The Effect of Math Modeling on Student's Emerging Understanding

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Abstract

This study investigated the effects of applying mathematical modeling on revising students' preconception of the process of optimizing area enclosed by a string of a fixed length. A group of 28 high school pre-calculus students were immersed in modeling activity that included direct measurements, data collecting, and formulating algebraic representation for the data. The lab conduct was enriched by scientific inquiry elements such as hypothesis stating and its verification. While 86% of the students (N=24) falsely hypothesized that the rectangular areas enclosed by a string of a fixed length will remain constant before engaging in the lab, the subsequent tasks of the modeling activity prompted the students to correct their ways of thinking. The study showed that the modeling processes provide ample means of revising students' perception to establish firm conceptual background for inducing a more rigorous algebraic approach to solving problems in math classes. Suggestions for further studies follow.

Keywords: Mathematical modeling; optimization; problem solving; high school.

Introduction

Misconceptions are defined as strongly held, stable cognitive structures that must be overcome or eliminated for students to achieve expert understanding (Hammer, 1996). Many researchers object the term misconception because from the student's viewpoint, the ideas are logical and instead the terms preconception, naïve theories, and alternative framework have been proposed (Sneider & Ohadi, 1998). Since the emphasis of this study is to help students uncover, by themselves, the mathematical underpinning of the optimization process and its correct interpretation, the author prefers to use the term *preconception* whose correctness will be attempted during the process of modeling.

Mathematical modeling (MM) has been described in many ways; for instance, Lesh and Harel (2003) defined MM as finding quantifiable patterns of a phenomenon and its generalization, while Blum, Galbraith, Henn, and Niss (2007) defined mathematical modeling as a process of "learning mathematics so as to develop competency in applying mathematics and building mathematical models for areas and purposes that are basically extra-mathematical" (p. 5). Although initiated several decades ago, MM recently gained substantial popularity in math research and education; in fact, in the Common Core Curriculum, MM is recognized as one of the eight fundamental standards (Porter, McMaken, Hwang, & Yang, 2011).

MM activities can be organized in many ways: by asking open-ended questions, by mathematizing given situations, by using simulations as a means of supporting context, or by solving word problems (Haines & Crouch, 2007). Thus MM serves as an embodiment of problem solving, and the learning methods are interwoven. A problem in mathematics is defined as a situation carrying open questions (Blum & Niss, 1991). As such, it usually follows four steps: understanding the problem, devising a plan, carrying out the plan, and looking back (Polya, 1957). MM offers a different perspective; it focuses learners on the process of transferring given information presented in a real-world situation by data gathering and model formulating and then requires the learner to use the model to develop new knowledge or solve context-related real-world problems (Crouch & Haines, 2004). By converting problems into the process of taking quantifiable data and modeling formulation, MM activities can "meet the individual abilities of (many) more students" than traditional teaching methods (Kaiser, 2007, p. 104). MM can be guided by different types of inquiries, ranging from deductively situated, authentic problem-modeling activities (English & Sriraman, 2010) to inductively organized inquiries that have students make plausible arguments that take into account the context (Lesh & Zawojewski, 2007).

Prior Research Findings

Optimization is a field of applied mathematics whose principles and methods are used to solve quantitative problems in mathematics and other disciplines including biology, engineering, physics, and economics (Rardin, 1997). Optimizing is a process of making the best possible choice from a set of candidate choices (Boyd & Vandenberghe, 2009). It consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set of computed values of the function. Optimization problems are very common in high school and undergraduate mathematics curricula, and they are an essential component of problem solving in calculus courses (National Council of Teachers of Mathematics [NCTM], 2000). Despite the wide applicability range, the process of optimization is not often investigated in mathematics education; in fact, a search for prior research findings using university search engines such as ProQuest Educational Journals, Science Direct, and Google Scholar returned only several such studies, whose summaries follow.

Troxell (2002) pointed out certain pitfalls in using technology while teaching optimization and suggested relying more on analysis and validation. Poon and Wong (2011) proposed an adoption of Polya's (1957) problem-solving model to investigate and solve optimization problems in geometry. They highlighted benefits of inter-disciplinary learning and investigation of multiple solutions while working on such problems. Schuster (2004) discussed the role of combinatorial optimization activities to solve optimization problems in high school mathematics and concluded that such activities provide opportunities not only for mathematical modeling of real world problems, but also for discovering, constructing, and investigating algorithms. The process of optimizing areas enclosed by a string of a fixed length was investigated by Brijlall and Ndlovu (2013), who found out that when taught by traditional methods, students used "isolated facts and procedures" (p. 16) while solving these problems, which showed their lack of understanding of underlying principles. They further suggested that "teachers needed to be aware of learners learning conflicts so as to reinforce the new concepts they encounter" (p. 17). Ledesma (2011) investigated how to identify a solid with a maximum volume built by cutting squares of various side lengths from a cardboard and folding the shape. She concluded that calculus students faced difficulties understanding the optimization process, which she considered a main obstacle in solving these problems. As a means of improving, she suggested strengthening the visualizing process by using simulations. A similar optimization problem was investigated by Lowther (1999). Although she intended the students to find the maximum value algebraically, the students chose to determine the answer using a method of trial and error. Lowther noted, "I was sure that someone would use an algebraic approach, but no one did" (p. 764). By not formulating algebraic function, this investigation did not help the students solve text problems, as was initially intended.

Although all of the above scholars provided valuable suggestions to improve understanding of the optimization processes, the idea of verifying students' preconceptions on these processes and applying MM to provide learners with tangible experiences was not proposed within this domain. While optimization problems can be classified as typical math problems, the process of transferring one dimensional geometrical object (length) into two dimensional geometrical object (area) is not that apparent. This study proposes an activity where this process is made more transparent to students.

Research Questions

While typical problems of optimization focus students' attention on finding unique solutions due to formulated algebraic representation that maximizes the scenario (e.g., see Demana, Waits, Foley & Kennedy, 2008; Stewart, 2007; Sullivan & Sullivan III, 2009), an MM activity seeks to diversify the process by having students investigate possible outputs and identify a pattern before formulating an algebraic representation. The purpose of the study reported herein was to determine if immersing students in an MM activity would help them with understanding the algebraic underpinning of optimization. In this line, the following research questions were formulated:

- What are the students' preconceptions on the output of area enclosed by a string of a fixed length?
- Will modeling environment help learners understand the process of optimization?
- Which of the MM stages—data collecting, table generating, graph plotting, or model formulating—have the most significant effect on understanding optimization?

Theoretical Framework of the Treatment Design

The theoretical framework of the study was supported by constructivist learning theory that suggests learner's construction of their own knowledge based on received impulses (von Glasersfeld, 1995). This theory has been selected because constructivism is suited for activities that focus on meaning – making, concept construction, or elucidation of alternative concepts (Fergusen, 2007) which encompass the process of finding an optimum of a quantity due to given constrains. Minstrell and Smith (1983) have recommended several teaching methods and learning settings for eliminating student's misconceptions.

Experiment or experience that will allow students testing their thinking is one of learning setting. While typical problems on optimization focus students' attention on finding unique solutions due to formulated algebraic representation that maximizes the scenario (e.g., see Demana, Waits, Foley & Kennedy, 2008; Stewart, 2007; Sullivan & Sullivan III, 2009), an MM activity seeks to diversify the process; have students investigate possible outputs, identify a pattern before formulating an algebraic representation and finding its optimum value. Considering research recommendations and MM learning outputs, modeling was selected as framework for the activity design.

Development of the Modeling Process

MM usually follows a cycle (e.g., see Blum & Leiss, 2007; Geiger, 2011; Lesh & Lehrer, 2003; Yoon, Dreyfus, & Thomas, 2010). Although the MM activity in this study intended to follow a general modeling process in which a real-world problem was abstracted, mathematized, solved, and evaluated, special attention was given to having students hypothesize the problem's output before attempting to solve it and then revise or confirm their hypotheses after eliciting and validating the mathematical model. It was anticipated that inserting this element of scientific inquiry (Hestens, 2013) would not only help assess the research questions but also allow the students to realize the importance of mathematical processes in modifying or confirming their prior thinking. Since the activity involved quantification, the hypothesis had a twofold nature: qualitative and quantitative.

The quantitative part, called a prediction (see Figure 1), involved having students predict a numerical output of their investigations, and the qualitative part involved having them hypothesize a general property or output of their investigations. While a typical modeling activity is initiated by asking learners to write a problem statement (Lesh & Zawojewski, 2007), in this study's activity, the problem was formulated and the students' initial task was to hypothesize possible outputs. Another element of the activity design was to decide about the type of inquiry.

There were two main options to consider, deductive and inductive, which are the main reasoning inquiries used in science, mathematics, and engineering (Prince & Felder, 2006). As deductive inquiry characterizes the process of reasoning from a set of general premises to reach a logically valid conclusion, inductive inquiry is a process of reasoning from specific observations to reach a general conclusion (Christou & Papageorgiou, 2007). In this activity, students were given a string of a fixed length and asked to formulate various rectangles and search for patterns; thus, an inductive inquiry was adopted. All of the design constraints were guided through the scheme depicted in Figure 1.

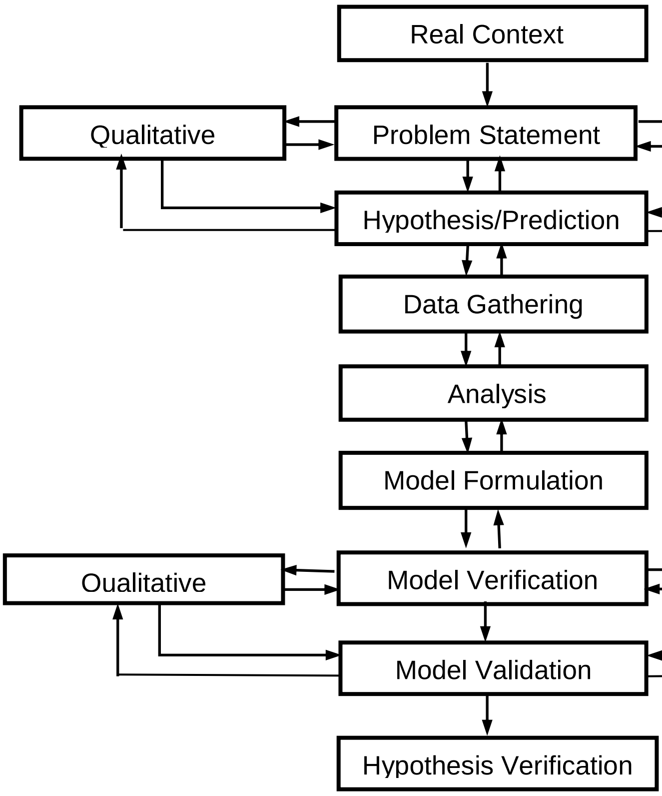

Figure 1. Adopted modeling process based on Sokolowski (2015)

The real context, in this activity, will be provided by material and described scenario that will be followed by a problem statement. Note that the multiple arrows indicate stages when students can decide to modify or revise concluded patter on algebraic model. Such established theoretical framework satisfied also the six principles for problem-solving activities developed by Lesh and Kelly (2000).

Context Development

The following problem was used as a contextual basis for converting into a modeling activity: "Among all the rectangles, whose perimeters are 100 ft, find the dimensions of the one with a maximum area" (Demana, Waits Foley, & Kennedy, 2007, p. 184). Converting the problem into a modeling activity required inducing all of the elements of the modeling cycle described in Figure 1 as well as considering research recommendations concerning inducing metacognitive changes in students' minds. For instance, Bonotto (2007) suggested that well designed MM activity (a) emphasizes the strong interactivity of the instructional techniques, and (b) eliminates ready-made solution processes. Bearing the adopted modeling process and the recommendations, an outline of instructional support was developed.

Methods

This study can be classified as an empirical one-group quasi-experimental (Shadish, Cook, & Campbell, 2002). Randomization of participants was not possible due to a low school population where the study was conducted. Quasi –experimental study shares though many similarities with experimental design.

Participants

A group of 28 students (12 females, 16 males, M_{age} = 16.5 years, age range: 16-17 years) enrolled in a pre-calculus course in a suburban high school was assigned as a treatment group. The group of students was selected due to availability. 25% of the group constituted minorities.

Evaluation Instruments

Analysis of students' hypotheses, assigned prior the lab conduct, along with verification of the hypotheses, reflections, and survey were used to evaluate learners' prior thinking and the effects of the treatment. A content analysis (Hsieh & Shannon, 2005) that characterized a systematic classification process of coding and identifying themes and patterns was used to evaluate students' free responses.

Treatment, Setup, Materials, and Procedure

A physics lab was the designated site for the activity because the students could be seated in groups, which allowed for collaboration and discussions. There were seven groups of four students. The treatment lasted for one class period (55 minutes). Each group was given a 74 cm long string, a Styrofoam board of 40 cm by 60 cm, four pins, a metric ruler, and a set of French curves to sketch a smooth curve for generated data. Each student was given instructional support in the form of a lab outline. The instructor initiated the activity by explaining that the students would investigate the area enclosed by a string of a fixed length and pointing out the importance of reading the problem statement and hypothesize the answer. The students were asked to state their hypothesis individually, without discussing the context with group partners. As suggested by previous research (Lesh & Lehrer, 2003), the instructor then took on the role of a guider, providing suggestions when questions arose rather than offering direct solutions. Although the students took data in groups, each student was responsible for completing the lab analysis individually. Students then began collecting data by formulating various rectangles due to prearranged lengths (see Figure 2).

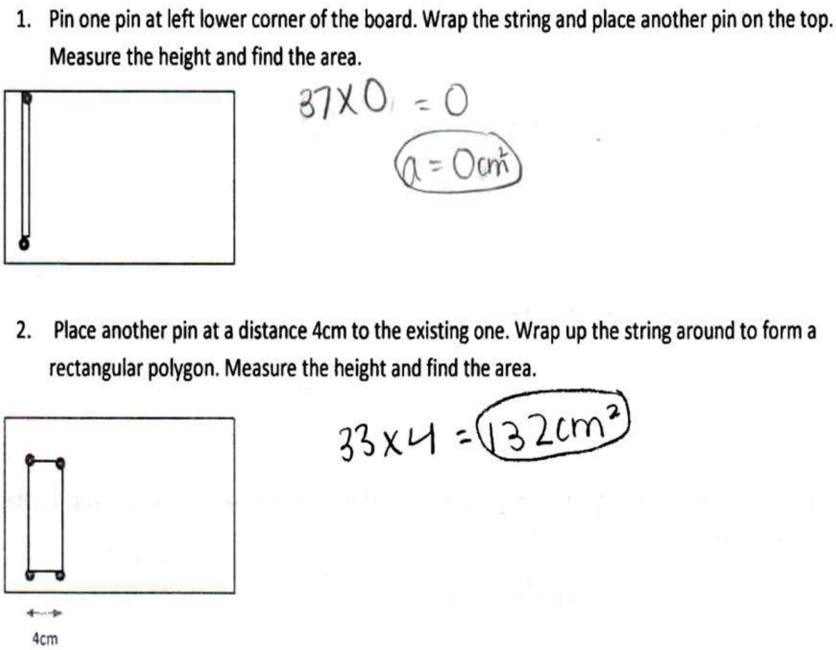

Figure 2. The process of data taking

In order to systematize the process, the participants recorded their data in a table (see Figure 3):

Length of the base	Height of the Rectangle	Area of the Rectangle
0 cm	37cm	0cm^2
4 cm	33cm	132cm^2
10 cm	27cm	270cm^2
16 cm	21cm	336cm^2
20 cm	17cm	340cm^2
25 cm	12cm	300 cm^2
30 cm	7cm	210 cm^2
35 cm	3cm	105cm^2

Figure 3. Generated table of values

The lengths of the bases of the rectangles were prearranged. The students measured resulting heights and computed area for each variation. The analysis of the data was initiated by generating the area of the rectangular polygons versus height graph (see Figure 4).

Figure 4. Area versus height graph

Note, in the Figure 4, the students called the height width which did not change the data interpretation. Having the points plotted and the graph sketched, the students identified its general algebraic structure. There pre-calculus students did not have difficulties with identifying underpinning function as quadratic. The next stage referred to formulating the function equation for the graph and discussing the graph properties such as the extreme value, its coordinates and interpretation (see Figure 5).

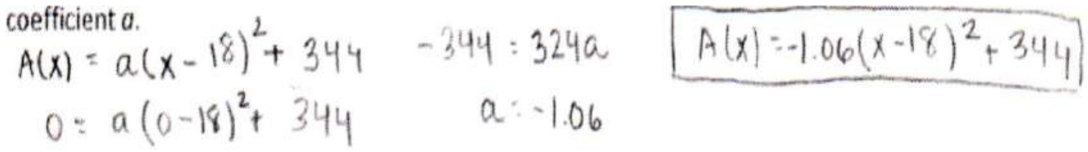

Figure 5. The process of formulating the area function

This student identified the height of 17 cm for which the area was a maximum and equal to 344 cm^2. She further used this coordinate to compute the leading coefficient, *a,* of the parabola.

Once the model was developed, it underwent a twofold validation process—firstly by using the elicited function to further identify and confirm the maximum area, and secondly by using graphing technology to generate the graph and calculate the maximum value. After the validation process was finalized, the students were asked to reflect on their hypotheses. Lab reflections and survey completions concluded the activity.

Results and Discussion

The analysis of students' hypotheses revealed that the majority (N=24, 86%) claimed that the magnitudes of the areas enclosed by a string of a fixed perimeter would remain unchanged. Following are examples of students' hypotheses:

- The area will not change because the total length of the string is the same.
- The area will not change because it is enclosed by a length that cannot change.
- The area will not change because the length of the string as a whole never changes.

Representative responses of students who claimed that the area would change were as follows:

- The area will change because if you have a perimeter of 30 cm, then, for example, 5 cm x 10 cm = 50 cm^2, 8 cm x 7 cm = 56 cm^2, etc.
- The area will change because it is dependent on its dimensions, which will change.

After the lab was completed and the model confirmed, the above-mentioned majority group of students agreed that the area enclosed was changing and that it reached a maximum value when the formulated polygon was a square of about 17 cm by 17 cm, and the remaining students (N=4, 14%) confirmed their hypotheses.

Students were asked to answer the following reflection question after the experiment: "Describe your learning experience while working on the lab. What did you learn?" In their responses, 54% of the students (N=15) referred to their falsely stated hypotheses and commented on revision of this preconception that was prompted by the lab outcomes. Specific verbal responses were clustered and quantified into three categories, as illustrated in Table 1. Sample student responses are provided following the table.

Table 1. Summary of Students' Verbal Reflections

Category	Frequency of Response /Percent
Commented on revising their hypothesis	15 (54%)
Commented on properties of heights and formulation of the maximum area	11 (39%)
Made general comments	2 (7%)

The majority of the students reflected on their incorrect hypotheses, indicating that the lab helped them discover the correct process. Responses included the following:

- The lab taught me that even though something is of a fixed length, it can be made into different shapes that will result in different areas.
- On the lab my original hypothesis was wrong so I learned that even with the same size of the string there are certain shapes and dimensions that will have a larger area.
- While working on the lab, I learned that a fixed string can have various areas depending on its length and width.
- I learned that even though the perimeter between two rectangles is the same, the areas do not have to be the same. They can be different.
- I proved my hypothesis about the string correct. I learned that the area of the string changes when the base length is changed.
- I learned that if you have a fixed perimeter, then it does not mean you have a fixed area. I never truly thought about it.

The majority of the students pointed out the interface of transitioning from one-dimensional geometrical object (length of the string) to a two-dimensional geometrical object (area). They noted that the product of the lengths can take different values despite a constant perimeter.

Eleven of the students (39%) described the behavior of the elicited function, focusing on the maximum area and on the properties of the dimensions that produced the maximum area. Responses included:

- We learned that changing bases and heights changes the area but in a pattern creating a parabola. The area increase reaches the max and then the area begins to decrease.
- I learned that a perfect square will have the greatest amount of area. I also learned how functions apply to the real world.
- It was fun and interesting working with classmates. I learned that to get the maximum area, the length and width of the rectangle should be as close in measurement as possible.
- I learned that the area increases as the width and height become more similar.
- I learned that as the width got bigger, the area would increase then reach its peak, then slowly decrease, making a parabola.

General comments describing the learning experience were provided by two students (7%), as follows:

- I learned how to apply the equations in real-world situations. This helped me understand some of the purposes for these equations and rules.

Since mathematical modeling is organized by sequentially organized phases, the question of interest was what phase of the MM process had the most significant impact on changing students' perception on the optimization process. The following multiple choice survey questions sought to provide insight on this inquiry: "What phase of the lab convinced you the most that the areas enclosed by a fixed length perimeter were not constant? Circle the phase that applies to you: sketching the area vs. length graph, measuring the lengths of rectangles and calculating the areas, or finding the function equation and its maximum value and generating a table of values." The order of the written phases did not follow the actual order of the activity during the lab in order to reduce the chance of bias. The question was addressed to all students, even to those whose hypotheses were correct. Their responses are summarized in Table 2.

Table 2. Summary of Modeling Phases That Affected Students' Change of Thinking

Phase Description	Frequency of Response/Percent
Sketching area vs. length graph	3 (11%)
Measuring the lengths of the rectangles and calculating the areas	8 (28%)
Finding the function equations and the maximum area	2 (7%)
Generating a table of values	15 (54%)

The table shows that most of the students concluded the pattern and revised how the area changed during systematizing their data in a table of values, followed by the phase of sketching area versus length graph. It is to note, that all of the processes are part of scientific modeling (Schwarz & White, 2005). This finding further supports the process of blending scientific modeling with mathematical modeling advocated by Sokolowski (2015).

Conclusions

The results support the research hypothesis that immersing students in MM activities can help revise their preconceptions and help them understand the underlying principle of mathematics. The study also revealed that the tangible experiences of taking measurements and generating a table of values were the phases that altered the students' thinking the most. Thus, the systematic way of gathering data and subsequent analysis led the students to discover an embedded principle of optimization. One can conclude that inducing elements of measurement that are typical of the scientific inquiry process (Hestenes, 2013) benefits learners in mathematics classes as well. This finding further supports integrating scientific modeling with mathematical modeling in one coherent structure advocated by Sokolowski (2015).

This study also supports the conclusion reached by other scholars (e.g., see Brijlall & Ndlovu, 2013; Ledesma, 2011) that the source of difficulties with optimization problems might not necessarily be rooted in the mathematization of the processes but in the difficulty of understanding the underlying mathematical principle. The ultimate extension of this study would be to investigate whether the lab experience helped students with formulating the solution processes to solve textbook problems on optimization. More specifically, another study might explore how to use reached conclusions to induce more rigorous mathematical apparatus to solve typical textbook problems on optimization, or might try to determine if removing the students' misconception is sufficient to have them succeed with problem solving. It seems that both of the venues are worthy undertaking.

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