Adding some perspective to de Moivre's theorem: Visualising the n-th roots of unity

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Introduction

An n-th root of unity, where n is a positive integer (i.e., $n = 1,2,3,\ldots.$), is a number z satisfying the equation

$$z^n = 1 \tag{1}$$

Traditionally, z is assumed to be a complex number and the roots are usually determined by using de Moivre's theorem adapted for fractional indices. The roots are represented in the Argand plane by points that lie equally pitched around a circle of unit radius. The n-th roots of unity always include the real number 1, and also include the real number –1 if n is even. The non-real n-th roots of unity always form complex conjugate pairs. This topic is taught to students studying a mathematics specialism (ACARA, n.d., Unit 3, Topic 1: Complex Numbers) as an application of de Moivre's theorem with the understanding that the roots occur in the complex domain.

Meanwhile, in the Cartesian plane, a closely related topic deals with the solution of polynomials (ACARA, n.d., Unit 2, Topic 3: Real and Complex Numbers). Consider the most general case in which

$$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0.$$

The roots are found by setting $y = 0$ and solving for x, which can be carried out easily when $n = 1$ and $n = 2$ but becomes significantly more challenging for higher order values of $n \geq 3$. However, there is one special case which is amenable to solution for any value of n, namely the one in which all the coefficients equal zero except $a_n = 1$ and $a_0 = -1$; then, the following reduced monic polynomial results:

$$x^n - 1 = 0 \tag{2}$$

This also leads to the n-th roots of unity, although the location of these n-roots relative to the Cartesian plane is commonly misunderstood (see Stroud (1986, Programme 2, Theory of Equations); he (incorrectly) places the complex conjugate roots in the Cartesian plane at one of the turning points of a cubic equation.) In fact, Equations (1) and (2) are essentially the same, since the possibility of complex solutions should most definitely be entertained for Equation (2). However, although both topics deserve to be taught as synonymous problems, the connection between them is rarely made.

The aim of this paper is to demonstrate visually the connection between the reduced polynomial $y = x^n - 1$ in the Cartesian plane and the resulting n-roots which invariably appear in the Argand plane. There is no contradiction here: the reader will find a three-dimensional surface representation of Equation (2) provides the full link between both the Cartesian and Argand planes, and illustrates not only the location of the roots in relation to the original equation but also shows why they occur with conjugate pairings. Examples will be provided for the cases $n = 3$, $n = 5$ and $n = 8$ which will be sufficient to illustrate the general pattern that emerges. The approach adopted here is a natural extension of the surface visualisation techniques first presented by Bardell (2012) for quadratic equations.

de Moivre's method

Students studying a mathematics specialism as a part of the Victorian VCE (Specialist Mathematics, 2010), HSC in NSW (Mathematics Extension in NSW, 1997) or Queensland QCE (Mathematics C, 2009) will be familiar with de Moivre's theorem and its applications to complex numbers (see the short discourse by Bardell (2014) for further details). In order to find the n-th roots of unity, the value of unity itself has first to be expressed as a complex number, z, in polar form thus:

$$z = \cos\,(2\pi k) + i\sin\,(2\pi k) \text{ where } k = 0, 1, 2, \ldots\ n-1 \text{ and } i = \sqrt{-1}\,.$$

The possibility of n-multiple solutions is accounted for by the inclusion of the integer k. Now apply de Moivre's theorem using the fractional index $\frac{1}{n}$ to find the n-th roots of z.

$$z^{\frac{1}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) \text{ where } k = 0, 1, 2, \ldots\ n-1. \qquad (3)$$

It only remains to substitute chosen values of n into Equation (3) to determine the roots of specific cases.

The cube roots of unity: n = 3

By using de Moivre's Theorem from Equation (3),

$$\sqrt[3]{z} = z^{\frac{1}{3}} = \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right),\ k = 0, 1, 2.$$

When $k = 0$ $\quad \sqrt[3]{z} = 1$

when $k = 1$ $\quad \sqrt[3]{z} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

and when $k = 2$ $\quad \sqrt[3]{z} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

These then are the three roots of unity. If these are plotted in the complex (Argand) plane they appear equally spaced at 120° intervals around a circle of unit radius centred at the origin O (see Figure 1).

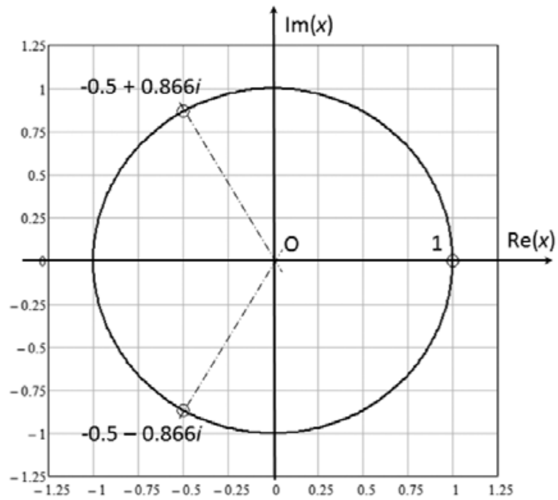

Figure 1. Plot in the Argand plane showing the three cube roots of 1.

Now consider the corresponding polynomial $y = x^3 - 1$. A simple plot of this equation in the Cartesian plane is shown in Figure 2. Although three roots are expected, only one real root is visible in this plot, occurring at $x = 1$. It is a matter of some conjecture exactly where the remaining two roots are located. An indication is given in Figure 2, based on the *a priori* knowledge that the real part of both complex conjugate roots is –0.5. Making sense of what this really means is another matter altogether! Note: The turning point of this curve is found by setting $\frac{dy}{dx} = 3x^2 = 0$, which gives $x = 0$. Hence it is by no means a given that the complex conjugate roots occur at a turning point, as inferred by Stroud (1986).

In this particular case, since the two other roots are known to be complex and always to occur as a conjugate pair, this implies that the values assigned to x in Equation (2) do not have to be limited to the set of real numbers but rather could—and possibly should—include complex values. Whilst this is implicitly obvious, since no restriction was ever placed on x or y at the outset,

it is rarely presented as an explicit proposition. Perhaps one reason why this is seldom considered is simply one of expedience—constructing a plot using generally complex values of x is not for the faint-hearted!

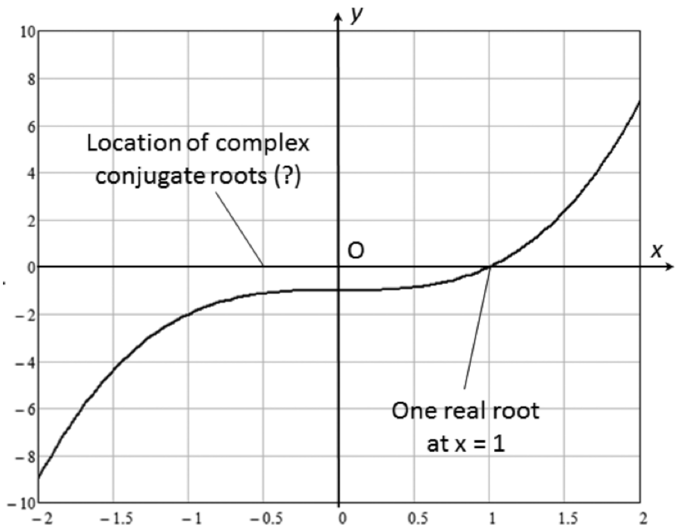

Figure 2. Plot in the Cartesian plane of the equation $y = x^3 - 1$.

Nevertheless, if this line of reasoning is pursued and x is allowed to take the generally complex value $G + iH$, then

$$y = x^3 - 1 \text{ becomes } y = (G + iH)^3 - 1 \tag{4a}$$

Rather than solve this equation directly using de Moivre's theorem (as explained previously), there is much profit to be gained from expanding and simplifying the complex form of this equation, viz.

$$y = G^3 + 3iG^2H - 3GH^2 - iH^3 - 1 \tag{4b}$$

The expansion of Equation (4a) is easily accomplished using Pascal's triangle and remembering that $i^2 = -1$, $i^3 = -i$ and so forth. Since y must also be generally complex, say of the form $A + iB$, then it is possible to equate the real (Re) and imaginary (Im) parts of both sides of Equation (4b).

Equating Re parts: $A = G^3 - 3GH^2 - 1$ (5a)

Equating Im parts: $B = 3G^2H - H^3$ (5b)

Equations (5a) and (5b) describe the generalised complex form of Equation (4a). Each equation represents a three-dimensional surface (the actual classification of this, and the higher-order surfaces presented herein, is beyond the scope of this paper) in which the ordinate value A or B can be plotted over a grid of points in the Argand plane defined by the H and G

axes. By calculating both individual surfaces defined by Equation (5a) and Equation (5b) for a range of complex x values, i.e., a range of corresponding G and H values, the true and most general form of Equation (4a) can be visualised.

Clearly, from the definition of a root to occur when $y = 0$, this implies that both $\text{Re}(y)$ and $\text{Im}(y)$ must simultaneously be zero. In other words, the location and nature of the roots will be defined where the two surfaces for $\text{Re}(y)$ ($\equiv A$) and $\text{Im}(y)$ ($\equiv B$) have a common intersection with a horizontal plane positioned at zero altitude.

In the three-dimensional surfaces that follow, the horizontal plane contains the $\text{Re}(x)$ ($\equiv G$) and $\text{Im}(x)$ ($\equiv H$) axes, thus forming the Argand plane, whilst the vertical axis represents either $\text{Re}(y)$ ($\equiv A$) or $\text{Im}(y)$ ($\equiv B$) depending on which surface is being investigated. It is understood that the GA and GB planes represent the Cartesian plane for each surface.

All the three-dimensional plots presented in this paper were constructed using *Mathcad* (2007), which is one of many VCE/QCE/HSC-approved computer algebra systems available to schools, and fully commensurate with the following ACARA (n.d., Rationale) stated aim: "to develop students' capacity to choose and use technology appropriately".

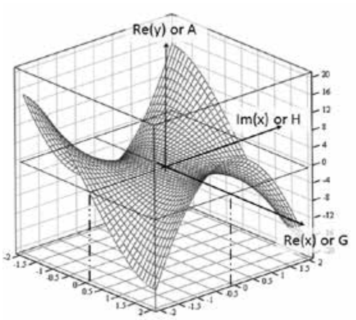

Figure 3(a). Surface A: The real part of the complex representation of $y = x^3 - 1$.

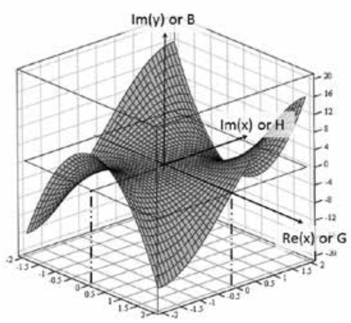

Figure 3(b). Surface B: The imaginary part of the complex representation of $y = x^3 - 1$.

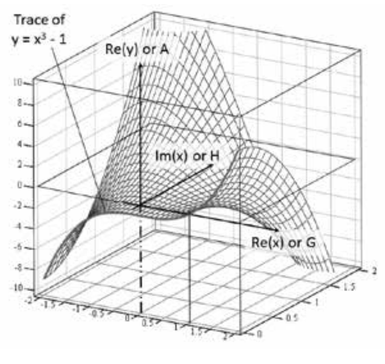

Figure 4. Section taken through Surface A at $H = 0$, revealing the trace of the original cubic equation $y = x^3 - 1$

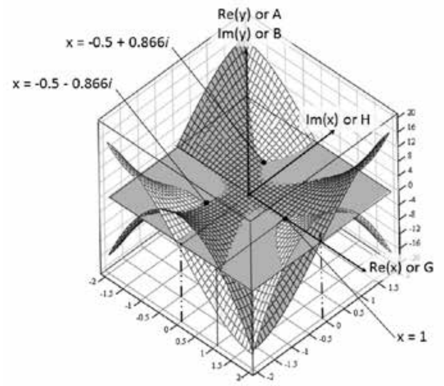

Figure 5. The intersections of Surfaces $A = B = 0$, revealing the location of the three roots of $y = x^3 - 1$.

The three-dimensional surfaces A and B, plotted over the range $-2 \le G$, $H \le 2$, are shown in Figure 3(a) and 3(b) respectively. Both surfaces are known colloquially as 'monkey saddles' on account that a monkey sitting astride the

G-axis of surface A, or the H-axis of surface B, has a place for its two legs and tail. The principal GA plane of surface A, which is its plane of symmetry, is perpendicular to the principal HB plane of surface B. HB is also the symmetry plane of surface B.

Upon setting $H = 0$ in Equation (5), the following results: $A = G^3 - 1$, and $B = 0$. In other words, the trace of the purely real cubic polynomial, as first presented in the Cartesian x–y plane in Figure 2, is found in the GA plane (see Figure 4). This confirms the GA (or GB) plane is in fact the Cartesian plane. It is observed that in the GA plane the locus of points represented by $B = 0$ is in fact a straight 'nodal' line coincident with the G-axis. This means the surface B has no influence at all on the Cartesian plane, which explains why the single curve shown in Figure 2 provides a full description of $y = x^3 - 1$. This also explains why it is reasonable to expect any real root(s) to lie somewhere along the G-axis. This observation holds for all reduced polynomials of the form $y = x^n - 1$.

With reference to Figure 5, the definition of a root to occur when $y = 0$ implies that both Re(y) and Im(y) must simultaneously be zero. In other words, the location and nature of the roots will be defined where the two surfaces for Re(y) ($\equiv$ A) and Im(y) ($\equiv B$) have common intersections with a horizontal plane positioned at zero altitude. For the present case with $n = 3$, three roots will be expected, and hence three unique points of intersection should be observed when the surfaces $A = B = 0$. These are clearly shown in Figure 5. These points of intersection are, of course, none other than the three points plotted in the Argand diagram in Figure 1.

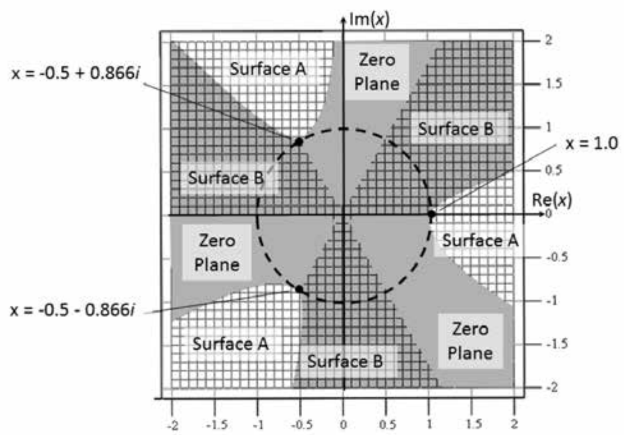

Figure 6. Plan view of Figure 5 showing the location of the cube roots of unity. The dashed line shows De Moivre's circle superimposed.

This is confirmed from a plan view of these surface intersections, as shown in Figure 6. The unit circle centred at the origin is superimposed, and passes through each point where $A = B = 0$; an immediate association with de Moivre's solution shown in Figure 1 is hence made. The ability to visualise the connection between the two approaches is now established and allows much insight to be gained. For instance, by extending the solution space of

Equation (2) to now include complex numbers, it is evident that the original x–y plot in the Cartesian plane is merely a two-dimensional 'slice' of a much more general three-dimensional surface. Also, by virtue of the symmetry present in both surfaces A and B, and the way these symmetry planes intersect each other at right angles, it is evident why the complex conjugate roots have to occur in pairs equi-distant from the G-axis, i.e., behind and in front of the original Cartesian x–y plane! Returning to Figure 2, it should now be clear that the two, somewhat tentatively placed, complex conjugate roots are indeed located at $x = -0.5$, but occur a distance 0.866 units behind and in front of the graph as it is currently plotted in two-dimensions.

The fifth roots of unity: n = 5:

By using de Moivre's theorem from Equation (3),

$$\sqrt[5]{z} = z^{\frac{1}{5}} = \cos\left(\frac{2\pi k}{5}\right) + i\sin\left(\frac{2\pi k}{5}\right),\ k = 0, 1, 2, 3, 4.$$

When $k = 0$, $\quad \sqrt[5]{z} = 1$
when $k = 1$, $\quad \sqrt[5]{z} = 0.309 + 0.951i$
when $k = 2$, $\quad \sqrt[5]{z} = -0.809 + 0.588i$
when $k = 3$, $\quad \sqrt[5]{z} = -0.809 - 0.588i$
and when $k = 4$, $\quad \sqrt[5]{z} = 0.309 - 0.951i$

These then are the fifth roots of unity. When plotted in the complex (Argand) plane they appear equally spaced at 72° intervals around a circle of unit radius centred at the origin O. Again, because n is odd, only one single real root appears at $x = 1$.

The polynomial $\quad y = x^5 - 1$ becomes $y = (G + iH)^5 - 1$. $\quad$ (6a)

Expanding and simplifying the complex form of this equation gives:

$$y = G^5 + 5iG^4H - 10G^3H^2 - 10iG^2H^3 + 5GH^4 + iH^5 - 1 \tag{6b}$$

The surfaces A and B are again found from equating the real (Re) and imaginary (Im) parts of both sides of Equation (6b):

Equating Re parts: $\quad A = G^5 - 10G^3H^2 + 5GH^4 - 1 \quad$ (7a)
Equating Im parts: $\quad B = 5G^4H - 10G^2H^3 + H^5 \quad$ (7b)

Surfaces A and B are plotted in Figures 7(a) and 7(b). Each surface now exhibits five primary 'lobes' equi-spaced around the origin, again with symmetry planes GA for surface A and HB for surface B. When these surfaces are superimposed, and a zero plane introduced, five unique points

of intersection are found, as shown in Figure 8. A true plan view of the surface intersections $A = B = 0$ is shown in Figure 9, with the unit circle superimposed to indicate the solution found from using de Moivre's theorem. Finally, a section taken at $H = 0$ through surface A will reveal the original quintic equation as it would appear in the Cartesian x–y plane.

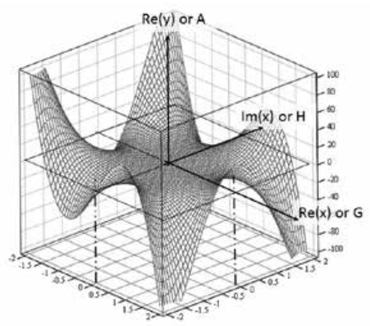

Figure 7(a). Surface A: The real part of the complex representation of $y = x^5 - 1$.

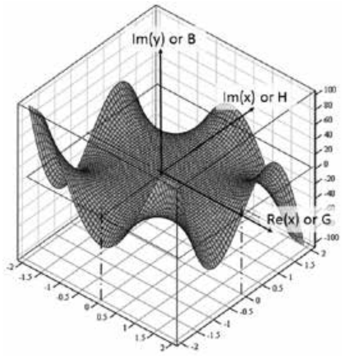

Figure 7(b). Surface B: The imaginary part of the complex representation of $y = x^5 - 1$.

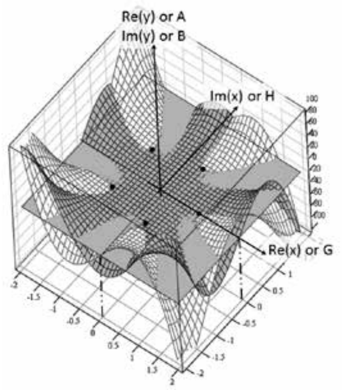

Figure 8. The intersections of Surfaces $A = B = 0$*, revealing the location of the five roots of* $y = x^5 - 1$*. The roots are indicated by black dots.*

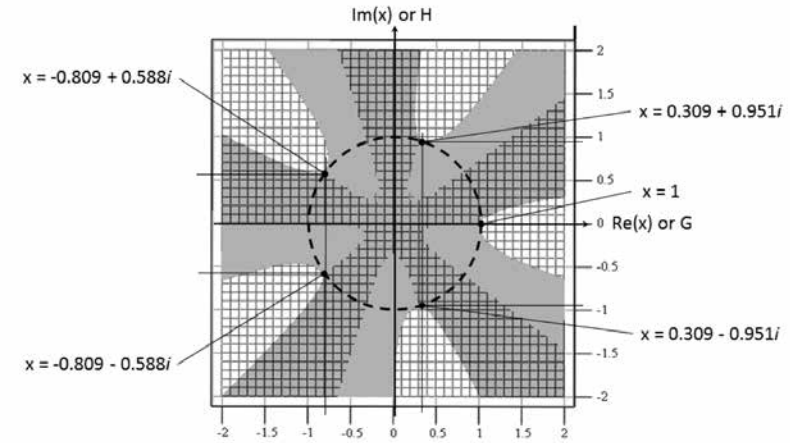

Figure 9. Plan view of Figure 8 showing the location of the fifth roots of unity. The dashed line shows De Moivre's circle superimposed.

The eighth roots of unity: n = 8

By using de Moivre's theorem from Equation (3),

$$\sqrt[8]{z} = z^{\frac{1}{8}} = \cos\left(\frac{2\pi k}{8}\right) + i\sin\left(\frac{2\pi k}{8}\right),\ k = 0, 1, 2, \ldots 7.$$

These eight roots appear in order as follows: 1, $0.707 + 0.707i$, i, $-0.707 + 0.707i$, -1, $-0.707 - 0.707i$, $-i$, $0.707 - 0.707i$. When plotted in the complex (Argand) plane they appear equally spaced at 45° intervals around a circle of unit radius centred at the origin O. Now, because n is even, real roots appear at $x = \pm 1$.

The polynomial $\quad y = x^8 - 1$ becomes $y = (G + iH)^8 - 1$ (8a)

Expanding and simplifying the complex form of this equation gives:

$$y = G^8 + 8iG^7H - 28G^6H^2 - 56iG^5H^3 + 70G^4H^4 + 56iG^3H^5 - 28G^2H^6 - 8iGH^7 + H^8 - 1 \qquad (8b)$$

The surfaces A and B are again found from equating the real (Re) and imaginary (Im) parts of both sides of Equation (8b):

Equating Re parts: $A = G^8 - 28G^6H^2 + 70G^4H^4 - 28G^2H^6 + H^8 - 1$ (9a)

Equating Im parts: $B = 8G^7H - 56G^5H^3 + 56G^3H^5 - 8GH^7$ (9b)

Surfaces A and B are plotted in Figure 10(a) and 10(b) over the range $-1.5 \le G$, $H \le 1.5$. (This curtailed range has been necessary to restrict the vertical scale to a sensible value). Each surface consists of eight primary 'lobes' equi-spaced around the origin, where a central plateau is evident. Again, the principal symmetry planes are found to be GA for surface A and GB for surface B. When these surfaces A and B are superimposed, and a zero plane introduced, eight unique points of intersection are found, as shown in Figure 11. A true plan view of the surface intersections $A = B = 0$ is shown in Figure 12, with the unit circle superimposed to indicate the solution found from using de Moivre's theorem. Finally, a section taken at $H = 0$ through surface A will reveal the original octavic equation as it would appear in the Cartesian x–y plane.

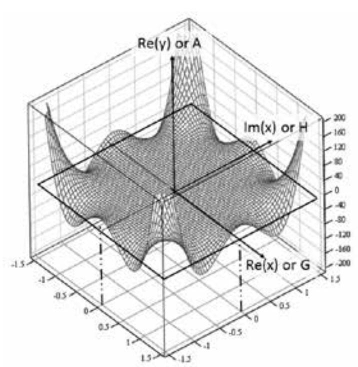

Figure 10(a). Surface A: The real part of the complex representation of $y = x^8 - 1$.

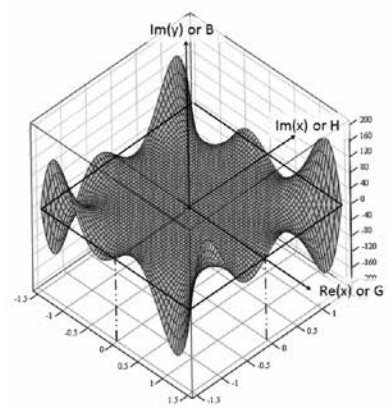

Figure 10(b). Surface B: The imaginary part of the complex representation of $y = x^8 - 1$.

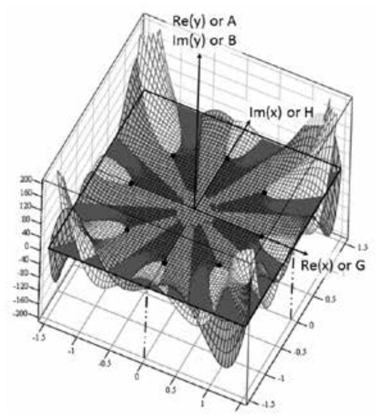

Figure 11. The intersections of Surfaces A = B = 0, revealing the location of the eight roots of $y = x^8 - 1$. The roots are indicated by black dots.

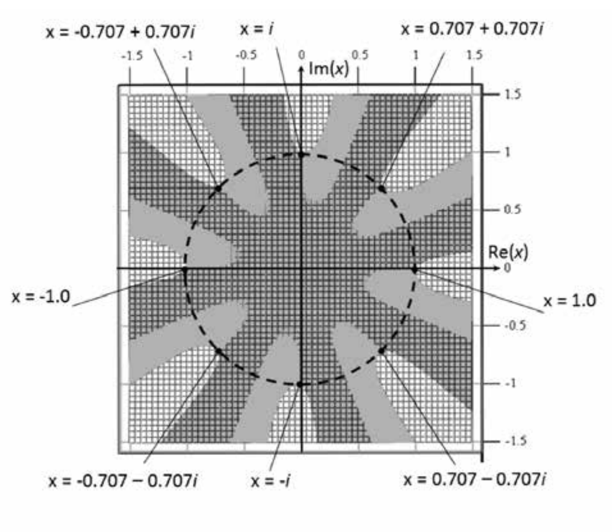

Figure 12. Plan view of Figure 11 showing the location of the eighth roots of unity. The dashed line shows De Moivre's circle superimposed.

Conclusions

This paper has provided the full and complete visual connection between the simple polynomial $y = x^n - 1$ presented in the Cartesian x–y plane, and its roots, which are invariably presented in the complex Argand plane. This link applies to any other polynomial and its roots—regardless of their nature—and it is hoped the surface visualisations presented herein have proved helpful in establishing this connection. All the real and imaginary surfaces presented here have exhibited strong symmetry properties about the $\text{Re}(x)$ and the $\text{Im}(x)$ axes respectively. It has been noted that for any given value of n, these principal symmetry planes are always arranged at right angles to one another. The nature of the surface intersections with the zero plane, and the intricate patterns thus formed, are observed to result directly from the symmetry forms shown. This explains why the complex roots have to occur as conjugate pairs, appearing the same distance in front and behind the Cartesian plane.

The approach adopted here should help students and teachers alike appreciate the rich interplay that exists between complex numbers and polynomial equations. The explanations provided should also help stimulate students' interest in complex numbers and assist in developing their ability to visualise such concepts in three dimensions, especially the orthogonal relationship between the Cartesian and Argand planes. It is anticipated this subject matter could easily form the basis of a classroom-based learning exercise in three-dimensional graphics using computer algebra systems that would amply satisfy some of the ACARA (n.d., Rationale) stated aims of Specialist Mathematics, namely "to develop students' understanding of concepts and techniques drawn from complex numbers, ability to solve applied problems using concepts and techniques drawn from complex numbers and capacity to choose and use technology appropriately".

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