academicJournals

Vol. 11(9), pp. 932-944, 10 May, 2016
DOI: 10.5897/ERR2016.2763
Article Number: 846445458579
ISSN 1990-3839

http://www.academicjournals.org/ERR

Educational Research and Reviews

*Full Length Research Paper*

# Using the wonder of inequalities between averages for mathematics problems solving

**Rachel Mogilevsky Shaanan[1] and Moshe Stupel Gordon[2]***

[1]The Religious College of Education, Haifa, Isreal.
[2]The Academic College of Education, Haifa, Isreal.

Received 15 March, 2016; Accepted 20 April, 2016

**The study presents an introductory idea of using mathematical averages as a tool for enriching mathematical problem solving. Throughout students' activities, a research was conducted on their ability to solve mathematical problems, and how to cope with a variety of mathematical tasks, in a variety of ways, using the skills, tools and experiences they have acquired. The article illustrates different mathematical averages, the relation between them, and the ability to use them as an important alternative tool for solving mathematical problems. Various examples for the use of the properties in the relations between the different averages were given for the following purposes: proving inequalities, solving algebraic and trigonometric equations, investigating functions and determining the properties of geometric shapes in the plane and in the space. Also presented was the methodological aspects of the use of averages and a minor research conducted among students, on the contribution to the knowledge of this tool for mathematical problem-solving.**

**Key words:** Problem-solving, different proofs, combinations of fields in mathematics, various mathematical averages.

## INTRODUCTION

The study attempts to integrate mathematical domains and properties as a useful tool in the course “strategy for solution of mathematical problems”, as part of the program of studies of the unit for mathematical education in the high school route of the “Shaanan” College for Teacher Training. The study attempt to teach mathematics through problem-solving working, based on the various mathematical averages and the relationship between them. The mathematical averages can be an important and powerful tool for solving mathematical tasks belonging to different fields: algebra, geometry, analytical geometry, trigonometry and also in differential mathematics.

### Problem-solving

The problem-solving "concept" has been a staple of school mathematics since the early 1980s. Its importance has been emphasized in documents that guide mathematics teaching and learning in various countries, and researchers have sought to better understand students' thinking and reasoning, to improve their problem-solving and, ultimately, their learning (Chunlian

*Corresponding author E-mail: stupel@bezeqint.net.

et al., 2014). Most mathematics educators agree that the development of students' problem-solving abilities is a primary objective of instruction, and how this goal is to be reached involves consideration by the teacher of a wide range of factors and decisions (Lester, 2013).

## Proofs in mathematics

Proof and reasoning are fundamental in solving mathematical problems: they help us make sense of the mathematics, aid in communicating mathematical ideas, and justify the validity of mathematical theorems (Martin et al., 2005). One would never doubt the importance of proof in mathematics in general, nor in school mathematics (Harel and Sowder, 2007). The roles of proof are to prove, to explain, and to convince (Hanna, 1990; Herch, 1993). Stylianides (2007) defines *proof* as "A *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community;
3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291)."

Stylianides uses this definition to analyze instruction involving proof and to illuminate possible actions teachers may take to support proving activities in their classrooms. Lo and McCrory (2009) propose that in order to understand proving activities in mathematics courses, a fourth element in the definition should be added:

4. The proof is relative to objectives within the context (*context dependence*) which determine what needs to be proved.

Rav (1999) indicates that a proof is valuable not only because it demonstrates a result, but also because it may display fresh methods, tools, strategies, and concepts that are of wider applicability in mathematics and open up new mathematical directions. In Rav (1999) view, proofs are indispensible to the broadening of mathematical knowledge and are in fact "the heart of mathematics, the royal road to creating analytic tools and catalyzing growth". As Rav's (1999) states in his thesis, "proofs, rather than the statement-form of theorems, are bearers of mathematical knowledge". Hanna and Barbeau (2008) examine Rav (1999) central idea on proof, and then discuss its significance for mathematics education in general and for the teaching of proofs in particular.

Hemmi and Lofwall (2009) after a study in which they explored mathematicians' views on the benefits of studying proofs, concluded that "all mathematicians in the study considered proofs valuable for students because they offer students new methods, important concepts, and exercise in logical reasoning needed in problem solving". It is thus no wonder that developing students' ability to prove and reason is one goal of the curricular standards in many countries. For example, one principle for American school mathematics is: "reasoning and proof should be a consistent part of students' mathematical experiences in prekindergarten through grade 12" (NCTM, 2000).

Learning mathematics is fundamentally about "acquiring a mathematical point of view", "developing mathematical reasoning", "learning to communicate mathematically", "making connections" in mathematics, and building "connections" with other disciplines and (among mathematical) experiences (NCTM, 2000). Because learning mathematics involves discovery, proof and reasoning are powerful ways of developing insight, making connections and communicating mathematically. National Council of Teachers of Mathematics (NTCM) underlines this fact by claiming:

"Being able to reason is essential to understanding" (NTCM, 2000).

This suggests that proficiency in mathematical proof and reasoning is an integral part of mathematics. Similar to NCTM, mathematics educational researchers also argue that they should not undervalue the role of argumentation and proofs in students' learning and support the idea of including this subject in the school curricula and in teacher education programs. For example, Ball et al. (2000), Dreyfus (2000) and Hanna (1996) support this position:

"Proof is central to mathematics and as such should be a key component of mathematics education. This emphasis can be justified not only because proof is at the heart of mathematical practice, but also because it is an essential tool for promoting mathematical understanding" (Ball et al., 2000, IX ICME). "Proof is at the heart of mathematics, and is considered central in many high school curricula" (Dreyfus, 2000). "It maintains that proof deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding" (Hanna, 1996).

## One problem, multiple solutions and proofs

Mathematics educators agree that linking mathematical ideas by using more than one approach to solving the same problem (for example, proving the same statement) is an essential element in the development of mathematical reasoning (NCTM, 2000; Polya, 1973; Schoenfeld, 1985). Problem-solving in different ways requires and develops mathematical knowledge (Polya, 1973), and encourages flexibility and creativity in the individual's mathematical thinking (Krutetskii, 1976; Silver, 1997; Tall, 2007; Leiken and Lev, 2007).

In addition to the specific roles of proof in mathematics, the study suggest that attempts to also prove a certain result (or solve a problem) using methods from several other different areas of mathematics (geometry, trigonometry, analytic geometry, vectors, complex numbers, etc.) are very important in developing deeper mathematical understanding, creativity, and appreciating the value of argumentation and proof in learning different topics of mathematics. The study approach of presenting multiple proofs to the same problem, as a device for constructing mathematical connections is supported by the study of Polya (1973, 1981), Schoenfeld (1988), NCTM (2000), Ersoz (2009) and Levav-Waynberg and Leikin (2009).

Very similar to the study notion of 'One problem, multiple solutions/proofs' is the idea of multiple solution tasks (MST) presented by Leikin and Lev (2007), Leikin (2009) and Levav-Waynberg and Leikin (2009). MSTs contain an explicit requirement for proving a statement in multiple ways. Leikin (2009) indicates that the differences between the proofs are based on using:

1. Different representations of a mathematical concept.
2. Different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic.
3. Different mathematics tools and theorems from different branches of mathematics or
4. Different tools and theorems from different subjects (not necessarily mathematics) (Stupel and Ben-Chaim, 2013).

Adding the concept of multiple solutions/proofs for one problem into the curriculum of mathematics studies, as well as MSTs, allows the development of connected mathematical knowledge not only for students, but for their teachers as well.

## The aesthetics and the beauty of mathematics

Many times over the last 50 years, important things have been said about the polyphonic aesthetics and beauty of mathematics. Also various studies in mathematics but mostly in mathematics education engaged on the importance and their contribution to the teaching profession. Here are a few quotes and sources for this subject:

"Mathematics is one of the greatest cultural and intellectual achievements of human-kind and citizens should develop an appreciation and understanding of that achievement, including its aesthetics and even recreational aspects" (NCTM, 2000).

"The mathematician's patterns, like the painter's or the poet, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics" (Hardy, 1940).

"Although it seems to us obvious that the aesthetics is relevant in mathematics education, the aesthetics also seems to be elusive when attempting to purposefully incorporate it in the mathematical experience" (Gadanidis and Hoodland, 2003).

"Mathematical beauty is the feature of the mathematical enterprise that gives mathematics a unique standing among the sciences" (Rota, 1997).

"Mathematicians who successfully solve problems say that the experience of having done so contributes to an appreciation for the power and beauty of mathematics" (NCTM, 1989).

Mathematicians and mathematics educators believe aesthetics should be an integral part of math class (Sinclair and Crespo, 2006; Sinclair, 2009; Dreyfus and Eisenberg, 1996).

## Various mathematical averages

In daily life, when one uses expressions such as "average age", "average salary" and "average height", everyone understands the meaning of the expression, since the average is an essential tool in statistics.

In mathematics, there are several types of averages, as elaborated in the present study. To differentiate between the different types of averages, each one is given a different name. To clarify the issue, it is specified where the different averages appear, what is the relation between them and how these relations are used for proving inequalities, for solving algebraic and trigonometric equations, for investigating functions and for determining the properties of geometrical shapes in the plane and in the space. Illustration of averages in different fields, proof of the relations between them and their utilization for solving problems, point to the beauty and completeness of mathematics.

The studies and the implementation of the material to the solution of various problems do not require complex mathematical tools, and therefore the spectacular material presented in this study is suitable not only for pre-service and in-service teachers of mathematics, but also for high-school and pre-high-school students. The study mainly presents mathematical tools for solving "by a different method", which is sometimes shorter and very simple. Many mathematicians dealt with proofs and applications of inequalities between averages; the study wanted to stress the effectiveness of using inequalities in cases in which usual ways lead to difficulties and complications.

## Definitions

For n positive numbers $a_1$, $a_2$, $a_3$, … ,$a_n$ the study define the following averages:

Arithmetic average: $\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n}$

Geometric average: $\sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot ... a_n}$

Harmonic average: $\dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}}$

The average of squares: $\sqrt{\dfrac{a_1{}^2 + a_2{}^2 + a_3{}^2 + ... + a_n{}^2}{n}}$

(Handbook of mathematical functions, 1964; Dragoslav and Mitrinović, 1970).

Averages appear often in geometry and algebra. It is only worthy to mention that the property that characterizes an arithmetic progression is the one according to which every term starting from the second is the arithmetic average of its two neighbors, and for this reason the progression is called an arithmetic progression.

Similarly, in a geometric progression with positive elements, each term starting from the second one is the geometric average of its two neighbors, and hence follows the name of the progression. When considering the sequence $1\ ,\ \frac{1}{2}\ ,\ \frac{1}{3}\ ,\ \frac{1}{4}\ ,\ ...$, it is easy to see that each term, starting from the second one, is the harmonic average of its two neighbors. This sequence is called a harmonic sequence, and the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} + ...$ that is called a harmonic series is very important in differential calculus. It is also easy to show that each sequence of numbers, whose terms are the inverses of the terms of an arithmetic progression, is a harmonic sequence.

## Additional examples for the harmonic average

### *Average velocity*

A bicycle rider cycles uphill at the steady velocity of $v_1$, and back downhill at the velocity $v_2$. What is the average velocity along the entire route?

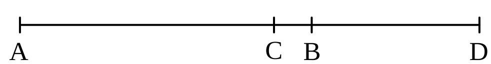
**Solution**

The study denote the travelled distance by S. The time it took to cycle uphill was $\frac{S}{v_1}$, and the time downhill was $\frac{S}{v_2}$. The study denote by V the average velocity, and obtain the equation $\frac{2S}{V} = \frac{S}{v_1} + \frac{S}{v_2}$. After some algebra, the study obtain that the average velocity of the bicycle rider is $V = \dfrac{2}{\frac{1}{v_1} + \frac{1}{v_2}}$, which is the harmonic average of $v_1$, and $v_2$.

### *Harmonic division of a segment*

The following are the definitions of harmonic division of a segment by Hardy et al. (1952*).*

1. If a point C that is located on the segment AB divides the segment so that the ratio $\frac{AC}{CB} = \frac{m}{n}$ is maintained, then the point C divides the segment AB by **an internal division** with the ratio $\frac{m}{n}$ .
2. If a point D that is located on the continuation of the segment AB satisfies $\frac{AD}{DB} = \frac{m}{n}$, then the point D divides the segment AB by **an external division** with the ratio $\frac{m}{n}$ .
3. If a point C divides AB by an internal division with the ratio $\frac{m}{n}$, and the point D divides the segment by an external division with the same ratio, then the points C




and D divide AB by a **harmonic division** with the ratio of $\frac{m}{n}$.

The word “Harmonic” that appears in the definition points to the special connection between the formed segments:

From the definition, $\frac{AC}{CB} = \frac{m}{n}$, the study obtain that:

$$\frac{AC}{AC + CB} = \frac{m}{m + n} \Rightarrow AC = \frac{m}{m + n} \cdot AB$$

From $\frac{AD}{DB}=\frac{m}{n}$, the study obtain that:

$$\frac{AD}{AD-DB}=\frac{m}{m-n} \Rightarrow AD=\frac{m}{m-n}\cdot AB$$

and using simple algebra, the study obtain:

$$\frac{2}{\frac{1}{AC}+\frac{1}{AD}}=\frac{2}{\frac{m+n}{m\cdot AB}+\frac{m-n}{m\cdot AB}}=\frac{2mAB}{2m}=AB,$$

which means that the segment AB is the harmonic average of the sections AC and AD. For example, in every triangle the bisectors of the internal and external angles of a vertex divide the base by a harmonic ratio, which equals the ratio between the lengths of the sides of the triangle.

**Theorem**

If $a_1$, $a_2$, $a_3$, …,$a_n$ are positive numbers, then the following set of inequalities holds:

$$\min\{a_1 , a_2 , a_3 , ... , a_n\} \le \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+...+\frac{1}{a_n}} \le \sqrt[n]{a_1\cdot a_2\cdot a_3\cdot ...a_n} \le$$
$$\frac{a_1+a_2+a_3+...+a_n}{n} \le \sqrt{\frac{a_1^2+a_2^2+a_3^2+...+a_n^2}{n}} \le \max\{a_1 , a_2 , a_3 , ... , a_n\}$$

and equality holds if and only if it is $a_1=a_2=a_3=...=a_n$. An algebraic proof of the system of inequalities can be found in study of Hardy et al. (1952*)*, Handbook of mathematical functions, 1964, Dragoslav and Mitrinović, 1970 and Niven (1981). A geometric proof for n = 2 can be found in in the study of Dragoslav and Mitrinović (1970) and Niven (1981).

## Applications of the relations between the inequalities of the averages

Throughout students' activities, when students were about to finish their teacher-training program at the college, a research was conducted on students' ability to solve mathematical problems, and cope with a variety of mathematical tasks, in a variety of ways, using the skills, tools, and experiences they have acquired. Some group discussions were conducted on various methodological aspects.

Following the presentation of the various features of inequality average, the students were asked to solve the tasks using the "new" tool, including explanations and clues. Finally, the new tool implementation strength was assessed, and an expectation of its future use in teaching was made.

## Extremum problems with several variables

The high school curriculum in mathematics permits one to solve extremum problems with a single variable. The use of average inequalities permits the solution of various problems with a single variable and with several variables without using differential calculus, which is one of the fascinating features of mathematics. Examples for the applications of the relations between the averages can be found in the study of Arbel (1990), Hardy et al. (1952), Kazarinoff (1961) and Niven (1981), however in order to put an emphasis on this tool, the study will present additional examples.

**Problem 1.1**

What is the value of the maximal surface area of a right cuboid with a diagonal whose length is a?

**Solution**

Let *x*, *y* and *z* be lengths of the edges that issue from one vertex of the cuboid. The diagonal of the cuboid and its edges satisfy the relation $x^2+y^2+z^2=a^2$ (the spatial Pythagorean theorem). The value of the surface area S is:

$$S=2(xy+xz+yz).$$

From the inequality of the geometric and arithmetic averages, one can write the following relations:

$$xy=\sqrt{x^2\cdot y^2}\le\frac{x^2+y^2}{2}$$
$$xz=\sqrt{x^2\cdot z^2}\le\frac{x^2+z^2}{2}$$
$$yz=\sqrt{y^2\cdot z^2}\le\frac{y^2+z^2}{2}$$

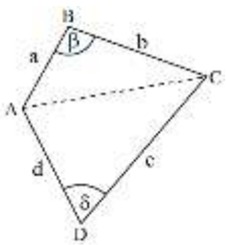
and hence, using the formula for the surface area and by adding the aforementioned inequalities, one can obtain:

$$S = 2(xy + xz + yz) \le 2 \cdot \frac{2x^2 + 2y^2 + 2z^2}{2} = 2a^2$$

The maximal area of $2a^2$ shall be obtained when $x = y = z$, for a cube.

### Problem 1.2

Prove that the area of some quadrilateral is not larger than a quarter of the sum of the squares of its side lengths. In which quadrilateral does equality hold?




### Solution

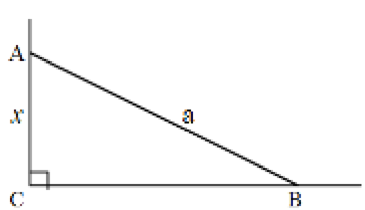
From the trigonometric formula for calculating the area of a triangle, and using the relation between the arithmetic and geometric averages, the study obtain:

$$S_{ABCD} = \frac{ab\sin\beta + cd\sin\delta}{2} \le \frac{ab + cd}{2} \le \frac{\frac{a^2+b^2}{2} + \frac{c^2+d^2}{2}}{2} = \frac{1}{4}\left(a^2 + b^2 + c^2 + d^2\right)$$

Equality holds when $a = b$ and $c = d$, and $\beta = \ \delta = 90°$, that is, the quadrilateral is a square.

### Problem 1.3

The ends of a given segment whose length is “a” move on the sides of a right angle. In which position of the segment will the area of the formed triangle be the largest?




### Solution

The study denote $AC = x$, and then $BC = \sqrt{a^2 - x^2}$.

Hence the area of the triangle is $S_{\Delta ABC} = \frac{1}{2} x \cdot \sqrt{a^2 - x^2} = \frac{1}{2}\sqrt{x^2\left(a^2 - x^2\right)}$.

From the inequality between geometric and arithmetic averages, the study obtain:

$$\sqrt{x^2\left(a^2 - x^2\right)} \le \frac{x^2 + \left(a^2 - x^2\right)}{2} = \frac{a^2}{2}$$

and then $S_{\Delta ABC} \le \frac{a^2}{4}$.

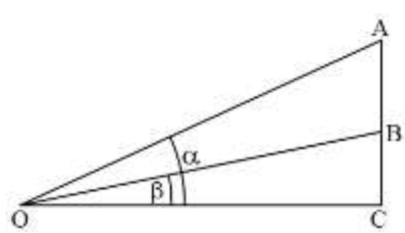
Equality holds only when $x^2 = a^2 - x^2$, in other words $AC = x = \frac{a}{\sqrt{2}}$ and also $BC = \frac{a}{\sqrt{2}}$.

***Conclusion**:* The area of the triangle $\Delta ABC$ shall be the largest when the formed triangle is not only right-angled, but is also an isosceles triangle.
***Note**:* The last problem can of course be solved using calculus.

The presented method does not require knowledge in differential calculus, but rather only the ability to apply and think creatively.

### Problem 1.4

A picture 60 cm long hangs on a wall. At what distance from the wall must one
stand in order to view the picture at the maximal angle of view?




### Solution

The study denote AB = 60 cm, the length of the picture;
O is a point at eye level above the floor, at the distance of *x* from the wall.
Find the distance OC, so that the angle AOB is maximal.

In addition, the study denotes:

$$AOC = \alpha \ , \quad BOC = \beta .$$

Using the trigonometric equality $\text{tg}\quad \text{AOB} = \text{tg}(\alpha - \beta) = \dfrac{\text{tg}\alpha - \text{tg}\beta}{1 + \text{tg}\alpha\ \text{tg}\beta}$ the study obtain that:

$$\text{tg}\quad \text{AOB} = \frac{\frac{\text{AC}}{x} - \frac{\text{BC}}{x}}{1 + \frac{\text{AC}\cdot\text{BC}}{x^2}} = \frac{60}{x + \frac{\text{AC}\cdot\text{BC}}{x}}.$$

The angle AOB obtains its largest value together with its tangent (increasing function). The value of the denominator satisfies the inequality:

$$x + \frac{\text{AC}\cdot\text{BC}}{x} \geq 2\sqrt{\text{AC}\cdot\text{BC}}.$$

Equality is obtained if and only if $x = \dfrac{\text{AC}\cdot\text{BC}}{x}$, in other words when $x = \sqrt{\text{AC}\cdot\text{BC}}$.

***Conclusion****:* When the point O is located at a distance of $\sqrt{\text{AC}\cdot\text{BC}}$ from the wall, the picture is viewed at the maximal angle of view.

**Problem 1.5**

Find the minimal value of the function $y = \dfrac{(x+1)(x+4)}{x}$ for x>0.

**Solution**

$$y = \frac{x^2 + 5x + 4}{x} = x + \frac{4}{x} + 5$$

From the inequality between the arithmetic and the geometric average, the study have $x + \dfrac{4}{x} \geq 2\sqrt{x \cdot \dfrac{4}{x}} = 4$, therefore $y \geq 9$. The minimal value of the function is obtained when $x = \dfrac{4}{x}$, in other words x=2.

Final answer: (2,9) is the point of absolute minimum of the function for x>0.

**Problem 1.6**

Find the maximal value of the function $y = \sqrt{\cos^2 x + a\sin^2 x} + \sqrt{\sin^2 x + a\cos^2 x}$ for a>0.

**Solution**

The study squares the two sides of the given function:

$$y^2 = 1 + a + 2\sqrt{(\cos^2 x + a\sin^2 x)(\sin^2 x + a\cos^2 x)}$$

Using the inequality between the geometric and arithmetic averages the study had:

$$\sqrt{(\cos^2 x + a\sin^2 x)(\sin^2 x + a\cos^2 x)} \leq \frac{\cos^2 x + \sin^2 x + a\sin^2 x + a\cos^2 x}{2} = \frac{1+a}{2}$$

In other words, $y^2 \leq 2(1+a)$, and equality is obtained if and only if

$$\cos^2 x + a\sin^2 x = \sin^2 x + a\cos^2 x,$$
$$\cos^2 x - \sin^2 x = a(\cos^2 x - \sin^2 x)$$
$$\cos 2x = a\cos 2x \Rightarrow \cos 2x = 0$$

$x = \dfrac{\pi}{4} + \dfrac{\pi}{2}k$ (k is a whole number)

Final answer: The points of absolute maximum of the function are: $\left(\dfrac{\pi}{4} + \dfrac{\pi}{2}k, \sqrt{2(1+a)}\right)$.

**Identity inequalities obtained from the average inequalities**

From the average inequalities one can produce a large number of other identity inequalities, which themselves constitute tools for multidisciplinary application. From the inequality between the arithmetic and geometric averages, it follows that for each pair of positive numbers (a, b) there holds:

$$\frac{a}{b} + \frac{b}{a} \geq 2 \qquad (2.1)$$

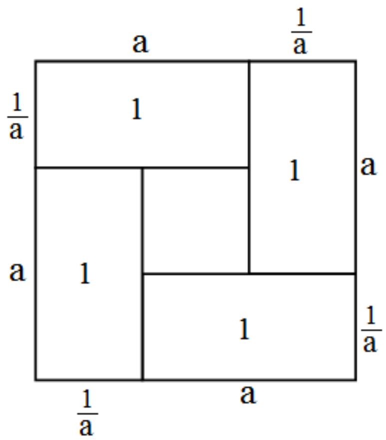
Equality holds only when a = b (this relations has uses in several branches of mathematics).

A particular case of the inequality (2.1) is that for each positive number a, there holds:

$$a + \frac{1}{a} \geq 2 \quad (2.2)$$

Equality in (2.2) holds only when a=1.

One of the geometric illustrations of inequality (2.2) is via the combination of four identical rectangles, each having one unit area, which are joined together to form a square, as shown below:



The area of the outer square is $\left(a+\frac{1}{a}\right)^2$, and it is larger than four area units. The inner square disappears when $a=\frac{1}{a}$, or $a=1$.

The study show several aspects whose proof is based on the use of average inequalities.

**Problem 2.1**

Prove that for each triplet $(a,b,c)$ of positive numbers, there holds:

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{a+c}{b}\geq 6$$

**Solution**

By decomposing each fraction, the study obtains:

$$\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{a}{b}+\frac{c}{b}=\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)\geq 6,$$

since each of the expressions in the brackets is equal or larger than 2 based on the identity in example 2.1.

**Problem 2.2**

Prove that for n positive numbers, $a_1$, $a_2$, $a_3$, … ,$a_n$, the following inequality holds:

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_n}{a_1}\geq n \qquad (2.3)$$

and equality holds only when $a_1=a_2=a_3=\ldots=a_n$.

**Solution**

By employing the average inequality for the numbers $\frac{a_1}{a_2}$ , $\frac{a_2}{a_3}$ , … , $\frac{a_n}{a_1}$, the study obtain:

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}\geq n\sqrt[n]{\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdot\ldots\cdot\frac{a_{n-1}}{a_n}\cdot\frac{a_n}{a_1}}=n$$

**Problem 2.3**

Prove the following numeric inequality: $\log_{30}{}^2 2+\log_{30}{}^2 3+\log_{30}{}^2 5>\frac{1}{3}$.

**Solution**

In accordance with the inequality between the arithmetic average and average of squares, the study had:

$$\sqrt{\frac{\log_{30}{}^2 2+\log_{30}{}^2 3+\log_{30}{}^2 5}{3}}>\frac{\log_{30}2+\log_{30}3+\log_{30}5}{3}=\frac{\log_{30}2\cdot 3\cdot 5}{3}=\frac{\log_{30}30}{3}=\frac{1}{3}$$

By squaring both sides and multiplying by 3, the study obtains:

$$\log_{30}{}^2 2+\log_{30}{}^2 3+\log_{30}{}^2 5>3\cdot\left(\frac{1}{3}\right)^2=\frac{1}{3}$$

**Problem 2.4**

Prove that in any triangle the following relation holds for the angles of the triangle $\alpha$ , $\beta$ , $\gamma$:

$$\sqrt{\operatorname{tg}\alpha\ \operatorname{tg}\beta}+\sqrt{\operatorname{tg}\beta\ \operatorname{tg}\gamma}+\sqrt{\operatorname{tg}\alpha\ \operatorname{tg}\gamma}\leq\operatorname{tg}\alpha\ \operatorname{tg}\beta\ \operatorname{tg}\gamma$$

When does the equality hold?

**Solution**

From the inequality between the geometric and the arithmetic averages, will obtain:

$$\sqrt{\operatorname{tg}\alpha\ \operatorname{tg}\beta}+\sqrt{\operatorname{tg}\beta\ \operatorname{tg}\gamma}+\sqrt{\operatorname{tg}\alpha\ \operatorname{tg}\gamma}\leq\frac{\operatorname{tg}\alpha+\operatorname{tg}\beta}{2}+\frac{\operatorname{tg}\beta+\operatorname{tg}\gamma}{2}+\frac{\operatorname{tg}\alpha+\operatorname{tg}\gamma}{2}=$$
$$=\operatorname{tg}\alpha+\operatorname{tg}\beta+\operatorname{tg}\gamma$$

From a well-known trigonometric relation, the study had:

$$\operatorname{tg}\gamma = \operatorname{tg}[\pi - (\alpha+\beta)] = -\operatorname{tg}(\alpha+\beta) = -\frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha \ \operatorname{tg}\beta}$$

By multiplying diagonally, the study obtained:

$$\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha \cdot \operatorname{tg}\beta \cdot \operatorname{tg}\gamma$$

Substitution of this relation proves the inequality.

Equality shall be obtained for:

$\operatorname{tg}\alpha = \operatorname{tg}\beta = \operatorname{tg}\gamma$, and since $\alpha+\beta+\gamma=\pi$, the study obtained that $\alpha=\beta=\gamma=\frac{\pi}{3}$, that is, the triangle is equilateral.

**Problem 2.5**

Prove that for $a>0$, $b>0$, $c>1$, the following relation holds:

$\frac{\log_c a + \log_c b}{2} \le \log_c \frac{a+b}{2}$. When does equality hold?

**Solution**

Based on the relation between the averages, $\frac{a+b}{2} \ge \sqrt{ab}$, and from the fact that for a base of $c>1$ the logarithmic function is an increasing function, the study obtained:

$$\log_c \frac{a+b}{2} \ge \log_c \sqrt{ab} = \frac{\log_c a + \log_c b}{2}$$

Thus proving the relation, equality holds when $a=b$.

**Solution of equations using the system of inequalities between the different averages**

The study presents a special kind of algebraic and trigonometric equations, for whose solution one can apply the inequality between the averages.

**Example 3.1**

Solve the irrational equation $\sqrt{5-x^2} + \sqrt{x^2+3} = 4$.

**Solution**

From the inequality between the arithmetic average and average of squares, one can write:

$$\frac{\sqrt{5-x^2}+\sqrt{x^2+3}}{2} \le \sqrt{\frac{(5-x^2)+(x^2+3)}{2}} = 2$$

And the equality $\sqrt{5-x^2}+\sqrt{x^2+3}=4$ holds only when $5-x^2=x^2+3$, that is, when $x=\pm 1$.

**Example 3.2**

Solve the following irrational equation:

$$\sqrt{x+1}+\sqrt{2x-3}+\sqrt{50-3x}=12$$

**Solution**

From the inequality between the arithmetic average and the average of squares, the study obtain:

$$\frac{\sqrt{x+1}+\sqrt{2x-3}+\sqrt{50-3x}}{3} \le \sqrt{\frac{x+1+2x-3+50-3x}{3}} = 4$$

In other words, $\sqrt{x+1}+\sqrt{2x-3}+\sqrt{50-3x} \le 12$.

Equality holds only when $x+1=2x-3=50-3x$.

The obtained system of the equation is contradictory, and therefore the original equation has no solution, even though the inequality:

$$\sqrt{x+1}+\sqrt{2x-3}+\sqrt{50-3x} < 12$$

has a solution for an *x* in the domain.

**Example 3.3**

Solve the trigonometric equation:

$$\sqrt{4\cos^2 x+1}+\sqrt{4\sin^2 x+3}=4.$$

**Solution**

From the inequality between the arithmetic average and the average of squares, one can write:

$$\frac{\sqrt{4\cos^2 x+1}+\sqrt{4\sin^2 x+3}}{2} \le \sqrt{\frac{4\left(\cos^2 x + \sin^2 x\right)+4}{2}} = 2$$

and the equality $\sqrt{4\cos^2 x+1}+\sqrt{4\sin^2 x+3}=4$ holds only when:

$$4\cos^2 x+1=4\sin^2 x+3 \;\Rightarrow\; \cos 2x=\tfrac{1}{2} \;\Rightarrow\; x=\pm\tfrac{\pi}{6}+\pi k$$

**Example 3.4**

Solve the equation $\frac{\sqrt{x-1}}{5}+\frac{5}{\sqrt{x-1}}=2$.

From the formula (2.1), having $\frac{\sqrt{x-1}}{5}+\frac{5}{\sqrt{x-1}}\ge 2$ for any $x>1$ (the domain of definition). Equality is obtained only when $\sqrt{x-1}=5$, in other words $x=26$.

**Example 3.5**

Find all the solutions of the system

$$\begin{cases} x_1+x_2+...+x_n=3 \\ \frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}=3 \end{cases}$$

**Solution**

Addition of the two equations yields the relation

$$\left(x_1+\tfrac{1}{x_1}\right)+\left(x_1+\tfrac{1}{x_2}\right)+...+\left(x_n+\tfrac{1}{x_n}\right)=6$$
(3.1)

From the relation (2.2), the value of each addend is larger or equal to 2, and therefore $n\le 3$.

If n = 3, equality holds only for $x_1=x_2=x_3=1$.
If n = 2, the study obtains the system of equations:

$$\begin{cases} x_1+x_2=3 \\ \frac{1}{x_1}+\frac{1}{x_2}=3 \end{cases} \Rightarrow \begin{cases} x_1+x_2=3 \\ x_1\cdot x_2=1 \end{cases} \Rightarrow x_1^{\,2}-3x_1+1=0 \Rightarrow \begin{matrix} x_1=\frac{3+\sqrt{5}}{2} \\ x_2=\frac{3-\sqrt{5}}{2} \end{matrix}$$

If n = 1, the study obtains a contradiction:

$x_1=3$ and also $\frac{1}{x_1}=3$.

In other words, the final solution is:

$$n=3 \;\Rightarrow\; x_1=x_2=x_3=1$$
$$n=2 \;\Rightarrow\; x_{1,2}=\frac{3\pm\sqrt{5}}{2}$$

**Example 3.6**

Prove that there are no positive numbers *x* and *y* that satisfy the following system of equations:

$$\begin{cases} x^{19}\cdot y^{92}=2^{1992} \\ 19x+92y=5752 \end{cases}$$

**Proof**

From the inequality between the arithmetic and geometric averages the study had:

$$\frac{19x+92y}{111}=\frac{x+x+...+x+y+y+...+y}{111}\ge \sqrt[111]{x^{19}\cdot y^{92}}$$

Therefore, $x^{19}y^{92}\le\left(\frac{5752}{111}\right)^{111}<52^{111}<\left(2^6\right)^{111}=2^{666}$,

and the study obtain a contradiction to the data in the first equation.

**Example 3.7**

Solve the equation:

$$2^{x^6}+2^{x^2}=2^{x^4+1}.$$

**Solution**

By using the inequality between the arithmetic average and the geometric average twice, sequentially, the study obtains:

$$2^{x^6}+2^{x^2}\ge 2\sqrt{2^{x^6}\cdot 2^{x^2}}=2^{\frac{x^6+x^2}{2}+1}\ge 2^{\sqrt{x^6\cdot x^2}+1}=2^{x^4+1},$$

which means that equality shall be obtained only when $2^{x^6}=2^{x^2}$, in other words:

$$x^6=x^2 \;\Rightarrow\; x=0\;,\; x=1\;,\; x=-1$$

***Conclusion**:* the equation has three solutions.

**A practical problems**

***Problem 4.1***

River boat should pass some distance against the direction of water flow of river. The operation costs per unit of time are proportional to the square forward speed. We need to find the speed of the boat so that the expenses will be minimal. River water speed is c kilometers per hour.

**Solution:**

Let ***x*** **km/h** be the ship's own speed (speed of standing water). Let $\ell$ be the distance it needed to pass.

The ship's expenses per hour is $\mathrm{k}x^2$, so expenses of all the way is $f(x) = \mathrm{k}x^2 \cdot \frac{\ell}{x-c}$ .

Let observe at

$$\frac{1}{f(x)} = \frac{x-\mathrm{c}}{\mathrm{k}\ell x^2} = \frac{1}{\mathrm{k}\ell} \cdot \frac{1}{x} \cdot \frac{x-\mathrm{c}}{x} = \frac{1}{\mathrm{k}\ell\mathrm{c}} \cdot \frac{\mathrm{c}}{x} \cdot \left(1 - \frac{\mathrm{c}}{x}\right)$$

By using the inequality of the geometric and arithmetic averages, one can write the following relations:

$$\sqrt{\frac{\mathrm{c}}{x} \cdot \left(1 - \frac{\mathrm{c}}{x}\right)} \le \frac{\frac{\mathrm{c}}{x} + \left(1 - \frac{\mathrm{c}}{x}\right)}{2} = \frac{1}{2}$$

and then $\frac{1}{f(x)} \le \frac{1}{4\mathrm{k}\ell\mathrm{c}}$, or $f(x) \ge 4\mathrm{k}\ell\mathrm{c}$.

Equality holds only when $\frac{\mathrm{c}}{x} = 1 - \frac{\mathrm{c}}{x}$ , i.e $x = 2\mathrm{c}$.

***Conclusion**:* The expenses of the ship are the smallest when its speed is twice the speed of the water flow in the river.

**Problem 4.2**

In pole scales an arm’s length are a and b when a>b. We weigh 2 kg of sugar by two ways:

1. We put the sugar to right palm of the scales and a 1 kg weight on the other.
2. We put the sugar to left palm of the scales and a 1 kg weight on the other.

Do you get 2 kg of sugar? And if not, who earned: seller or buyer?

**Solution**

Let X kg be the weigh of sugar in the first case and y kg be the weigh of sugar in the second case.

For scale balance it is needed that

$$\begin{cases} a \cdot 1 = b \cdot x \\ a \cdot y = b \cdot 1 \end{cases}, \text{ thus, } y = \frac{b}{a} \;; x = \frac{a}{b} \;;$$


$$x + y = \frac{a}{b} + \frac{b}{a} > 2 \qquad (a \ne b)$$

***Conclusion:*** The buyer profited.

**The methodological research aspect of the use of the properties of various averages**

The subject of various averages and the connections between them, was presented to students teaching mathematics training courses usually in the third year, in the course "Strategies of Math Problem Solutions", but also in courses like: "Integration of mathematical topics" and "Mathematical Enrichment Seminar", where different aspects of the averages are presented. Each of the three courses mentioned earlier is attended by approximately 15 to 20 students.

**Stage one**

In this stage, the students were asked which of them know the term 'Mathematical average ' and in which domains it appears. The students were told that there are a few kinds of averages. All students mentioned they know the term in connection to average salary and average speed, when the subject was taught during statistics studies. Some students mentioned that in an arithmetic progression each member is the average of the two members adjacent to it.

Some students mentioned the term 'Weighted average', and as an example noted the method used to calculate the average of their final exam grades, where each subject has its own weight. Only about 10% of the students mentioned the term 'Geometric average' in a geometric progression. Only about 5% knew that the arithmetic average is bigger than the geometric one, and they are equal only if the two numbers are equal. None of them have heard of any other types of averages.

The students were impressed by geometric proofs of inequalities between averages. They were surprised to know that average speeds are actually a harmonic average. It was new for them to know that harmonic division of segment connect to harmonic average. Students learned about the connections between names of progressions (arithmetic, geometric, harmonic) and their basic features (property of three successive members).

**Stage two**

In this stage, the students learned 5 different types of averages, examples of their usage and their relationship. To convince the order of the averages beyond numerical examples, algebraic and geometric proofs were brought for every two types of averages. Using two different mathematical fields for a proof helps to increase the importance of using different aspects of math, and demonstrate how different mathematical subjects are all interwoven.

**Stage three**

In this stage, the students were asked to solve the problems of this article, and some additional ones in the ways that are familiar to them. Some of problems could be solved in a conventional way (by using the regular and known methods taught in high-school or during academic studies). Then, the students were asked to solve the problems using properties of averages. The students having trouble (most of the students) were given hints. For example, they were told which inequality they should use, or how to break tasks data, or which of the previous problems they should refer to. After that process, they were given the nice, easy solution. All groups expressed great interest and desire to pass the knowledge on to their future pupils.

**Stage four**

In their homework, the students were asked to solve two relatively simple problems, when the aim was using averages in the process. Indeed, 80% of the students solved one problem correctly, and 55% solved both. 6 additional problems were given to them as homework and they were required to solve each one in the conventional way, and then using properties of averages. Some of the problems were given with hints.

**Stage five**

The problems that were given to the students gave impressive results. Some were introduced to the rest of the students. About 80% of the students successfully solved the problems on their own. Usually, the students learning two from the three courses mentioned earlier, and the subject of averages and their application is taught over 3 to 4 lessons only.

**Stage six**

At this point, the students were asked a few questions about the importance of obtaining knowledge on the subject, and here are a few statements gathered from their answers:

1. The subject promoted to expanding the toolbox of methods to solve math problems.
2. Elementary knowledge was received to help solve problems quickly and easily.
3. The new tools and their application show the beauty of math and how its fields all connect one to other.
4. The time dedicated to the subject was too short and it requires additional practice for better ability to use it in class.
5. The mathematics timetable is massive and it's doubtful that enough time can be dedicated to teaching the subject to high school students. For this reason, and considering the fact that good students often need additional challenges, it's better to teach the subject of averages during mathematical circles in schools.
6. Problems that were difficult to solve using algebra were easily solved using the properties of averages.
7. It was suggested that while teaching fractions to elementary students, it's a good idea to teach them that the sum of every two inverse positive numbers is always equal to or greater than 2, and let them practice this using numerical examples.

**General satisfaction from the course**

In addition to this study, feedback regarding this course was obtained via post-course assessment forms distributed by the college. The courses involving multiple solutions scored relatively well. The average assessment was 5.7 (out of 6), and trainees expressed fairly high satisfaction from the course and its management in the free-text section.

## CONCLUSION

There are different strategies for improving and solving problems in different areas of mathematics. The strategies are based on the use of standard mathematical tools that are suitable for the type of the task. Sometimes, the solution of the task is achieved by means of mathematical tools from one area, and in other cases by a combination of tools from different areas. The more tools exist in the toolbox, the higher is the chance of successfully dealing with the task. A “complete toolbox” and extensive mathematical knowledge permit one to find different methods of solution for the same task, and in particular to reach the fastest, simplest and clearest solution, thereby showing the beauty of mathematics. The “elegant” solution is usually brief, aesthetic and instructional and it is better than the “Standard” solution which is usually longer and is based on ordinary technique. The study presented a collection of various tasks from different fields in mathematics, whose solution is attained more easily by means of the special relations between the different averages, with the purpose of familiarizing ourselves with these relations and strengthening the use of them as a tool that in many cases allows one to achieve short, simple and very surprising solutions

## Conflict of interests

The authors have not declared any conflict of interests.

### REFERENCES


Arbel B (1990). Strategies for solving mathematical problems (in Hebrew), published by the Open University, Tel Aviv.

Ball DL, Hoyles C, Jahnke HN, Movshovitz-Hadar N (2000). ‘The teaching of proof’, *ICME IX Discussion Document*, Tokyo, Japan.

Chunlian J, Stephen H, Cai J (2014). Chinese and Singaporean sixth-grade students' strategies for solving problems about speed. *Educ. Stud. Mathe. 87*:27-50. DOI 10. 1007/s 10649-014-9559-x.

Dragoslav Dr, Mitrinović S (1970). "Analytic Inequalities"**,** Springer-Verlag, New York.

Dreyfus T (2000). ‘Some views on proofs by teachers and mathematicians’, In: A. Gagatsis (ed.) Proceedings of the 2nd Mediterranean Conference on Mathematics Education, The University of Cyprus, Cyprus. 1:11-25.

Dreyfus T, Eisenberg T (1996). On different facets of mathematical thinking. In: R. J. Sternberg, & T. Been-Zeen (Eds.). The nature of mathematical thinking. Mahwah, NJ: Lawrence Erlbaum. pp. 253-284

Ersoz FA (2009). Proof in different mathematical domains. ICME Study. 1:160-165.

Gadanidis G, Hoodland C (2003). The aesthetics in mathematics as story. Can. J. Sci. Math. Technol. Educ. 3(4):487-498.

Hanna G (1990). Some pedagogical aspects of proof. Interchange 21:6-13.

Hanna G (1996). The ongoing value of proof’, Proceedings of PME 20: Psychology of Mathematics Education 20th International Conference 2:21-34.

Hanna G, Barbeau E (2008). Proofs as bearers of mathematical knowledge. ZDM Math. Educ. 40:345-353.

Harel G, Sowder L (2007). ‘Toward comprehensive perspectives on the learning and teaching of proof’, In: F.K. Lester (ed.), Second handbook of research on mathematics teaching and learning: a project of the national council of teachers of mathematics, Information Age Publishing Inc., Greenwich, CT. pp. 805-842.

Hardy GH (1940). A mathematician`s apology. Cambridge, UK: Cambridge University Press. P 40.

Hardy GH, Littlewood JE, Polya G (1952). Inequalities, Cambridge University Press.

Hemmi K, Löfwall C (2009). Why do we need proof. Proceedings of CERME 6, January 28th - February 1st 2009, Working Groups. 2:201-210.

Hersh R (1993). Proving is convincing and explaining. Educ. Stud. Math. 24:389-399.

Kazarinoff ND (1961). Analytic Inequalities, New York; Holt, Rinchart & Winston.

Krutetskii VA (1976). The psychology of mathematical abilities in schoolchildren. J. Kilpatrick and I. Wirszuy (eds.), (translated from Russian by Teller, J.), The University of Chicago Press, Chicago.

Lester FK (2013). Thoughts about research on mathematical problem-solving instruction. Math. Enthusiast. 1(2):245-278.

Levav-Waynberg A, Leikin R (2009). Multiple solutions for a problem: A tool for evaluation of mathematical thinking in geometry’, In Proceedings of CERME 6, January 28th - February 1st 2009, Lyon, France. pp. 776-785.

Liekin R (2009). Multiple proof tasks: Teacher practice and teacher education’. ICME Study 19(2):31-36.

Liekin R, Lev H (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity, In: J.H. Wo, H.C. Lew, K.S. Park and D.Y. Seo (eds.), Proceedings of the 31st International Conference for the Psychology of Mathematics Education. The Korea Society of Educational Studies in Mathematics, Korea. 3:161-168.

Lo JJ, McCrory R (2009) ‘Proof and proving in a mathematics course for prospective elementary teachers’. ICME Study 19(2):41-46.

Martin TS, McCrone SMS, Bower MLW, Dindyal J (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. Educ. Stud. Math. 60:95-124.

National Council of Teachers of Mathematics (NCTM) (1989). Curriculum and Evaluation standards for school mathematics. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics, NCTM, Reston, VA.

Niven I (1981). Maxima and Minima without Calculus, Washington: The Mathematical Association of America.

Polya G (1973). How to Solve It: A New Aspect of Mathematical Method, Princeton University Press, Princeton, NJ.

Polya G (1981). Mathematical Discovery: On Understanding Learning and Teaching Problem Solving, Wiley, New York.

Rav Y (1999). Why do we prove theorems? Philosophia Math. 7(1):5-41.

Rota GC (1997). The phenomenology of mathematical beauty. Syntbase 111(2):171-182.

Schoenfeld AH (1985). Mathematical Problem Solving, Academic Press, New York.

Schoenfeld AH (1988). When good teaching leads to bad results: The disasters of “well-taught” mathematics courses’. Educ. Psychol. 23(2):145-166.

Sinclair N, Crespo S (2006). What makes a good problem? An aesthetics lens.In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), Proceedings of the 30th Conference of the International Group for the Psychology of Mathe. Educ. 5:129-136.

Silver EA (1997). Foresting creativity through instruction rich in mathematical problem solving and problem posing. ZDM 3:75-80.

Sinclair N (2009). The aesthetic as a liberating force in mathematics education. ZDM 41(1):45-60.

Stupel M, Ben-Chaim D (2013). One problem, multiple solutions: How multiple proofs can connect several areas of mathematics. Far East J. Mathe. Educ. 11(2):129-161.

Stylianides AJ (2007). Proof and proving in school mathematics. J. Res. Math. Educ. 38:289-321.

Tall D (2007). Teachers as Mentors to encourage both power and simplicity in active material learning’, Plenary Lecture at the. Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing, 17-19, March 2007, Abu-Dhabi.