the given conditions. From the interview, it seems that the reasoning sequence was new to the students. Also they showed a certain degree of flexibility and relied on the intrinsic properties of concepts involved in the task. Student M could not remember the proper terminology, but he sketched his "divergent and convergent functions" which were in fact figures representing curves being concave upward and concave downward. This enabled student N to give the "right name" for function property. When student N pointed out that discontinuity should be incorporated in the drawing, and it should be where the function was changing its shape from concave downward to concave upward, it had the effect of disturbing student M's reasoning sequence. He became puzzled by the outcome. Similarly, he expressed his doubts at the end of the solution process, because the graph did not look like any other graph he had ever seen.

When a conclusion was reached, student N verified their solution. His reasoning sequence had many characteristics of being creative, especially in terms of flexibility in the problematic situation of adjusting the figure according to the given conditions. From the interview it can be seen that the reasoning sequence was entirely new to the reasoners. Some parts of student M's reasoning sequence also have characteristics of being creative, for instance, when he reconstructed his knowledge for property of concavity. But his uncertainty with the final product of solving indicates a switch to familiar AR; seeking security in something familiar.

After the interview, the students said they like challenges and they would probably attempt to solve this kind of task in the exam. They just needed to "connect the dots" which indicates the use of more creative reasoning in the solution process.

Students B and J

Description:

In Task 2, students B and J first decided to draw the coordinate system, and then they read the rest of the task. As he was reading condition *a*. out loud student B noticed there were some points in the text and drew them in the coordinate system. The points he marked had the following coordinates (1,0) and (0,0). Student B said that condition *b*. was about intervals of increase and decrease of that function, and then he stopped, looking puzzled. During that time, student J was silent, examining the task. Since both students did not say a word for some time, the interviewer intervened asking questions about the given condition. Student B corrected his own drawing by marking point (0,1) instead of point (1,0), but he interpreted condition *b*. again in the same manner. This time student J got involved and corrected him:

I (interviewer): Are you sure you drew it correctly?

B: Well...no [corrects his drawing]

I: What about condition *b*.?

B: We had this for maximum and minimum. When second derivative is less than zero, than we have maximum. When second derivative is greater than zero, we have minimum.

J: No. This [condition] says when the function is concave upward and when the function is concave downward…

[silence]

I: Where the function is concave upward and where the function is concave downward?

J: On the left side of x-axis is concave upward and on the right side is concave downward.

[silence for a while]

Students B and J discussed what shapes of concave upward and concave downward looked like, trying to decide what shape parabola $y = x^2$ has. After this discussion, student J drew the graph of function that satisfied condition *b*. The students moved to condition *c*. However, this new information caused confusion when the students tried to incorporate it into other data they had.

[silence]
I: What does the third condition say?
J: That's critical point.
I: Where?
J: On the negative part of the x-axis.
I: What property does function have there?
J: It's concave upward
I: So what kind of point do we have there?
J: Hm … It's maximum.[puts the pencil down]
I: Is this the solution?
B: Yes.

Interpretation:
At the beginning, the students were reluctant to express their thinking out loud and to continue their reasoning, which can be inferred from the silent gaps in the process. However, we would not say that the students lacked the resources or knowledge needed for the tasks. The students used arguments that are mathematically founded to provide validity of their conclusions. Their reasoning was anchored in intrinsic properties of the components in the reasoning: the relation between the property of function being concave upward or downward and the shape of parabola to determine what shape a given function has, or the relation between the critical point and the shape of the function to conclude what extreme value is given. Here in Task 2, the students' reasoning had many characteristics of being creative mathematically founded reasoning, but the gentle guidance of the interviewer was needed to stimulate the reasoning sequence. At the end of the task, the students did not adjust their figure so that all conditions were met i.e. they did not verify their conclusion. The drawing represented the graph of the function with two extrema; having maximum on the left side of x-axis, and having minimum on the right side of x-axis.
After the interview, the students concluded that the task was not hard at all. Student J claimed that in the beginning the unfamiliar situation looked daunting, and student B agreed with student J. This indicates a change in their reasoning compared to Task 1.

Met- befores and Met-afters

Several negative met-befores and met-afters were detected in the knowledge of students D and P, which hindered their reasoning sequence. For example, student P interpreted $x_0$in $f'(x_0) = 0$ and-1 in$f'(-1) = 0$ as a zero point of the function. The calculation of zero points of a function is frequently performed in high school mathematics, wherefore student P disregarded a sign for the derivative and identified this expression with commonly seen expression $f(x_0) = 0$ and this triggered familiar AR. On the other hand, in their calculus course, the students learned about the concepts of critical points and extrema before the concept of point of inflection. Therefore identifying the expression $f'(x_0) = 0$ in Task 1 as a property for the point of inflection can be considered as a met-after which, we believe, influenced student D's reasoning.

We identified certain negative met-befores in the knowledge structure of students M and N. In Task 1, those were the interpretation of x_0 as zero point in $f'(x_0) = 0$ (student N), and placing the point $(x_0, f(x_0))$ in the first quadrant (student M). The first met-before is similar to the case of students D and P, and the latter can be connected with the presentation of many function graphs in textbooks and lectures. Usually, when the lecturer (either at university or secondary school) draws a graph of arbitrary function as an example to show a property, it is mainly placed in the first quadrant, or the major part of the graph is drawn there. In Task 2, student N identified the second derivative with properties of increase or decrease of the function. In the students' calculus course, usage of the first derivative for determining intervals of increase and decrease is taught before the second derivative and its connection with concavity. This met-before indicates that concepts related to the first and second derivatives were not properly understood. But, it seems that this mental construct did not have a negative effect on solving the tasks in calculus exams, because the students did pass the course.

Some negative met-befores were found in the knowledge structure of student B. At the end of Task 1, student B identified the graph of the function as the graph of well-known quadratic function. This interpretation of the obtained graph is not necessarily a problematic met-before, but it can certainly have a negative impact in other situations. Here we see another met-before as much more problematic and that is the need for a "formula" i.e. a concrete expression to help the student to draw "parabola". According to Tall (2006), seeking a function that a person has already met and the need for formula hinders the development of advanced mathematical thinking. Another negative met-before is the interpretation of condition b., namely connecting $f''(x) < 0$ with the property of increase or decrease of some function, which together with the interpretation of $f'(x_0) = 0$ as the derivative of the constant, indicates that the student did not quite understand the topic of the first derivative and related concepts of extrema and intervals of increase and decrease. However, the student did pass the calculus course with an average grade, indicating that these problematic met-befores were not evident in the reasoning that was required in the calculus exams.

Discussion and Conclusion

In this study, we wanted to investigate how average non-mathematics students reason when faced with non-routine calculus tasks and what characterizes their reasoning. Throughout their primary and secondary mathematics education, Croatian students frequently use imitative reasoning (Glasnović Gracin, 2011), and this is also the case in many university calculus courses. However, the students in this study had non-routine tasks implemented in their university calculus course. According to Hiebert (2003), students learn when they are given an opportunity to learn, so we expected that the incorporation of non-routine tasks into the course curriculum would have a positive effect and influence on the students' reasoning in terms of making it more creative. Even though we did not design the study as an experiment nor did we have a control group to which we could compare the effect of the implementation of non-routine tasks into the course, the results of this study can still provide valuable insights into students' reasoning.

First, we identified whether the tasks in the study were routine or not. The reasoning of students in this study showed that the tasks given to them in the interview do not belong in the category of routine tasks. On the contrary, the students had many difficulties, met many problematic situations when solving the tasks and did not identify them as the type of tasks they had previously been exposed to. These tasks did not represent problems in terms of Schoenfeld (1985), but can be characterized as moderately non-routine tasks. The students did not already know a method to solve the tasks, and the tasks were dependent on their background. Although students had learnt and used concepts that appeared in the tasks, not providing a concrete algebraic expression for the functions made the tasks problematic.

The results showed that the students' reasoning differed in the middle of the course and after passing the course. The students' reasoning had become more creative by the end of the course, although they still showed a tendency to be guided in the reasoning sequence. The absences of computation in tasks prevented students D and P from relying on procedures and using algorithmic reasoning to which they were accustomed. However, if we compare their reasoning as examined on two separate occasions, in the middle of the course (Task 1), and after the course (Task 2), we can see some positive shifts in the reasoning sequence. Reasoning in the second task is in some part local creative reasoning since the students did use argumentation and considered intrinsic properties of problematic components together. On both occasions, students M and N had some difficulties when solving the tasks, but their reasoning had many characteristics of being creative. They corrected each other, and consequently, produced a valid solution for the tasks. But the question that remained unanswered is whether students would be able to solve the tasks alone. When it comes to students B and J, student J showed many characteristics of creative reasoning even in Task 1. Student B's reasoning had improved by Task 2, but on both occasions the students needed supportive guidance.

On the other hand, what the students had met before and after learning a mathematical topic had a wide impact on their reasoning. Negative met-befores and met-afters prevented students from proceeding in solving the task, or led them to imitative reasoning. However, our intention is not to classify all met-befores and met-afters that might appear in the students' knowledge and that might inhibit creative reasoning, but to caution to their existence. The students in our study passed the calculus course, which indicates that those met-befores and met-afters did not prevent them from successfully solving tasks that required imitative reasoning. Here we based our conclusion also on the students' mid-term and final exams which we examined prior to each interview session.

When each interview was finished, the students commented on the tasks that they had been given. Besides students D and P in Task 1, other students were surprised that the tasks were not very difficult in the end, that they indeed contained concepts that they had learnt and used in the calculus course. In the interview after the mid-term exam, the students claimed that they probably would not even try to solve such a task in the exam. The main reason was fear of an unfamiliar situation and being accustomed to calculation, i.e. using familiar algorithmic reasoning. However, after passing the course, the students changed their attitude a bit. They said they might try to solve this kind of "task without calculation" if they encountered it in the exam. Even students D and P softened their attitude. We believe that the non-routine tasks implemented throughout the traditional calculus course had a positive effect, making a small shift in the student's reasoning. We believe that our findings are valuable since the study investigated the reasoning of average non-mathematics students in a traditional calculus course.

When faced with new situations, students tend to look for something familiar, and usually seek a remedy in the form of imitative reasoning, like searching the textbook for similar solutions or recollection of similar tasks (e.g. Boesen, Lithner & Palm, 2010; Haavold, 2011). Selden et al. (1998) pointed out that students lack tentative solution starts, i.e. general ideas for beginning the process of finding a solution, and that, together with mental constructs of met-befores and met-afters, provides a significant obstacle for creative reasoning. We argue that mathematics educators and lecturers should take this into consideration when teaching students. On the other hand, creative reasoning is beneficial in checking the quality of students' long-term knowledge. In imitative reasoning, students do not consider the intrinsic properties of the objects they are reasoning about, and frequently they rely on well-established procedure, mimicking, almost unconsciously, its every step (Lithner, 2012). Even though imitative reasoning provides a reduction of complexity in the course requirements, students do not construct appropriate meaning in such a process. The remedy is not avoidance of non-routine tasks, but quite the opposite, facing students with new situations. The non-routine tasks and creative reasoning can uncover negative met-befores and met-afters which students are oblivious to when they perform imitative reasoning. This uncovering is important for the sequencing courses that build upon previous courses, i.e. where new knowledge is building up previous acquired and mastered concepts.

But is it possible that non-routine tasks become more visible in a calculus course for non-mathematics students, so that they are not only part of the homework, but part of the exercise sessions? And to whom does this matter? There are no simple answers to these questions. In the calculus course that our participants took, the non-routine tasks were implemented mainly in the homework. The course syllabus is overloaded, and it is difficult to explicitly deal with non-routine task on regular basis. During the course, the students showed resistance toward non-routine tasks that required a greater investment of their time than the usual routine tasks. But we as educators argue that it does matter, because we want to build up a work force that is able to adapt to the demands of today's business and economy. We believe that flexible thinking and creative reasoning are part of this ability. There is also no simple answer to this question from the point of view of the students. Students, not only in this study program but in many other science and technical study programs, have many requirements in the courses more closely related to their profession. They usually lack the time for deeper engagement in mathematics, they want to pass the mathematics course and at the same time they would like to know how to apply the gained mathematical knowledge (Jukić Matić, 2014). But not engaging students in creative reasoning gives them the illusion of understanding and leaves them with the idea that mathematics is about "prescriptions", i.e. algorithms.

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