Exploring quadrilaterals to reveal teachers' use of definitions: Results and implications

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The differences between mathematical language and everyday language are explored in the mathematics education literature, demonstrating that mathematical definitions are negotiated and used differently, such as Landau's comparison of extracted definitions and stipulated definitions (as cited in Edwards & Ward, 2004) and how these differences influence students' thinking (Edwards & Ward, 2004; Zaslavsky & Shir, 2005). Not surprisingly, students' interactions with mathematical terminology may be more heavily influenced by their use of everyday language than by the role of definition within mathematics.

As teachers respond to students' developing use of definition, it becomes imperative to consider how teachers themselves might use definitions within mathematical tasks. Such an exploration is not new. Previous research has found that teachers may be challenged by the idea of equivalent or alternative definitions or that criteria used for selecting definitions for student use may not represent mathematical ideals (Keiser, 2000; Leikin & Winicki-Landman, 2000).

Less understood is how teachers' use of definition interplays with their selection, implementation, and assessment of tasks. For example, a teacher's own definition of a concept may hinder his or her ability to anticipate the value of a particular task for classroom use, or the limited understanding of *definition* versus the process of *defining* may result in teachers' exposing students to the former with little experience in the latter. Teachers use pedagogical criteria, such as what they may perceive as 'simpler' for students to understand, for making decisions related to definitions in the classroom (Leikin & Winicki-Landman, 2000), but these decisions may only serve to develop an understanding of a particular definition and may actually avoid the act of defining that may be more indicative and supportive of mathematical thinking.

As teachers respond to students' developing use of definition, then, it becomes imperative to consider how teachers themselves might use definitions within mathematical tasks. Previous research illuminates that teachers may be

challenged by the idea of equivalent or alternative definitions. In addition, the criteria used for selecting definitions for student use may not represent mathematical ideals, particularly in the case of minimality of definition (Leikin & Winicki-Landman, 2000).

Geometry provides numerous examples of activities that employ definition, such as constructions, proofs, and categorising geometric objects (de Villiers, 1998). In fact, the process of defining and the act of proof are treated similarly in the literature (Zaslavsky & Shir, 2005), as they are both related to the process of systematisation. Unfortunately, many students actually encounter these mathematical activities with intentions of verification rather than exploration in the classroom, resulting in superficial exposure to them. Furthermore, de Villiers suggests, students should properly encounter proof as an investigatory and exploratory technique and should do so at an appropriate time in their development, lest their ability to experience proof as a meaningful activity be endangered (2004).

Because of the axiomatisation inherent in Euclidean geometry, topics related to classifying shapes and their relationships are promising for engaging students in the act of defining, characterising, and comparing. In Australia, students' geometric reasoning progresses across the F–10 curriculum as they transition from simply recognising and classifying shapes, as in Year 1, to applying reasoning using congruence and similarity to proofs involving plane shapes, as in Year 10 (Australian Curriculum Assessment and Reporting Authority(ACARA), n.d.). The *Australian Curriculum: Mathematics* for senior secondary expands upon student understanding by expecting them to apply reasoning to problems related to shape, including investigating compound shapes and extending understanding of area and characteristics of two-dimensional shapes to three-dimensional shapes. To be successful, students require flexibility in defining and an awareness of how context affects the use of a particular definition.

Teachers in F–10 are primarily responsible for constructing students' foundation of proof and definition thinking. Their selections of tasks create the scaffold upon which students construct their initial notions of geometric objects. At the senior secondary level, however, understanding student thinking about geometric relationships and characteristics becomes essential for selecting appropriate tasks and for effective classroom questioning that engages students at their level of understanding. Thus teachers at the senior secondary level who understand how their students have developed notions of proof and have used definition are better equipped to identify and counter challenges students may have in further developing or applying their knowledge of justification and reasoning.

The van Hiele levels of geometric thought, which posit that students progress through a series of levels or stages of thinking about and understanding geometry, provide a useful framework with which to view geometric understanding, as shown in Figure 1. The theory suggests that learners assisted

by appropriate instructional experiences pass through levels of geometric thought. The student cannot achieve one level of thinking without having passed through the previous levels (van Hiele, 2006). Consequently, the van Hiele levels are an influential and necessary component when designing a coherent and developmentally appropriate geometry curriculum.

Theoretical (Level 3)	
Learning period 2	*Phases of Learning* Integration Free orientation Explication Bound orientation Information
Descriptive/Analytic (Level 2)	
Learning period 1	*Phases of Learning* Integration Free orientation Explication Bound orientation Information
Visual (Level 1)	

Figure 1. Van Hiele Levels of Geometric Understanding and Phases of Learning, Levels 1–3.

The levels begin with the simplest actions of recognition and progress through to formal deduction. In the initial level, visualisation, students identify, name, compare, and operate on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance. For example, a student may state, "It is a rectangle because it looks like a box" (van Hiele, p. 311). At the next level, descriptive, students analyse figures in terms of their components and relationships among components and discover properties/rules of a class of shapes empirically. However, properties are not ordered logically, "a triangle with equal sides is not necessarily one with equal angles" (p. 311). The next level, informal deduction, the student logically interrelates previously discovered properties/rules by giving or following informal arguments. In subsequent stages, like formal deduction, the last level appropriate for school mathematics, students attend to Euclid's "logical construction of geometry with its axioms, definitions, theorems, and proofs" (p. 310).

Because the van Hiele model is based on the idea that students' progression through the levels is more dependent on instruction than on maturation, it provides a structure on which instruction related to geometry can, and should, be framed. Selecting appropriate instructional activities supports students' progression through the phases of learning, thus developing their use of language and definition as they move through the levels of understanding (Teppo, 1991). Exploring activities that model such transitions may also support teachers in reflecting on the importance of designing instruction

with student understanding and developmental readiness in mind (Crowley, 1987; de Villiers, 2004) and maintaining a focus on definition and the act of defining.

The activities

Context

About 80 middle and secondary school teachers participated in the professional development sequence as a part of a larger professional development project. Sessions provided participants with half-day experiences in an inquiry-oriented classroom taught by a master teacher, in which they experienced examples of inquiry lessons as both student and teacher. The remainder of the day consisted of content-rich activities that explored mathematics and pedagogy in depth and sought to reveal and develop teachers' mathematical understanding and practices.

We developed a sequence of professional development activities that sought to reconnect teachers with their own mathematical ways of thinking, namely of reasoning using definition. We used the van Hiele levels of geometric thought to select and sequence activities and as a lens through which to view teacher work. In this article, we present descriptions of the activities and a rationale for their selection along with insight gained from teachers who completed the activities.

The collection of initial tasks for use during the content activities focused on components of mathematical definition or of reasoning with definitions. Teachers worked in small groups to explore each task, and discussions were facilitated across the whole group. We selected the sequence of activities to explore components of geometric thought related to quadrilaterals at van Hiele levels 2 and 3. Data collected during the activities included artifacts of teachers' work and notes taken during observations of small groups and discussions by whole groups.

Content

Four initial activities adapted from *Geometric Structures: An Inquiry-Based Approach for Prospective Elementary and Middle School Teachers* by Aichele and Wolfe (2008) provided teachers with explorations employing geometric reasoning. The first provided four geometric conjectures and asked groups to reason through the value of the conjecture. For example, Conjecture 1 stated, "A diagonal of a rectangle is a line of reflection for the rectangle."

In the second activity, teachers responded to the question "Possible or Not?" by evaluating five statements for their potential to create possible figures with a given description. For example, Statement 3 directs participants to determine the possibility of creating a "quadrilateral with exactly one right

angle." Teachers explored the statements, creating example sketches where possible or providing reasoning for the impossibility of others.

In the third activity, teachers explored true or false statements, providing examples or counter examples along with their reasoning. For example, Statement 2 states that "The diagonals of an isosceles trapezium are equal." (Note, in the actual professional development in North Carolina, USA, the term trapezoid rather than trapezium was used).

Finally, in the fourth activity teachers considered under what conditions given statements are true. For example, they might determine whether the diagonals of a quadrilateral are perpendicular. These statements required teachers to write a statement or collection of statements that would clarify under what conditions the initial statement would be true.

These initial activities engaged teachers in geometric thinking at van Hiele levels 2 and 3. Following these initial activities, teachers were requested to generate a list of quadrilaterals and then to construct a representation of a hierarchy relating the quadrilaterals to one another. This activity was intended to generate some dissonance as teachers recognised that there were two definitions of trapezium and also to hone in on characteristics of "good" definitions. The hierarchy activity engaged teachers at van Hiele level 3.

Two days later, we began a conversation on area and prompted teachers to generate proofs of why the area formulas for various quadrilaterals are what they are. We selected the trapezium area formula as a culminating discussion point, allowing us to use area as an inheritable characteristic that would prompt teachers to reconsider their hierarchy diagrams. It is worth noting that our purpose in these activities was not to evaluate the correctness of teachers' responses or to generate an agreed-upon definition. Instead, we wanted to elicit the perspectives that teachers brought to the use of definitions and to characterise those perspectives. From a pedagogy standpoint, we also sought to model the use of what might seem to be non-typical classroom activities as a means to facilitate rich mathematical discussion supportive of students' mathematical practices.

Observations and implications for professional development

Teachers wanted one definition for each object

Teachers seemed to struggle with the nature of mathematically defining geometric objects. Mathematical definitions are used and learned in different ways from those of non-technical language, but the idea that mathematical definitions may be stipulated rather than extracted (Edwards & Ward, 2004) seemed to challenge teachers. This became particularly apparent with the definitions of kite and of trapezium.

When defining the kite, most teachers, relying on memorised definitions from curriculum materials or those discovered through quick online searches, provided definitions such as "four-sided polygon, 2 pairs of adjacent, congruent sides" or "quadrilateral with no parallel sides, consecutive sides have equal length, diagonals are perpendicular." Others struggled with defining the kite as they simultaneously tried to orient it among the other quadrilaterals. For example, one group that initially organised quadrilaterals into convex and concave classifications, went on to divide the convex quadrilaterals into three groups: no parallel sides, 1 pair of parallel sides, or 2 pairs of parallel sides. For this group, the "no parallel sides" group was also labeled "outcasts." From this location on their tree diagram, these teachers placed the kite with the notation that it is "not really classified by parallel sides" and punctuated it with a sketch of a sad face. While this group began with an idea for classifying that arguably could have resulted in an accurate tree diagram, their lack of an alternative definition for kite—one that did not rely on parallel sides—proved to derail their classification scheme.

The existence of multiple definitions of the trapezium has been of interest to mathematics educators for some time (e.g., Craine & Rubenstein, 1993; de Villiers, 2004; Popovic, 2012), the distinction between the two being whether the trapezium possesses at least or exactly one pair of parallel sides. Most teachers in this professional development session were unaware of the existence of two definitions. In this case the definitions are not just alternative definitions—such as defining an equilateral triangle by its congruent sides or by its congruent angles—but are actually different in what might be included as examples of each. Here again the initial step of classifying all quadrilaterals based on the characteristic of parallel sides proved problematic as this key characteristic seemed unstable.

Teachers overlooked the importance of necessary and sufficient conditions for defining

While early activities in the professional development sequence were intended to 'flex' teachers' definitions, they seemed to return in subsequent explorations to those with which they were most comfortable. When pressed to evaluate their definitions based on the necessary and sufficient characteristics, teachers were successful in editing their work. For example, one group initially defined a square as "a quadrilateral with four equal sides and four equal 90 degree angles, diagonals are of equal length, opposite sides parallel." After revision, this group retained only the first piece of the definition, that a square is "a quadrilateral with four equal sides and four 90-degree angles." It is not clear whether this group found only this portion of the definition necessary and sufficient or whether they might have reconstructed their definition based on one of the other initial characteristics. Other groups explored the specificity of terms within definitions to determine in what ways they might incorporate necessary and sufficient information more economically. Beginning with the

definitions of a square as “a parallelogram with all sides and angles congruent,” one group edited their version to instead define a square as “a quadrilateral with all sides and angles congruent.” An example of an initial hierarchy and an edited one can be seen in Figures 2 and 3, respectively.

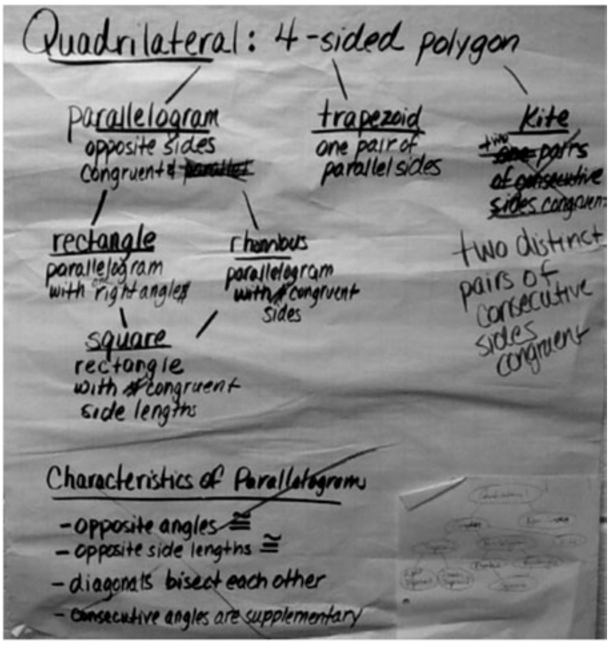

Figure 2. An example of an initial hierarchy map.

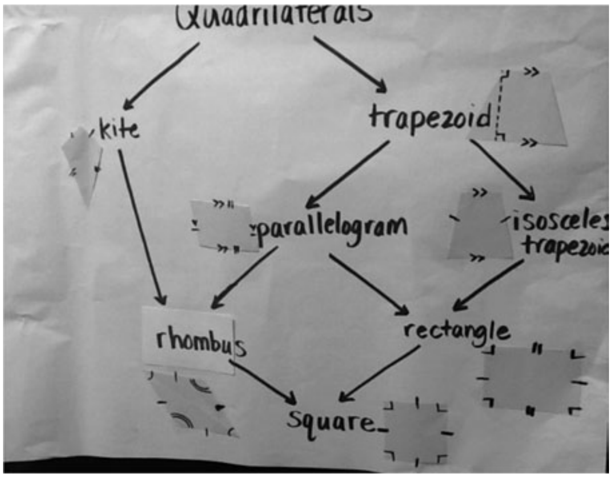

Figure 3. An example of an edited hierarchy map.

Obviously this line of thought was guiding teachers towards establishing a hierarchy of the quadrilaterals in which referring to the classification above would carry with it inheritable characteristics. Few teachers, however, approached the task of defining quadrilaterals in this manner until prompted to evaluate and edit their definitions.

Throughout the discussion, we were unsure whether teachers valued economy as a characteristic of a definition or as a way to evaluate students’ thinking of mathematical definition. Groups who had defined quadrilaterals

by giving a characteristic true of all four sides or of all four angles wrestled with whether providing the characteristic as true of one side or of one angle was sufficient, such as stating that a square is a parallelogram with four right angles versus a parallelogram with one right angle. Here the content of the definition is not an issue of economy; the definitions are the same length. Instead, the teachers attempted to verify whether a quadrilateral might exist that was a parallelogram with one right angle but was not also a square. Such struggles called to mind that teachers may also be challenged by discerning what is necessary or sufficient in a particular instance of a definition as well.

Teachers did not naturally tend to think hierarchically when developing, evaluating, or editing definitions

Teachers' tendency to devalue the economy of definition may be reflective of a lack of hierarchical thinking when defining geometric objects. Thinking hierarchically may be a hallmark of van Hiele Level 3, as these activities require the use of logical deduction and more advanced interactions between concepts and images (Fujita & Jones, 2008), but our teachers did not clearly display proficiency with doing so. However, whether this tendency is indicative of a lack of mathematical understanding or of a priority to consider the didactical purposes of a definition, we are unsure.

Zaslavsky and Shir (2005, p. 317) summarise the roles of definitions in mathematics as

1. introducing the objects of a theory and capturing the essence of a concept by conveying its characterising properties,
2. constituting fundamental components for concept formation,
3. establishing the foundation for proofs and problem solving, and
4. creating uniformity in the meaning of concepts, which allows us to communicate mathematical ideas more easily"

The preference seemingly given to didactical concerns by teachers demonstrates their view of definitions as a means to communicate. It is not clear whether teachers recognise the other roles of definitions or simply value the role of communication more than the others.

If a teachers' purpose in offering a definition is simply to communicate something about an object, he or she may seek what appears to be the most uncomplicated definition for students. That is, a teacher may wish to select the definition that uses what he or she perceives as the simplest terms and that offers the least challenge to students' existing conceptual understanding. However, by doing so a teacher may inadvertently dismiss important characteristics of a figure or overlook important connections to previous and future mathematics.

Implications

Teachers' understandings and use of definitions have immediate implications for both their practice and for the work of mathematics teacher educators. We suspect that teachers may dismiss examples or tasks that do not fit with their own particular use of a definition or may be less likely to anticipate students' use of definitions during the implementation of a task. Doing so will likely fail to transition students through the van Hiele levels effectively. For example, examining hierarchical classifications, such as of quadrilaterals, provides experiences that may support the transition of learners from van Hiele level 2 to 3. The classification process requires learners to both control an image (level 2) and to examine its properties (level 3) (Fujita & Jones, 2008). The potential of such an activity, though, may be overlooked if the acts of defining and of negotiating a definition are not also strongly valued. In fact, the teachers engaged in these explorations noted how their implementation of classroom tasks was likely limited by their use of simplistic but rigid definitions rather than the incorporation of the act of defining.

Negotiating definitions in small groups or as a whole group requires that teachers have flexible understandings of concepts and can effectively navigate the evolving definitions offered by students. A criticism of standards-based curricula is that while they may provide teachers support in anticipating students' strategies in response to tasks, they may not provide guidance related to the justification related to these strategies (Grant et al., 2009). This includes definitions as used in students' justifications, implying that teachers may not be able to rely on curriculum materials to provide guidance in developing students' mathematical practices via definitions and defining. Consequently, intentionality in negotiating definitions must be included in professional development for in-service teachers and coursework for pre-service teachers.

Much remains to be understood about the interactions of teachers' instructional decisions and students' development of definitions and concepts in geometry (Jones, Fujita, & Kunimune, 2012). We propose that engaging teachers in professional development that incorporates a direct focus on the importance of and roles of mathematical definition contextualised within authentic teacher tasks enfranchises them to participate in an investigation of such interactions. The field of geometry, with its well-defined van Hiele levels and varied uses of definition, holds particular promise for offering a concrete foundation for the exploration of the intersection of teachers' instructional practices and student learning.

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