Making connections

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The idea of facilitating learning by providing ways for students to make connections between different parts of mathematics and between mathematics and the rest of reality is widely acknowledged to be a good one.

A search of the *Australian Curriculum* (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.) documents using the phrase "making connections" reveals several instances in the mathematics courses as well as in many of the other disciplines. The idea of 'connectedness' is one of four dimensions in Queensland's Productive Pedagogies framework (Queensland Government Department of Education and Training, 2015). Further afield, the idea forms the basis of the long-running Connected Mathematics Project, a curriculum development and teacher support initiative from Michigan State University (n.d.).

In a paper presented at the 2008 Annual Conference of the Mathematics Education Research Group of Australasia, Abigail Sawyer makes the well-supported claim that "there is broad consensus that in order to become numerate, students must become competent in perceiving the connections between mathematics and other forms of knowledge and between mathematics and their lived experience" (Sawyer, 2008, p. 429).

Sawyer goes on to explain that strategies to implement the overarching idea of making connections are not always straightforward and may work differently for different groups of students.

Nevertheless, in the same vein, it seems reasonable to claim that learning has occurred if and only if a connection has been made in the mind of the learner between otherwise isolated concepts or between fragments of mental or physical action (as when we learn to play a musical instrument). Indeed, the frequently heard comment, "When are we ever going to use this?" can be read as an admission by the student that a connection has not yet been made and the mathematical facts that the teacher has been trying to impart remain a disconnected and hence irrelevant jumble.

This article aims to illustrate, via a moderately rich task, a process of making connections, not between mathematics and other activities, but within mathematics itself, between diverse parts of the subject. In general, the connections between mathematical ideas appear to be the stuff of deep understanding.

The following mathematics has been explored since the time of the ancient Greeks and results surrounding it have long been known, but novel connections are still possible when the material happens to be unfamiliar, as may be the case for a learner at any career stage. The geometrical configuration to be explored, now known as 'Ford circles' after Lester R. Ford Sr (1886–1967), is related to ideas about mutually tangent circles that were studied by, among others, Apollonius of Perga in the third century BC and by René Descartes in the 17th century (Wikipedia, 2015).

This exposition, admittedly neither concise nor elegant, is intended rather to conjure the thoughts of a hypothetical mathematician attempting to find and explain some connections, in the process exploring some lines that turn out to be unproductive and making observations that are really non sequiturs, before eventually achieving some success. This approach contrasts with the manner of writing in typical mathematical research articles and in many text books, which tends to present only polished statements and proofs, the finished results of a hidden process of discovery.

Euclidean geometry

A line is drawn tangent to two circles. A third circle is to be drawn tangent to the line and tangent to each of the first two circles, as shown in Figure 1. How is the radius of the third circle related to the radius of the other two and where is its centre?

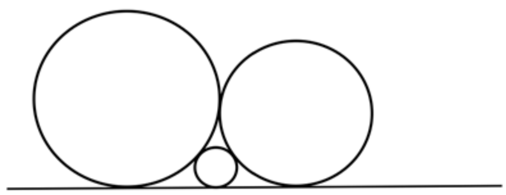

Figure 1. Three circles tangent to each other and to a line.

To proceed, decisions are needed about what prior knowledge in geometry can be brought to bear, what construction lines might be helpful and what reference point should be chosen by which to locate the centre of the third circle.

The reader may have considered adding the construction lines and labels as in the Figure 2. The point of tangency of the left-hand circle with the line has been chosen as a reference point.

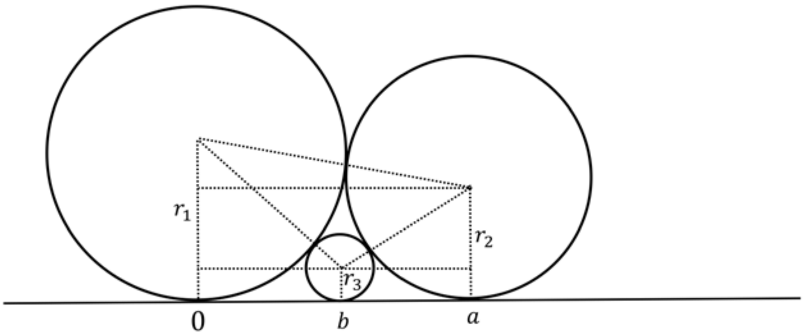

Figure 2. Constructions for locating the centre of the middle circle.

Right-angled triangles have appeared because any radius meeting a point of tangency is perpendicular to the tangent. This looks correct but it is true because it is consistent with previously encountered fragments of knowledge about the angle in a semicircle and angles on the same arc and a limit process; or maybe we recall Euclid's *reductio ad absurdum* proof (ProofWiki, 2014).

The right-angled triangles suggest Pythagoras and some algebra enters the discussion. There are connections between the radii and the horizontal distances of the centres of the three circles from the origin. To begin,

$$(r_1+r_2)^2-(r_1-r_2)^2=a^2$$

which simplifies to $a=2\sqrt{r_1\cdot r_2}$. Similarly, $b=2\sqrt{r_1\cdot r_3}$ (which locates the centre of the small circle) and $a-b=2\sqrt{r_2\cdot r_3}$. By substitution, we have $\sqrt{r_1\cdot r_2}-\sqrt{r_1\cdot r_3}=\sqrt{r_2\cdot r_3}$ and this is equivalent to the nice result,

$$\frac{1}{\sqrt{r_3}}=\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_2}}$$

One might mention parenthetically that Pythagoras and other Greek mathematicians catalogued several different *means*. In this case we have, after rearrangement,

$$\sqrt{r_3}=\frac{1}{2}\left(\frac{2}{\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_2}}}\right)$$

That is, $\sqrt{r_3}$ is half the harmonic mean of $\sqrt{r_1}$ and $\sqrt{r_2}$. It remains to be seen whether this connection will turn out to be useful.

Now, suppose further circles are added to the diagram (see Figure 3). Each new circle is tangent to two parent circles and also to the line. As before, we might ask whether anything can be said about the sequence of locations of the centres of the circles, starting from the left-most circle.

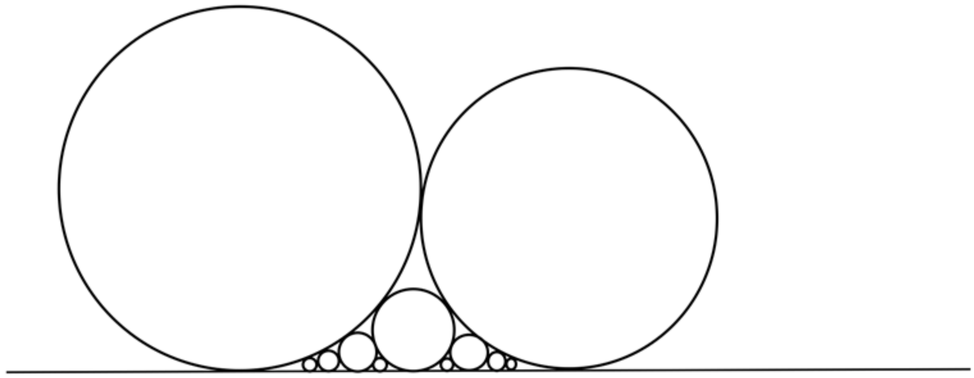

Figure 3. Further circles added, each tangent to two circles and the line.

Graphs

To keep track of the infinitely many circles, a possible strategy is to consider instead a representation consisting of vertices and edges.

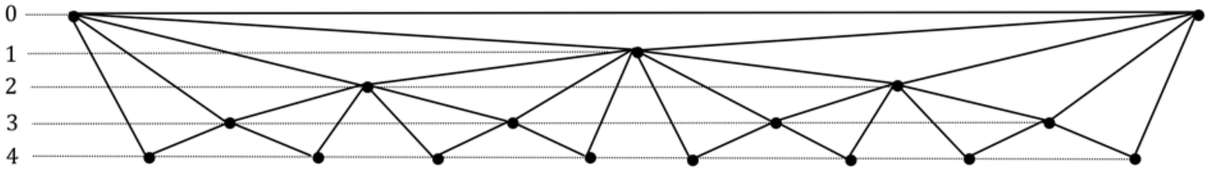

Figure 4. Representation of the circles by a graph.

Each vertex represents a circle. An edge between vertices indicates that the corresponding circles are tangent to one another.

The graph partially drawn above represents four generations of circles after the original pair. Further generations can be appended but even with this somewhat simpler representation, the picture will soon become crowded or very large. In fact, by the nth generation there are 2^{n-1} children.

The vertices can be numbered in a systematic way. One way to do this is to go from left to right within each generation of vertices, as in Figure 5. The fact that a numbering exists confirms that the infinite set of circles is *countable*. Thus, we have a connection between Euclid and Cantor.

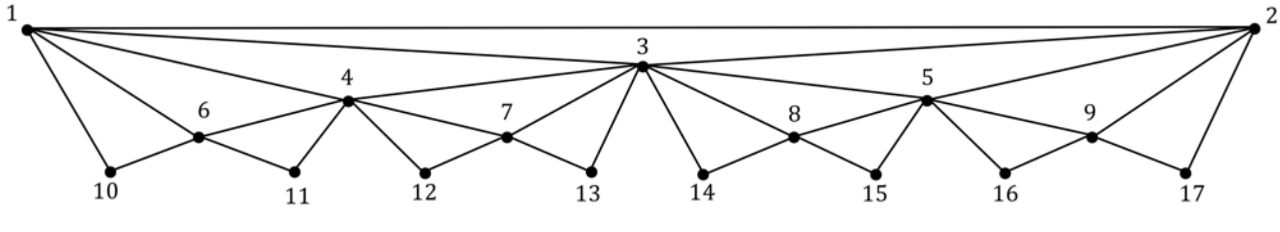

Figure 5. Counting vertices.

The graph shows the connections between parent and children circles but it does not yet have the numerical information about the radii of the circles or about the horizontal distances of each centre from the origin.

The relation developed above

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

makes it possible to calculate recursively the radii. These quantities can then be assigned as *weights* to the vertices. Similarly, weights for the edges come from the fact that the distance $d_{i,j}$ between adjacent vertices i and j, is given by $d_{i,j} = 2\sqrt{r_i r_j}$, corresponding to the horizontal distance between the centres of the ith and jth adjacent circles.

Paths

By inspecting the graph of the first four generations of circles, it is not hard to find both Euler and Hamilton paths, where we take one of the two original parent vertices as the starting point. For each type of path, an induction argument is possible showing that such a path exists for every finite number of generations.

There is an Euler path, for example, that sweeps alternately from left to right and from right to left through the edges belonging to each successive generation, as shown in Figure 6.

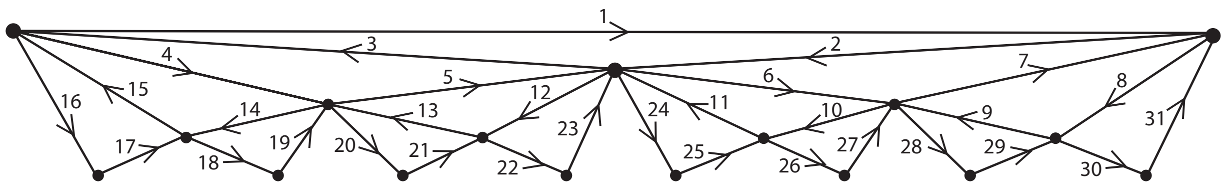

Figure 6. Euler path.

For the inductive step, observe that if there is an Euler path for a graph with n generations and the nth row of edges sweeps from left to right (or right to left) then constructing another row of edges (and vertices) sweeping the opposite way will produce the graph with $n + 1$ generations.

The fact that there is always a Hamilton path that begins with the left-most original parent vertex and goes consistently from left to right, can be seen when the vertices are renumbered, as shown in Figure 7.

Formulated in terms of the graph, the goal is to find a pattern, if one exists, of distances from the first vertex to each of the vertices along this Hamilton path.

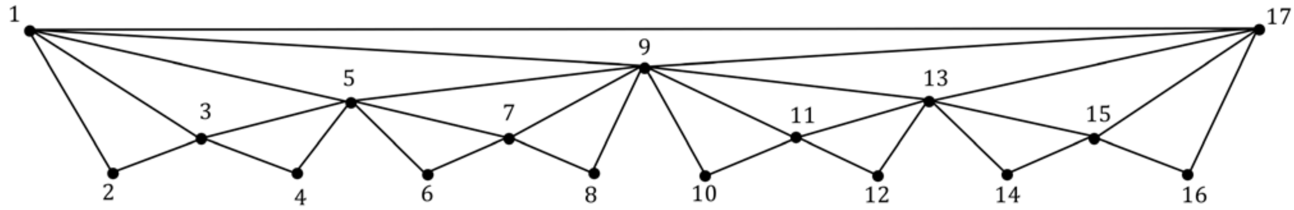

Figure 7. Hamilton path.

Distance

As mentioned above, the distance $d_{i,j}$ in the graph between adjacent vertices i and j is defined to be $d_{i,j} = 2\sqrt{r_i r_j}$, where r_i and r_j are the weights of the vertices. It is clear that $d_{i,j} = 0$ is impossible because the vertex weights are always positive (although they can be as close to zero as we like). This is confirmed by considering the diagram with the circles: no two circles can be arranged one directly above the other.

As an aside, we note that the distance measures along the Hamilton path constitute a metric in the standard sense if we define $d_{i,i} = 0$ for every i. It is true that $d_{i,j} = d_{j,i}$ for any pair of distinct vertices i and j. Also, for any three vertices i, j and k, the triangle inequality $d_{i,k} \le d_{i,j} + d_{j,k}$ holds, with equality when vertex j is the child of vertices i and k. The requirement that $d_{i,i} = 0$ if and only if vertices i and j are identical is satisfied, the 'if' part holding by definition and the 'only if' part holding vacuously.

The distances along the Hamilton path between pairs of vertices, adjacent or otherwise, are just the sums of the intervening edge weights. These total distances are of interest because they correspond to the total horizontal distance of each circle from the origin. By reference to the Ford circle diagrams or to the expressions for edge weights in the graph, it is clear that the distance of a vertex from the origin is independent of the actual route taken so long as progress is consistently from left to right in the graph. To obtain these distances, the vertex weights seem to be needed.

With the relation

$$\frac{1}{\sqrt{r_k}} = \frac{1}{\sqrt{r_i}} + \frac{1}{\sqrt{r_j}}$$

where vertices i and j are parents to vertex k, the weight of each vertex can be written in terms of the weights of its parents, whose weights can then be expressed in terms of the grandparent weights. Eventually, the weight of any vertex can be written as a linear combination of the weights of the initial and final vertices.

This rapidly becomes tedious and cumbersome. A simplification is called for.

Matrices

Suppose that by dogged persistence a large number of vertex weights has been calculated. The sums of the intervening edge weights that give the distance of a vertex along the Hamilton path can be expressed in matrix form. Numbering the weights according to their consecutive positions in the Hamilton path we have, in this very small example:

$$2\begin{bmatrix} \sqrt{w_2} & 0 & 0 \\ \sqrt{w_2} & \sqrt{w_3} & 0 \\ \sqrt{w_2} & \sqrt{w_3} & \sqrt{w_4} \end{bmatrix}\begin{bmatrix} \sqrt{w_1} \\ \sqrt{w_2} \\ \sqrt{w_3} \end{bmatrix} = 2\begin{bmatrix} \sqrt{w_1 w_2} \\ \sqrt{w_1 w_2}+\sqrt{w_2 w_3} \\ \sqrt{w_1 w_2}+\sqrt{w_2 w_3}+\sqrt{w_3 w_4} \end{bmatrix}$$

While this makes the calculations systematic, it does nothing to advance the project. Again, we try another approach.

Numbers

From the relation
$$\frac{1}{\sqrt{r_k}} = \frac{1}{\sqrt{r_i}} + \frac{1}{\sqrt{r_j}}$$

concerning adjacent circles, or equivalently

$$r_k = \frac{r_i r_j}{\left(\sqrt{r_i}+\sqrt{r_j}\right)^2}$$

it is apparent that r_k is a rational number either if r_i and r_j are both squares of rational numbers or if $r_i = r_j$.

In the special case of the original parent circles having equal radii, r_1 say, the child circle must have radius

$$\frac{r_1}{4}$$

Then, the two circles in the next generation will each have radius

$$\frac{r_1}{9}$$

A pattern emerges. The relation between parent and child radii shows that if parent circles have radii

$$\frac{r_1}{a^2} \text{ and } \frac{r_1}{b^2}$$

where a and b are integers, then the child radius is

$$\frac{r_1}{(a+b)^2}$$

Moreover, the grandchildren will have respective radii

$$\frac{r_1}{(2a+b)^2} \text{ and } \frac{r_1}{(a+2b)^2}$$

The denominators will always contain a squared factor.

For convenience, we could put $r_1 = \frac{1}{2}$. This choice sets the distance to 1 between the original parent centres. All other cases with equal sized original parents would involve only a rescaling. Thus, the radii of the circles are in the set

$$\left\{\frac{1}{2}, \frac{1}{8}, \frac{1}{18}, \frac{1}{32}, \dots\right\}$$

That is, $$r_n = \frac{1}{2n^2} \text{ for } n \in \mathbb{N}$$

Suppose two of the circles, with radii

$$\frac{1}{2n^2} \text{ and } \frac{1}{2m^2}$$

are touching. Then the horizontal distance between their centres is

$$d_{n,m} = 2\sqrt{\frac{1}{2n^2} \cdot \frac{1}{2m^2}} = \frac{1}{nm}$$

Suppose further that the two circles are located at distances $\frac{p}{q}$ and $\frac{r}{s}$ along the line, with $\frac{r}{s} > \frac{p}{q}$. Then, $\frac{r}{s} - \frac{p}{q} = \frac{1}{nm}$ or equivalently,

$$\frac{rq - sp}{sq} = \frac{1}{nm}$$

This will certainly be possible if $n = s$ and $m = q$, or $n = q$ and $m = s$. That this is also a necessary condition is not immediately obvious.

If the fraction $\frac{rq-sp}{sq}$ is written in lowest terms, we must have $rq - sp = 1$ and $sq = mn$.

Now, if $rq - sp = 1$, then s and q are coprime. Otherwise, a common factor would divide the 1 on the right, which is impossible. Then, since s and q are coprime we only need only show that m and n are also coprime, and then the fundamental theorem of arithmetic would compel the conclusion that in order for $sq = mn$, we must have $s = m$ and $q = n$, or, paired the other way, $s = n$ and $q = m$. (This is because there is essentially only one way to factorise numbers sq and mn into primes.)

The previously abandoned idea of the graph with the Hamilton path turns out to be useful after all. The vertices have now been labelled with the number that is squared in the denominator of the weight fraction as shown in Figure 8.

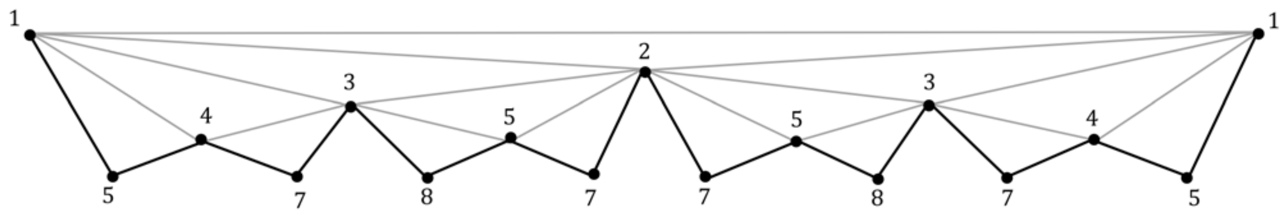

Figure 8. Hamilton path with vertices relabelled.

It is easy to check from the diagram that in the Hamilton paths when there are one, two, three or four generations of circles, the squared parts of adjacent vertices have no common factors. Assume for the moment that in all generations up to the kth, the squared parts of adjacent vertices are coprime. A new generation and a new Hamilton path is produced by appending child vertices between each pair of adjacent vertices in the Hamilton path. Thus, in each case the parents have radii

$$\frac{1}{2a^2} \text{ and } \frac{1}{2b^2}$$

and the child has radius

$$\frac{1}{2(a+b)^2}$$

where a and b have no common factors. It follows that the children have no factors in common with either parent. Thus, again in the $(k + 1)$th Hamilton path the squared parts of the weights of adjacent vertices have no common factors.

By induction, we see that the squared numbers in the denominators of adjacent circle radii are always coprime. That is, referring to the discussion above, m and n are coprime.

We conclude that if two of the circles, with radii

$$\frac{1}{2n^2} \text{ and } \frac{1}{2m^2}$$

are tangent, then the difference in their locations is given by

$$\frac{r}{n} - \frac{p}{m} = \frac{1}{nm}$$

for some choice(s) of r and p.

Do r and p exist and how can they be found? Using arguments similar to those above it is clear that r and p must have no common factors, and neither do the pairs (r, n) and (p, m).

The Euclidean algorithm

Tangency requires that $rm - pn = 1$. The Euclidean algorithm for finding the greatest common divisor is applicable. It guarantees that given two integers m and n that have 1 as their greatest common divisor, the integer coefficients r and p exist and can be found by working through the algorithm in reverse.

Thus, circles with radii

$$\frac{1}{2m^2} \text{ and } \frac{1}{2n^2}$$

with m and n coprime, are tangent when located at rational points $\frac{r}{n}$ and $\frac{p}{m}$ along the line for choices of p and r such that $mr - np = 1$.

The mathematician Lester R. Ford Sr after whom the configuration of circles is named reached a conclusion along these lines (Ford, 1938).

There is more. The relation $mr - np = 1$ calls to mind a property of Farey sequences, named after but not first discovered by a geologist, John Farey.

Farey sequence

Suppose two adjacent circles along the Hamilton path are located at points measured from the origin $\frac{r}{n}$ and $\frac{p}{m}$ respectively. The child of these parents is located at $\frac{x}{y}$. Then we have $ry - nx = 1$ and also, $mx - py = 1$. Thus, from $ry - nx = mx - py$ we obtain

$$\frac{x}{y} = \frac{r+p}{n+m}$$

The location of the child is the mediant of the fractions $\frac{r}{n}$ and $\frac{p}{m}$ that locate the parents. In this way, the same process that produces generations of circles makes Farey sequences.

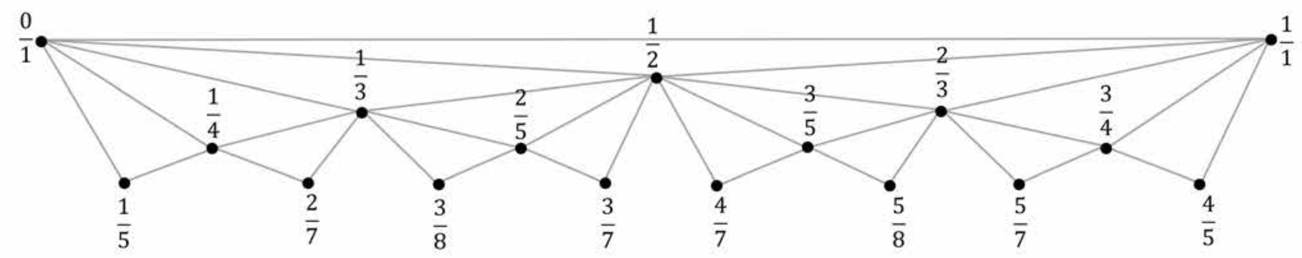

Figure 9. Graph showing vertices located by the mediants

Area

If one were to ask what area the Ford circles cover, one would need to think about whether every rational number is in a Farey sequence and how many fractions there are that have each possible denominator. For example, in the generations of circles displayed above, the area would be something like:

$$\frac{\pi}{4}\left[2\times\left(\frac{1}{1^2}\right)^2+1\times\left(\frac{1}{2^2}\right)^2+2\times\left(\frac{1}{3^2}\right)^2+2\times\left(\frac{1}{4^2}\right)^2+4\times\left(\frac{1}{5^2}\right)^2+\ldots\right]$$

And we might observe that the coefficients of the fractions, after the first, are given by the Euler φ function of each denominator—another curious and fascinating connection between geometry and number.

The topic is not exhausted, although the hypothetical mathematician may well be, at least temporarily, and may wish at this stage to run the findings past a colleague to check for omissions, lapses in clarity and logical errors.

The conclusion I wish to suggest is that seemingly innocent mathematical fragments can have connections to many related ideas. If a teacher is in possession of a broad subject knowledge then the likelihood seems high that it will be possible to draw out useful connections in the classroom or in well-designed projects and assignments. For this reason I claim that an ever-widening subject knowledge is of utmost importance in a teacher's program of professional development.

The connected ideas in this illustration may not all be particularly suitable for the secondary classroom. Better and more appropriate examples can be substituted. Thus, creativity too is a plank in the edifice of good teaching.

Finally, the illustration makes the point that doing mathematics is not in the first instance a tidy and precise activity. Learners should always be free to make mistakes and to try strategies that may not in the end be productive. In reality, the tidying-up and polishing comes after a possibly arduous process of individual and collaborative exploration.

References

Australian Curriculum, Assessment and Reporting Authority [ACARA]. (n.d.). *Australian Curriculum: Mathematics*. Retrieved from http://www.australiancurriculum.edu.au/mathematics/

Ford, L. R. (1938). Fractions. *The American Mathematical Monthly, 45*(9), 586–601. Available: http://www.jstor.org/stable/2302799

Michigan State University. (n.d.). *Connected mathematics project*. Retrieved from https://connectedmath.msu.edu/

ProofWiki (2014). *Radius at right angle to tangent*. Retrieved from https://proofwiki.org/wiki/Radius_at_Right_Angle_to_Tangent

Queensland Department of Education and Training. (2015). http://education.qld.gov.au

Sawyer, A. (2008). Making connections: Promoting connectedness in early mathematics education. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 429–436). Brisbane: MERGA. Available: www.merga.net.au/documents/RP512008.pdf

Wikipedia (2015). *Ford circles*. Retrieved from https://en.wikipedia.org/wiki/Ford_circle