DYNAMIC VISUALIZATION AND SOFTWARE ENVIRONMENTS

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ABSTRACT
This paper will examine the use of software environments and dynamic visualization in mathematics education. This examination will be based on theoretical papers and research reports from substantial literature on visualization. How the term “visualization” is used in mathematics education field is also discussed. The thesis of this essay is that visualization and visual reasoning play vital roles in mathematical thinking. Therefore, the software environments could be integrated in mathematics teaching to foster a greater understanding of mathematical concepts.

INTRODUCTION

In this essay, the use of software environments and dynamic visualization in mathematics education will be discussed. There will be three parts to this discussion. Firstly, a general definition for ‘visualization’ will be sought. The underlying reason for such an inquiry stems from the fact that dynamic visualization is a subcategory of visualization, and the most of the issues about visualization are also important for dynamic visualization. After stating a definition of visualization, the importance of visualization in a general sense will be considered, and finally the discussion of dynamic visualization and software programme, which are believed to be the most suitable dynamic visualization environments, will be presented.

Dynamic visualization and dynamic software environments are mostly regarded as the same things, since it is almost impossible to create dynamic images on static mediums such as paper or blackboard. However, by saying that they are regarded as the same things, it will not be claimed that dynamic images can only be created on external mediums, since it is well known fact that some people can create such environments in their heads; they can move, shrink, rotate figures in their minds, and can see changes or unchanged relationships before and after the variation of figures, and by using this they can reason about mathematical concepts.

SEEKING A DEFINITION

Since the relationship between visualization and mathematical performance has been an area of interest for a number of researchers concerned with mathematics education for many years, there is a substantial literature in which relationships between visualization, mental imagery, and mathematical performance have been investigated (e.g. Bishop, 1980; Lean & Clements, 1981; Tall & West, 1986; Presmeg, 1986, 1989, 1992; Yerushalmy and Chazan 1990; Zimmerman and Cunningham 1991; Goldenberg, 1992; Dreyfus, 1991; Tall, 1991; Klotz, 1991; Shama & Dreyfus, 1994; Drake 1996; Zazkis, Dubinsky and Dautermann 1996; Hazzan & Goldenberg, 1997). In the mathematics education literature, although there is a large body of work on visualization and visual thinking, there is not a common consensus for the terminology used in this field. For instance, Drake (1996) uses the terms visualization and imagery interchangeably. Guttierez (1996) discusses how various people in mathematics education use the terms visualization, visual image, and mental image differently:

"There is no general agreement about the terminology to be used in this field: It may happen that an author uses, for instance, the term "visualization" and another uses "spatial thinking", but we find that they are sharing the same meaning for different terms. On the other hand, a single term, like "visual image", may have different meanings if we take it from different authors. Such an apparent mess is merely a reflection of the diversity of areas where visualization is considered relevant and the variety of specialists who are interested in it " (p. 4)

This state of confusion, which Guttierez mentions, is not the only problem that readers may face while reading various works on visualization. It should be pointed out that some more other complications exist in this area, and these are the reflections of the difficulties that researchers face while describing what visualization is and where this action occurs. For example, the problem whether visualization is just an act of looking at pictures or drawings on external medium, or must involve internal processes of a person can also be considered as an apparent mess of this area. In other words, whether visualization occurs purely in the mind or outside the person is one of the complicated issues that researchers try to specify precisely. For instance, Presmeg (1986) thinks that visual image is in the mind: “A visual image is a mental scheme depicting visual or spatial information”. (p. 42), and she claims that this mental scheme can exist with or without the presence of the external object being visualized. On the other hand, Zimmerman and Cunningham (1991) define visualization, at one point, as “the process of producing or using geometrical or graphical representations of mathematical concepts, principles, or problems, whether hand drawn or computer generated.” (p. 1), and at another point they claim that “…visualization is the process of forming images (mentally, or with paper and pencil, or with the aid of technology)" (p. 3). These two different views imply different notions about visualization: first one says that looking at pictures drawn on some external medium constitutes visualization in and of itself, and on the contrary, second one implies that visualization involves some sort of mental processes. Zazkis, Dubinsky and Dautermann (1996) also try to solve this complicated issue of what visualization is and where it occurs, and provide a working definition of visualization that is more precise than the others given in the literature. The following definition of visualization considers the range of different processes involved in visualization:

Visualization is an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper, chalkboard, or computer screen, of objects or events which the individual identifies with object(s) or process(es) in her or his mind. (p. 441)

This definition does not restrict visualization to either the individual's mind or some external medium; on the contrary it restricts 'visualization to constructions that transform between mental and other media.', furthermore, this definition also dissociates itself 'from other forms of construction of mental images based entirely on other mental images in the absence of external media.' (Zazkis, Dubinsky and Dautermann 1996 p. 441). Put another way, this definition of visualization gives importance to the establishments of connections between internal constructions and external medium, thus the place of where the visual image is become unimportant.

THE ROLE OF VISUALIZATION IN MATHEMATICS

Many mathematicians and mathematics educators claim that visualization and visual reasoning play vital roles in mathematical thinking (e.g. Lean & Clements, 1981; Presmeg, 1989; Zimmerman & Cunningham, 1991; Davis, 1993; Shama & Dreyfus, 1994) Furthermore, as Fennema (1979) mentions, "…some mathematicians have claimed that all mathematical tasks require spatial reasoning." (Cited in Lean & Clements, 1981, p. 267). For instance, Lean & Clements (1981) quote from McGee (1979) that "…H.R. Hamley, an Australian mathematician and psychologist, wrote that mathematical ability is a compound of general intelligence, visual imagery, and ability to perceive number and space configurations and to retain such configurations as mental pictures." (p. 267).

On the other hand, some authors point out that visual thinking alone cannot be enough for doing mathematics; it can only be '*a precursor'* and '*a complement'* to analytic thinking. Therefore they regard visualization as an alternative and powerful resource for students learning mathematics (Dreyfus, 1991; Goldenberg, 1992; Tall, 1991; Hazzan & Goldenberg, 1997). For instance, Tall (1991) points out that although visualization might give powerful source of ideas in the early stages of development of the theory of some mathematical concepts, it may also be a hindrance for doing mathematics since pictures may often suggest false theorems. Presmeg (1986) also discusses the advantages and disadvantages of visualization by developing a list of kinds of visual imagery used by students: 'Concrete, pictorial imagery', 'Pattern imagery', 'Memory images of formula', 'Kinaesthetic imagery', 'Dynamic (moving) imagery'. She writes that over-reliance on a single diagram may bring about inflexible thinking which prevents the recognition of a concept in a non-standard diagram, and this may introduce false data. Yerushalmy and Chazan (1990) research on the ways software environments can help to mitigate some of the disadvantages of visual imagery, particularly over reliance on a single diagram.

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Tall & West (1986) mention; dynamic representations of mathematical processes may enable "the mind to manipulate them in a far more fruitful way than could ever be achieved starting from static text and pictures in a book." (p. 107). Therefore, as many other authors point out *dynamic visualization* can be a very powerful tool to gain a greater understanding of many mathematical concepts or it can be a resource to solve mathematical problems. (Harel & Sowder, 1998; Goldenberg, Lewis & O' Keefe, 1992; Presmeg, 1986; Tall &West, 1986). Although these authors use different terms for dynamic visualization: 'dynamic imagery' (Presmeg, 1986); 'dynamic representations' Tall & West, 1986; 'viewing a triangle [geometrical object] as a dynamic entity' (Harel & Sowder, 1997); 'dynamic reasoning, dynamic visualization, or dynamic imagery', (Goldenberg, 1992): it should be acknowledged that they share the same meaning. This claim is put forward by keeping in the mind the definition of visualization provided by Zazkis, Dubinsky and Dautermann (1996), which points out that visualization is an act of construction of transformations between external media and individual's mind. Dynamic visualization is also such an act, but in this case this act constitutes moving pictures in the mind, or on some external medium 'which the individual identifies with object(s) or process(es) in her or his mind.' In other words, the peculiar property of dynamic visualization is that individuals who possess this ability can reason about the essential properties of moving, shrinking, and rotating figures, which appear on the screen or, in their mind, and thus they can solve the mathematical problems. There are research evidences too, which show that some students spontaneously develop this ability to move figures in their heads, to stretch and shrink them, to rotate them, to see them interact, and hence solve problems by using this kind of technique. (Harel & Sowder, 1998; Presmeg, 1986; Goldenberg, 1992). For instance, in their study of students' proof schemes, Harel and Sowder (1998) report how one of the students solve the following problem:

Question: Show that the sum of the measure of the interior angles in a triangle is 180°.

They (ibid) describe Amy's solution, which contains dynamic events, through static pictures and words as follows:[1] The student (Amy) solves this question by using her dynamic visualization ability[2]; she rotates the two sides *AB* and *AC* of a triangle Δ*ABC* in opposite directions through the vertices *B* and *C*, until their angels with the segment *BC* are 90°, and by doing this she transforms the triangle *ABC* into the figure *A'BCA"* (Figure 1a, b). She then performs the reverse of her previous rotation and transformation, and recreates the original triangle (Figure 1c). By doing these, she can identify that the lost angles *x* and *y* from the 90° *B* and *C* are gained back in creating the angle *A* (Figure 1d).

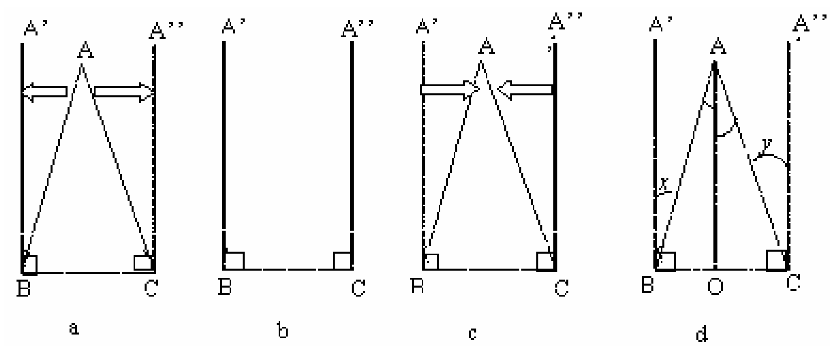

Figure 1 A student's solution for proving the sum of the measure of the interior angles in a triangle is 180°.

Since this kind of reasoning is used very infrequently among students. (Presmeg, 1986; Goldenberg, 1992), it might be considered as an "extraordinary" thinking style. However; the relevant specialty of this thinking style is not that it is an unusual way of reasoning, rather its importance is that students who have this ability can readily recognize its efficacy when they could use it. (Harel & Sowder, 1998; Goldenberg, 1992). And, as Harel & Sowder (1998) point out, they are fully able to *anticipate* the results of the transformations of the figures they make in their minds hence they can *deduce* the solutions of the problems, moreover these kinds of operations are intended the *generality* aspect of the conjectures rather than a particular figure. After all saying these, persons (who believe that visual is not mathematical) might still argue that the above solution cannot be accepted as a mathematical solution since it does not contain any range of formal representations in which the information forms a sequence of verbal expressions. In this respect they may be right, but this is not withstanding the importance of these kind of reasoning. Since, this kind of reasoning is regarded as a key to the analytical proof schemes that encompass mathematical proofs by Harel & Sowder (1998):

Key to the analytical proof scheme is the *transformational proof scheme:* the creation and transformations of general mental images for a context, with the transformations directed toward explanations, always with an element of deduction. (original italic, p. 276)

Students like the one mentioned above, as Goldenberg (1992) points out, learn to create such dynamic mental images and "perform such experiments in their heads without (curricular) experiences doing similar experiments with their hands and eyes." (p. 2). If students are given suitable technologic tools, then they can develop the ability to carry out such experiments. In other words, technology can foster students' dynamic visual reasoning ability. And fortunately, lots of software has been developed in the last twenty-five years that intended to contribute to students' abilities to visualize in mathematics. (*Cabri Geometer, Geometer's Sketchpad, DynaGraphs, Geometric Supposer, Graphic Calculus...)*. Therefore, emergence of such software has inevitably influenced mathematical education, and it will continue to influence and change the ways mathematics have been taught as a result of the growing interests mathematics educators are showing in tools like these. Furthermore, these software environments also add new research areas in mathematics education. For instance, some researchers attempt to describe how work in 'dynamic visualization environments (DVEs)' can contribute to the understanding of geometrical concepts, or some other mathematical concepts such as functions, derivatives, etc. (Tall, 1986; Klotz, 1991; Hazzan & Goldenberg, 1997).

For example, Tall (1993) point out that software environment could allow the students to manipulate the picture and relate its dynamically changing state to the corresponding concepts. It therefore has the potential of improving understanding. Software could be used for laborious constructions whilst the student can focus on specific relationships. This Tall (1993) terms the *principle of selective construction*.

In order to focus on new ideas and to suppress the processes that may cause the cognitive burden, it may be possible to get the computer to carry out the processes which are not desired to be the focus of attention. The educator may provide the learner with an environment in which the learner can focus on selected constructions, whilst other constructions which are not to be the focus of attention are performed by the computer (p. 391).

For instance, in the British SMP 16-19 syllabus, the structural stability of the Newton-Raphson process is explored visually on the computer before the numerical formula for the procedure is introduced.

The DVEs like *Cabri, Sketchpad,* and *Graphic Calculus* provide visual constructions that can allow students to use, explore, and come to understand the algorithm –building geometric objects or calculus concepts and displaying them in a certain way- before they know the particular mathematical properties or theorems involved. In other words these software environments allow students to discover rules or theorems by themselves using experimental or inductive methods. Students can use these tools to find rules and make hypotheses, and then they may attempt to prove their hypotheses.

Although, even very old textbooks use diagrams to introduce axioms and theorems, these diagrams are static. Therefore, before presenting some geometric theorems verbally or by static diagrams, which rarely invite experimentation and exploration, visual representation of the theorem may be constructed dynamically in a certain way, and then students may be invited to explore this representation. As an example of this may be as following[3]:

Theorem: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.

Before stating this theorem verbally, students may create a dynamic construction with software (such as *Sketchpad)* in which they can observe the midpoint of a hypotenuse. Students may track the midpoint of the hypotenuse as dragging the highlighted point along the base of the triangle, and notice that the path looks like a circle around the right angle vertex and if it is so, then it is some fixed distance R from that vertex (Figure 2). The value of R can be found with the help of dynamic visualization: "As the hypotenuse is moved continuously until it lies directly on the horizontal leg, the half of it that extends from its midpoint to the vertical leg moves to lie directly on a radius of the circle: R is the half the hypotenuse" (Goldenberg 1992, p. 7). Another way is to fix the hypotenuse and drag the other corner (maintaining the right angle), then the circle and its radii can be seen.

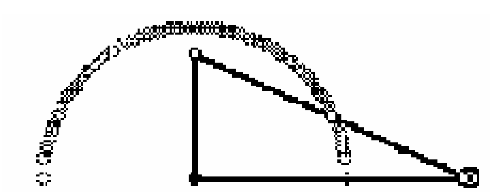

Figure 2. Proving the theorem that states that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle by using computer.

As can be seen from the above example, the most salient feature of software environments is that with them it is possible to produce *dynamic geometric visuals* which can be adjusted by *dragging* certain points (or other objects) around the screen in a continues manner while observing how the entire construction responds dynamically. In some cases, certain relationships among the figure's components may change while others remain invariant. In other cases, dragging of a point may cause a rotation or transformation of the entire figure. In this way students can focus on new ideas and suppress the processes that may cause the cognitive burden. As Hazzan and Goldenberg (1997) point out, dynamic dragging can shift one's attention to how things work, "by giving one a strong feeling of operating a smoothly functioning mechanical device –one in which the mathematical objects behave as if they were physical, obeying the laws of mechanics and conforming to our intuitions about movement in a continuous space-…" (p. 264).

In addition to these benefits; the drag-mode can also be a mediator between the concepts 'drawing' and 'figure' (Laborde & Laborde, 1995). Fischbein (1993) defines a figure as 'the construct handled by mathematical reasoning in the domain of geometry', and a drawing as not 'the geometrical figure itself, but a graphical or a concrete, material embodiment of it'. (p. 149). Figure is more abstract and drawing is more concrete and more close to physical environment. The diagram is simply the (fixed) picture after it has been drawn. For example, a triangle is a geometric figure which is defined as having three angles and three sides; on the other hand the diagram of a triangle is the representation of the figure. The diagram alone is not always sufficient to express its definition. That is, by just looking at the diagram, we cannot easily deduce the definition of the figure.

Laborde and Laborde (1995) point out; passing from drawing to figure can be enhanced by geometry programs by making explicit definition of the referent of the geometrical object which user draws on the screen. Therefore, software environments provide an opportunity for students to sense the difference between the figure and drawing by dragging his or her drawings which are constructed in different ways. However, the type of the geometry that evolves out of these computer environments can be considered as somewhat different from traditional geometry (Laborde & Laborde, 1995; Hölzl, 1996). If the geometry is viewed as only the body of stated axioms, permitted operations and proven theorems then proving theorems on the computer screen would be counted as one of acceptable formal proof methods.

Another critical issue in this new approach may be that what is the value of these tools to students who are called non-visualisers. Non-visualisers are defined as persons who prefer non-visual methods of problem solving or more clearly these persons tend to solve problems using analytical thinking in the cases when both methods can be used (Presmeg, 1986a). Here, it should be argued that this definition is different from saying that non-visualisers are persons who can *never* think of pictures. In the first case the *tendency* towards choosing one of the methods non-visual or visual is differentiating non-visualisers from visualisers; therefore this definition implicitly says that non-visualisers also can think of pictures but they are reluctant to use this kind of thinking. The reason of this reluctance may be multi-faceted as Eisenberg and Dreyfus (1991) point out: visual methods are more difficult to teach and learn, and there is a belief that visual is not mathematical. As a result, most students and teachers do not tend to use visual methods and hence students who are outstanding in mathematics appeared to be non-visualisers. And thus people argue that non-visual processing in mathematics is better than visual methods. More clearly, it should be claimed that software environments may also of use to those students who are called non-visualisers. Furthermore, as Presmeg (1992) points out that in order to construct rich understanding of mathematics, people need to possess integrated visual and analytic thinking, therefore prepared visual software environments may be used to help this integration accomplished.

CONCLUSION

In this paper, the use of software environments is discussed by the help of some theories such as "Principle of Selective Construction", the distinction between "figure" and "diagram", "transformational proof scheme". In order not to advertise any of the commercial software, clear examples of software such as screen shots were not provided. Rather, common properties of such software such as dragging, moving, shrinking, rotating was discussed in order to point out the importance of dynamic visualization. Dynamic visualization which can only be possible in software environment make easier to see changes or unchanged relationships before and after the variation of figures. Therefore, software environment has the potential of improving understanding. Software could be used for laborious constructions whilst the student can focus on specific relationships (Tall, 1993).

From the arguments made, it could be claimed that dynamic visualization can be a very powerful tool to gain a greater understanding of mathematical concepts or it can be a resource to solve mathematical problems. Software environments can make it possible to produce such dynamic images which cannot be produced on static mediums. Moreover, software environments can be a very valuable tool to provide opportunities for investigation, experimentations of geometrical as well as some other mathematical concepts. Therefore, a view that software environments should be integrated into mathematics education where possible is held by this paper.

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END NOTES

[1] As Goldenberg, Lewis, O'Keefe (1991) point out that describing a dynamic event or reading it through static mediums such as paper and words which are incapable of dynamic visualization may cause problems for both authors and readers. "Therefore, the dynamics of which we [authors] speak must be created by you [readers], in your [readers'] head, as you [they] read, just as students of mathematics (for lack of dynamic representations) have been forced to all along." (ibid, p. 1)

[2]Harel & Sowder (1998) do not exactly use the term dynamic visualisation ability; rather they use the term transformational proof scheme.

[3] This example is adapted from Goldenberg (1992, p. 6)