Assessing Developmental Students’ Number Sense: A Case Study
Parveen Ali, Indiana State University of Pennsylvania

Abstract
The data for this study were gathered from an assignment consisting of 10 number sense related mathematics problems completed in an algebra course at developmental level. The results of the study suggest that a majority of developmental mathematics students use routine algorithmic procedures rather than mathematical reasoning to solve problems. They lack quantitative judgment or estimation skills. Only a small percentage of students utilize benchmarks to mentally compare numbers and are unable to use numbers in convenient ways to simplify calculations. A socio-constructive approach to teaching that encourages multiple procedures for problem-solving and invites students to invent and discuss ways to solve a numerical situation is recommended in this study. Educators are encouraged to question, facilitate, and engage rather than model solutions to their students.

Mathematics proficiency is becoming increasingly important, especially for the technological careers of the 21st century. As a result, employers are demanding higher levels of mathematics skills from their employees in their workplaces than in the past. In addition, mathematics and science-oriented jobs will have the highest rate of growth and tend to yield higher salaries. Therefore, students with limited mathematics proficiency may have limited career opportunities (Lago & Diperna, 2010). However, many students perceive mathematics as a subject that is not creative and is disconnected from reality, consisting of facts and symbols that need to be memorized and have no value (Maclellan, 2012; Silver, 1989). These students do not perceive that meaningful ways of mathematics learning increase logical thinking and problem-solving skills. As increasing numbers of students are entering colleges, a vast majority of them are leaving high school underprepared for college-level work especially in mathematics. The rising number of college students enrolling in developmental mathematics courses suggests a need to reexamine the mathematics curriculum and the learning environment that developmental educators and secondary school teachers provide for these students. Most of these students are taught to follow routine algorithms to derive answers and are lacking quantitative thinking skills, including number sense, the skills of which provide the basic foundation for higher order mathematical skills and concepts (Lago & Diperna, 2010).

What is Number Sense?
Number sense is a holistic construct that is difficult to define (Yang & Wu, 2010). It is not a fixed entity that a student either has or does not have but rather a process that develops and matures with experience and knowledge (Reys & Yang, 1998; Sood & Jitendra, 2007). Some other definitions of number sense include “the ability to quickly understand, approximate, and manipulate numerical quantities” (Wilson et al, 2009, p.124) and “a term which encompasses several skills related to a ‘common sense’ about numbers” (Gay & Aichele, 1997, p. 27). Students with good number sense have the ability to use numbers in flexible ways to make mathematical judgment and to develop useful strategies for handling numbers and operations. "Those who view numbers in this way continually utilize a variety of internal ‘checks and balances’ to judge the reasonableness of numerical outcomes. When an outcome conflicts with the perceived expectation, the person revisits the mathematical situation to externally view it, often through another lens, attempting to resolve the conflict” (Reys et al., 1999, p. 61).

Students with strong number sense understand how to:
- use numbers in flexible ways when adding, subtracting, multiplying or dividing;
- use benchmarks to make mathematical judgments;
- make mental calculations and reasonable estimations;
- make predictions;
- understand numerical relationships between mathematical concepts, facts and skills;
- recognize unreasonable answers. (Faulkner, 2009; Dunphy, 2007; Gersten & Chard, 1999; Gersten et al., 2005; Lago & Diperna, 2010; Malofeeva et al., 2004; Reys et al., 1999).

These students are capable of transferring their mathematical knowledge and skills to a broad range of quantitative tasks. They find mathematics everywhere, not only in school.
“Number Sense” continued

Purpose of the Study
An increasing number of studies have focused on fostering number sense among students. Most of these studies have examined improving instructional models for teachers, early intervention efforts, and ways of improving instruction. The targeted populations in these studies included elementary students, students with learning disabilities and low income, and middle school students (Bobis, 2008; Cain, 2009; Dunphy, 2007; Faulkner, 2009; Lago & Diperna, 2010; Malofeeva et al., 2004; Sood & Jiterdra, 2007; Wilson et al., 2009). In addition, cross-cultural studies have been conducted including those in sites such as Taiwan, Japan, Australia, and Sweden (Reys & Yang, 1998; Yang & Wu, 2010; Reys et al., 1999). Because of the importance of number sense skills, it is essential that developmental mathematics educators investigate how to improve the learning environments of their struggling students in a way that increases number sense. Therefore, the primary purpose of this study was to analyze the number sense of students enrolled in a developmental mathematics course offered by the developmental studies department of a Pennsylvania university. The secondary purpose of this study was to apply the results of the analysis to improve instructional practices of developmental mathematics educators in fostering the growth of number sense in their students.

Sample
Subjects in this study were students enrolled in a developmental mathematics course titled Elements of Algebra. Students were placed in this course based on their standardized Compass mathematics test scores, which they took during their freshman orientation. They scored above 30 in Pre-Algebra and between 0 and 26 in Algebra.

They were instructed to solve 10 number sense related problems in the mathematics lab during the spring semester of 2010. The author developed these problems after reviewing literature on number sense. The students solved these problems in the mathematics laboratory without using calculators. It should be noted here that students in these mathematics classes were required to attend at least 10 tutorial/mathematics laboratory sessions during the semester, and their attendance in these sessions was included in the grade calculation. Participation in this study was voluntary. Students received bonus points and one mathematics lab attendance grade to encourage participation in this study. In addition, participating students provided demographic information which is summarized in the following table:

Findings
Student responses to the 10 number-sense-related basic arithmetic problems are described below. Students were also instructed to explain their responses in writing.

Q1. What is three tenths more than 0.52?
This question was designed to investigate whether the students understood the positional notation that characterizes our base-10 number system, their word names and basic mathematics vocabulary. Their responses are summarized in Table 2.

<table>
<thead>
<tr>
<th>Response type</th>
<th>Number</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct answer</td>
<td>12</td>
<td>41</td>
</tr>
<tr>
<td>Incorrect answer</td>
<td>17</td>
<td>59</td>
</tr>
</tbody>
</table>

Table 2: Demographic Information

<table>
<thead>
<tr>
<th>Gender/Race/Generation</th>
<th>Distribution</th>
<th>Sample</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gender</td>
<td>Male</td>
<td>14</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>Female</td>
<td>15</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>29</td>
<td>100</td>
</tr>
<tr>
<td>Race</td>
<td>White</td>
<td>15</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>African American</td>
<td>10</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>Hispanic</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Asian</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>Generation in College</td>
<td>First</td>
<td>17</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>Second</td>
<td>7</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>More than second</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>Did not answer</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
“Number Sense” continued

Out of 29 students, only 12 (41%) students answered the question correctly \((0.52 + 0.3 = 0.82)\). Eight of them did not offer any explanations. One student explained, “I would first convert 3/10 into a decimal and then I would add that decimal to the one given to get my sum.” Some of the particular incorrect responses included the following: “you multiply 0.52 by three”; “you have to divide by 10”; “I personally don’t even remember how to do this. I do remember how 2 is in ones column & 5 is in the tenth column.” Another student wrote “0.52 + 0.03 = 0.55.”

It appears from analysis of the above that most of these students had a weak understanding of the place value recognition of numbers in base 10. Some of these students most likely were not familiar with very basic mathematics vocabulary, such as “more.”

Q2. Is \( \frac{1}{3} - \frac{5}{7} \) the same as \( \frac{3}{7} - 2 \)?

Why or why not?

This problem was designed to investigate whether the students had a conceptual understanding of fractions and flexibility with numbers, especially fractions to simplify calculation. Table 3 provides a summary of their responses.

<table>
<thead>
<tr>
<th>Response type</th>
<th>Number</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>7</td>
<td>24</td>
</tr>
<tr>
<td>No</td>
<td>22</td>
<td>76</td>
</tr>
</tbody>
</table>

Seven out of 29 (24%) students concluded “yes” to this question. Out of the seven, one student followed a routine algorithm for subtraction of fractions and concluded that they are equal.

Two students provided explanations: “I am not allowed to use a calculator. I think yes it’s the same because \( \frac{5}{7} \) rounded to 2 is the same.” Another student wrote, “Yes they are equal to each other. I do not know how to explain why, they are just are?”

Out of the 22 students who responded “no,” most incorrectly subtracted the fractions.

It appears from this study that most of the students struggled with both conceptual and procedural understanding of fractions. Moreover, they were most likely not used to using numbers in flexible ways to make their computations simpler. Research demonstrates that there is a correlation between mastering fractions and performance in advanced mathematics courses. Students who struggle with fractions most likely will not graduate from college and will have fewer career opportunities (Jordan et al., 2013; Seigler & Pyke, 2013).

Q3. Which is larger: \( \frac{13}{24} \) or \( \frac{19}{39} \)?

This problem was intended to explore the students’ conceptual understanding of rational numbers, especially whether they were able to compare the size of rational numbers by using benchmarks. Table 4 represents the summary of their responses.

<table>
<thead>
<tr>
<th>Response type</th>
<th>Number</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>13/24 is larger</td>
<td>23</td>
<td>79</td>
</tr>
<tr>
<td>19/39 is larger</td>
<td>5</td>
<td>17</td>
</tr>
<tr>
<td>Equal</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Out of these, 23 (79%) were correct in stating that \( \frac{13}{24} \) is larger than \( \frac{19}{39} \); 14 of them did not provide any explanations. Seven students explained their answers:

“I would initially say that \( \frac{13}{24} \) is larger because the bottom of the fraction is smaller.”

Q4. Find the best estimate of \( \frac{1125}{0.98} \).

This problem was designed to identify students’ understanding of the meaning and effect of operations (identify more or less than 1125). The students’ responses to this problem are summarized in Table 5.
He then added 22.55 to 1125 and wrote the estimation as 1147.55. Three students used paper and pencil division. Out of these three students, only one student successfully came up with the correct quotient. Two could not divide them properly. Very poor performance of students was observed with this item. A large proportion of the students were most likely lacking knowledge of the relative effects of mathematical operations on numbers, especially with decimal numbers. They were also deficient in reasonable estimation skills. They most likely did not recognize that multiplication does not always yield a larger result, and division does not always yield a smaller number.

Q5. How many decimal numbers are there between 4.3 and 4.4?
The primary purpose of this problem was to investigate whether the students understand that there are numbers between other numbers. There is a space between 4.3 and 4.4; a goal of this problem was to see whether students can relate that space with numbers. The responses are summarized in Table 6.

<table>
<thead>
<tr>
<th>Response type</th>
<th>Number</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reasonable estimation (number sense-based)</td>
<td>8</td>
<td>28</td>
</tr>
<tr>
<td>Did not answer</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>Wrong estimation</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>Did not understand Question</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Paper pencil/calculator/explanation without answers</td>
<td>13</td>
<td>45</td>
</tr>
</tbody>
</table>

Eighteen students added all the numbers together and wrote “11446” without estimating. Also, another four tried to add all of the numbers together and came up with wrong answers. Four participants estimated without any explanation. The remaining three students used number sense to estimate:

6000 + 4000 + 2000 + 10 = 12000, “I just rounded every number and added them.” (This student ignored 10 in her final estimation.)

5575 + 3882 + 1982 = 11439—11,000—11,400—11,440 (This student ignored 7 and wrote three estimations.)

5575 + 3882 = 10357 + 1989 = 12,346 = 12,000 (she ignored 7)

The majority of students in this sample tended to heavily rely on computational techniques that had been taught in their schools. They did not even use standard algorithms to estimate the addition. Also, comparatively, a large portion of the students added all numbers together. It appears from this study that a large proportion of the students are most likely lacking good estimation skills. They also demonstrate weak quantitative judgment about the relative size of numbers, which depends on situational context. The same number can refer to a lot or a little, depending on situational contexts. Gay & Aichele (1997) explained, “Understanding a number as a quantity of a specific magnitude and being able to judge how it compares to another number is a basic to number sense” (p. 27).

7. Which is larger: 75 ÷ 0.025 or 75 ÷ 0.25?
This item attempted to examine whether or not students understood the meaning of operations by decimal numbers, and whether or not they had the ability to make a magnitude comparison between these two quotients. Their responses are summarized in Table 8.
Although slightly more than 50 percent of students responded that 75/0.025 is larger than 75/0.25, only four of them provided number-sense-based explanations, such as: “75/0.025 is larger because it’s a smaller decimal”; “The extra zero makes it a smaller number.” Thirteen participants responded that 75 ÷ 0.25 is larger. Eight of the students were unable to explain their answers. Some of the ambiguous explanations from the students who stated that 75/0.25 is larger than 75/0.025 were: “Because 0.25 is larger than 0.025”; “Because the 0 in front of the 2 matters, it’s a lower number”; “75 ÷ 0.25 because it only goes out to the hundredths place.”

It appears from the students’ responses that a large proportion of the sample share conceptual misunderstandings about division. Also, the absence of connection between understanding and rules was evident. They are most likely missing the concept of how many 0.025 and 0.25 they could make out of 75 (the relative size of the divisor). They may not understand that numerals are used to stand for quantities, and operations are for actions on quantities (Hicbert, 1989).

Q8. Estimate 48 percent of 500.
The main purpose of this particular item was to test the students’ conceptual understanding of the percent symbol and its fractional representation. Additionally, this problem was designed to explore whether the students were capable of using benchmarks to simplify calculation. Table 9 represents the findings.

Thirteen participants estimated the number from 235 to 250. Among them, one notable explanation was, “Half of 500 is 250. 48% is of 100. There is 5, 10 so 2% less x 5 = 10. 240.” Five students multiplied and came up with the exact answer. One student did not answer the question. The rest of the participants’ estimations were not realistic. Some of them multiplied the two numbers incorrectly.

It appears from the responses that the majority of the students in this sample were lacking estimation skills, which limit their ability to assess the reasonableness of an answer. They tended to rely solely on rules and procedures to arrive at an exact answer rather than estimating. The responses from the students also uncover that a large proportion of them are most likely not familiar with using benchmarks that can be applied to simplify calculations.

9. Which is larger?
A) \( \frac{1}{5} \times \frac{1}{8} \times \frac{1}{2} \times \frac{1}{4} \)
B) \( \frac{1}{5} \times \frac{1}{32} \)
The goal of this item was to test whether the students were familiar with the flexibility of substituting different representations of numbers for a quantitative judgment. Table 10 summarizes their responses.

Out of the 10 correct responses, only four students used number-sense-based strategies to make the correct quantitative discrimination between the two statements. They clarified that 1/2 is larger than 1/5. Two multiplied the fractions together to make the statement, and four did not write any reasoning for their answers.

In contrast, 18 students thought that 1/5 * 1/8 * 1/2 * 1/4 is larger than 1/5 * 1/32. Out of these 18, 10 stated that A is larger than B because when they multiplied the fractions together the denominator of A is larger than B. Eight students did not have any further clarifications for identifying A > B. Two students indicated the two statements are equal.

The above analysis on this item suggests that these students struggle with quantitative judgment, especially with regard to fractions. They most likely prefer to follow a routine algorithm and are incapable of using numbers in flexible ways to simplify comparison.
“Number Sense” continued

10. Is A equal to B? Why or why not?
A) $4 \times 18 \times 5 \times 12$
B) $5 \times 9 \times 24 \times 4$

This problem was also designed like the previous one to investigate the students’ quantitative judgment with respect to whole numbers. Another purpose was to test whether they could use meaningful and flexible ways to identify different representations of a number for mental calculation. Table 11 presents a summary of findings.

<table>
<thead>
<tr>
<th>Response type</th>
<th>Number</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal</td>
<td>15</td>
<td>52</td>
</tr>
<tr>
<td>Not equal</td>
<td>14</td>
<td>48</td>
</tr>
</tbody>
</table>

Out of these 15 correct responses, only two students demonstrated good number-sense-based explanations: “Yes, they are equal. Both have 4 and 5, then A has 18 and 12, B has 9, 24. Multiply one from each by 2 = same.” “Yes, they are because 19*12 and 9*24 both equal 216 and 5*4 are in both.” Another response was: “A = 4*320, B = 4*320, A = B. Eight students multiplied and arrived at the correct answer. Out of the 14 incorrect responses, eight multiplied inaccurately. The remaining six students gave various reasons for their conclusion. Interestingly, three students stated that the answers are not equal due to multiplying different numbers.

It appears that some students were skilled in paper-and-pencil computations, especially with whole numbers, but not skilled in their use of non-computational approaches that rely on number sense. However, their performance on this item was better than the previous item, which dealt with fractions.

The study further investigated the performance of students based on their gender, race and generation in college. The 10 number sense-based problems were graded out of 100 which revealed that male and female students did not differ significantly in their performance on number-sense test items (female students’ average score = 24% and male students’ average score = 25%). White students performed slightly better than other races (white students’ average score = 27%, 15 students; other races average score = 24%, 14 students).

Interrestingly, second or multi-generation college students did not perform better than first generation college students (first generation average score = 29%, 17 students; and multigenerational average score = 18% 11 students; one student did not answer the question).

Conclusion
The researcher conducted this study in a department that offers two developmental mathematics courses based on students’ standardized Compass test scores. The lower level of these two courses is “Basic Mathematics,” which reviews arithmetic skills and their applications. The higher level of developmental mathematics course is “Elements of Algebra” which reviews basic algebra. The students in this study were enrolled in the algebra course entitled Elements of Algebra which teaches symbolic mathematics and abstract reasoning skills, and which requires a solid conceptual understanding of basic mathematics.

However, after analyzing students’ work from the sample in this study, it appears that they most likely do not have a solid foundation of basic arithmetic skills, especially with regard to number sense. They struggle with the most fundamental concepts of the base ten numeration system and place value. Significantly, a large percentage of them demonstrated the following characteristics: poor understanding of rational numbers; inability to make good quantitative judgments; lack of reasoning skills; reliance solely on rule-based procedures to arrive at an answer; deficiency in estimation skills; and inability to perform mental calculations. Only 29 students participated in this study. However, based on this sample, male and female students did not differ significantly in their performance on 10 number-sense problems. Data also suggest that white students performed slightly better than other races, and second or multi-generation college students did not perform better than first generation college students in this study.

Why do most of these students lack number sense? It may be a direct consequence of curriculum, especially in secondary schools. Students understand mathematics depending on the way they were taught, which may have been mainly through procedures and habits (Faulkner, 2009; Palha et al., 2013). Written rule-driven computation limits mathematical thinking and understanding and hinders development of number sense. Dunphy (2007) found too much emphasis on assessment in public schools and almost no discussion of content. Too often students have been taught how to derive a correct answer but have not been encouraged to explain the logic behind the solution (Wu, 2011). Other contributing factors include classroom practices that do not provide or emphasize mental computation and verbal reasoning or that do not teach mathematics in the context of real life. Also, many instructional practices do not include providing appropriate feedback to students (Gersten & Chard, 1999).
“Number Sense” continued

What can developmental mathematics instructors do to promote number sense among their students? Developmental mathematics classes tend to cover too many topics in one semester, which may result in less student engagement. With this in mind, developmental mathematics educators need to revisit their mathematics curriculum. It is suggested that covering fewer topics in depth may improve students’ conceptual learning outcomes. It is also suggested that instructors include more classroom practices that encourage students to develop multiple procedures to solve problems and invite students to invent these multiple procedures. Classroom environments should emphasize mental computations and verbal reasoning for these computations, and educators should invite their students to share and discuss these mental computations and solutions with other students. Instructors should be encouraged to question, facilitate, and engage rather than model solutions to their students. In addition, it is apparent from this study that there is a need for more collaboration and discussion among mathematics educators, curriculum developers, and textbook writers, test developers and researchers to find ideas and ways that may improve number sense among developmental mathematics students. There is a new trend of redesigning developmental mathematics to achieve better learning outcomes for students, and continued research on this matter is vital. Some colleges in the nation are adapting self-paced computer aided instruction (emporium models) for this redesign. Time will tell whether this emporium model improves students’ quantitative ability or number sense.

Limitations
This study is based on students who took a developmental algebra course at a rural university in Pennsylvania. Its representativeness is limited. Caution should be exercised in generalizing the results. However, the data suggest that the mathematical problems in this study may be useful in assessing and monitoring developmental mathematics students’ level of understanding in number sense.

References
DOI: http://dx.doi.org/10.5038/1936-4660.5.2.3
Available at: http://scholarcommons.usf.edu/numercy/vol5/iss2/art3
“Number Sense” continued


Dr. Parveen Ali is an associate professor in the Department of Developmental Studies at Indiana University of Pennsylvania in Indiana, Pennsylvania.