A fascinating application of Steiner's Theorem for Trapezium: Geometric constructions using straightedge alone

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Based on Steiner's fascinating theorem for trapezium, a seven geometrical constructions using straightedge alone are described. These constructions provide an excellent base for teaching theorems and the properties of geometrical shapes, as well as challenging thought and inspiring deeper insight into the world of geometry. In particular, this article also mentions the orthic triangle and proves its special property, and shows some other interesting constructions, such as, for example, how to construct a circle's diameter using straightedge alone and having only a segment with its midpoint. In addition, it is enhanced by aspects of the historical background of geometric constructions, including reference to "impossible constructions." Application of the material presented in college or high school can enhance students' appreciation of the elegance, beauty, and fascination of mathematics. Through such "adventures," students will be encouraged to further pursue geometric problems and explore various methods of problem solving, especially those concerned with geometric constructions.

Introduction

The motivation for this article stems from the presentation of a geometrical problem (from a high school textbook) to pre-service mathematics teachers during a course in plane geometry given in an Israeli teacher's college for prospective high school teachers. The purpose of the course was to deepen their knowledge in this topic, and included the construction of geometrical forms incorporated with the history of geometric constructions.

The problem presented was the following: The sides of trapezium $ABCD$ meet at point F. A line segment, FN, passes through the intersection of the diagonals, E (see Figure 1). Prove that $AM = MB$ and that $DN = NC$.

Note: A trapezium is a quadrilateral with one pair of opposite sides parallel.

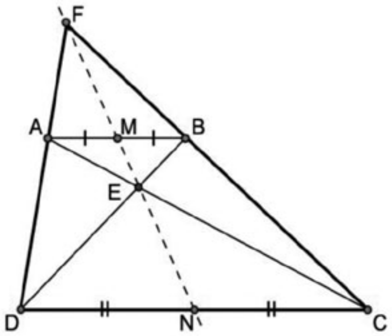

Figure 1. A trapezium ABCD with FN passes through the intersection point of the diagonals, E.

The problem is solved by building an auxiliary figure (which generally is the key to the solution)—passing a line parallel to the base of the trapezium ABCD through the meeting point of the diagonals, using similar triangles, and the *intercept theorem*, also known as *Thales' theorem* (not to be confused with another theorem with that name). It is an important theorem in elementary geometry about the ratios of various line segments that are created if two intersecting lines are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles. However, the segment *FN* that passes through the midpoints of the trapezium's bases, the meeting point of the diagonals and the meeting point of the trapezium's continuations of the sides, immediately reminded the authors of Steiner's theorem, which in fact states the results of the problem above.

The Steiner Theorem for Trapezium (Jakob Steiner, 1796–1863)

For every trapezium *ABCD*, the following four points—the midpoints of each base, the point where the diagonals cross, and the point of meeting of the continuation of the sides—are on the same line.

The students noted that Steiner's proof was comparable to the solution of their problem (the proof of which is given below) and thus were stimulated to continue researching the use of the Steiner theorem for the trapezium, which ultimately led to an interest in general geometric constructions according to the rules of ancient Greek mathematics, and building geometric figures using a straightedge only.

The aim of this article is to point out the value of developing pre-service teachers', in-service teachers', and students' understanding of geometric constructions in general along with their ability to construct geometric figures using only a straightedge, along with giving them the historical perspective of the development of geometric constructions to increase their interest and fascination with mathematics.

Importance of geometric constructions

The rules of ancient Greek mathematics allow geometric constructions using only a compass and straightedge. A straightedge is defined as infinite in length, without markings, and with only one edge. A compass can be used to inscribe circles, but cannot be used to transfer measurements.

There are several reasons why geometric constructions are considered important. First, geometric constructions are a part of the mathematical heritage and have been a popular part of mathematics throughout history. This popularity may possibly be attributed to the ancient Greeks' introduction of the following four famous geometric constructions which have been proven impossible to construct many hundreds of years later (see, for example, Heath, 1981):

- *Doubling the cube:* Given any cube, find a geometric construction for a cube with twice the volume of the given cube.
- *Trisecting the angle:* Given any angle, find a geometric construction for an angle that is one-third the measure of the given angle.
- *Squaring the circle:* Given any circle, find a geometric construction for a square of equal area to the circle.
- *Inscribing a regular heptagon in a circle:* Construct a regular heptagon (seven-sided polygon) with compass and straightedge only.

Although the Greeks did believe that these constructions were impossible, they were not able to actually prove their impossibility, and these problems drew the attention of many famous mathematicians over the centuries, until they were finally proven to be impossible in the nineteenth century. The proofs of impossibility of two of the constructions came from Pierre Wantzel (1814–1848) as indicated by Suzuki (2008). In 1837, Wantzel proved the impossibility of duplicating the cube or trisecting an arbitrary angle using his theorem that if r is a constructible number, it must be the root of an irreducible polynomial of degree 2^n. In essence, this is the basis of the Galois Theory that was derived at that same time. Then, in 1882, Karl Ferdinand Von Lindemann (1852–1939) proved the impossibility of the third problem by showing that since π is transcendental, no equation of any degree with rational coefficients can have π as a root (Berggren, Borwein & Borwein, 1997), and consequently, squaring the circle is impossible.

The impossibility of constructing a regular heptagon was actually solved earlier, in 1790, by Carl Friedrich Gauss (Childs, 2009; Krizek, Luca & Somer, 2001). His proof involved the Fermat primes (the only ones known are 3, 5, 17, 257 and 65 537). Gauss proved that an n-sided regular polygon can be constructed by compass and straightedge if and only if n is equal to a power of 2, and, possibly, multiplied by distinct Fermat primes. Obviously, the regular 7-gon is not such a number, as well as 9-gon or 11-gon.

However, what is even more important to understand is that, through their investigations of these impossible geometric constructions, mathematicians were able to achieve many remarkable feats that contributed much to the development of geometry. For example, the dead end to which the Greeks arrived trying to solve the above problems inspired them to attempt to invent more technologically advanced tools to enable them to perform the required constructions. Second, "geometric constructions can reinforce proof and lends visual clarity to many geometric relationships" (Sanders, 1998, p. 554). Third, geometric constructions "give the secondary school student, starved for a Piagetian concrete-operational experience, something tangible" (Robertson, 1986, p. 380). Another reason is presented by Pandisico (2002) who claimed that "unless constructions simply ask students to mimic a given example, they promote true problem solving through the use of reasoning" (p. 36).

The reason for exploring the history of geometric constructions is argued by Lamphier (2004) who said that "to fully understand a topic, whether it deals with science, social studies, or mathematics, its history should be explored. Specifically, to fully understand geometric constructions, the history is definitely important to learn" (p. 1). Swetz (1995) also believes that history can supply the why, where, and how for many concepts that are studied.

The authors of this manuscript believe that geometric constructions, through the various methods of their solution, encourage thought, creativity and originality, while presenting opportunities for integrating solutions and developing unique strategies. Indeed, when dealing with a construction problem, interesting solutions that emphasise the beauty of mathematics are often revealed. Complicated construction problems constitute a powerful challenge which contributes to thought development, and allows the student to implement the important properties of known geometric shapes in the Euclidean or projective plane, and thus diversify the process of studying geometry. Geometric constructions expand the student's comprehension of mathematics.

Incorporating geometric construction into the curriculum

The topic of geometry is always serving as one of the major topics across the school mathematics curriculum all over the world. Geometry is an inspirational part of mathematics tending to engender mathematical thinking, and geometry can be useful in developing ideas of 'proofs'. In addition, inspecting the Australian Curriculum, Assessment and Reporting Authority (ACARA) documents, in each geometry program for high school grades, the topic of geometric constructions is included. Given the value of geometric constructions, their incorporation into geometry instruction is very important, especially for pre- and in-service mathematics teachers and for high-level students in the upper grades of secondary schools. In fact, trying to learn geometry without geometric construction is like trying to learn chemistry or biology without laboratories. Presenting geometric constructions together with their historical contexts enlivens the topic. The basic knowledge and skills needed to perform geometric constructions encourage students to discover and explore geometric relationships and interpret geometric concepts and theorems. Geometric construction can also help teachers transform the static and confusing array of definitions and theorems typically found in geometry textbooks into an active and exploratory investigation of geometric relationships (Lamphier, 2004).

A very helpful tool for encouraging and motivating teachers to incorporate geometric constructions within the framework of their mathematics curriculum is dynamic geometric software (DGS) such as *The Geometer's Sketchpad* or *Geogebra*. The ACARA document, *Shape of the Australian Curriculum: Mathematics* (May 2009) emphasises the role of digital technologies in teaching mathematics. It states that "digital technologies allow new approaches to explaining and presenting mathematics, as well as assisting in connecting representations and thus deepening understanding" (6.5.1). Then, continues to indicate that "digital technologies can make previously inaccessible mathematics accessible, and enhance the potential for teachers to make mathematics interesting to more students, including the use of realistic data and examples" (6.5.2). In fact, currently, the introduction of DGS and the rapid progress of the technology and its availability in education, produces an environment that enables both students and teachers to explore geometric relationships dynamically and to create very complex and yet very precise geometric constructions and diagrams. Obviously, static constructions lack the strong impact of dynamic constructions. However, one must be aware that while DGS helps to visualise a relationship, it does not provide formal proof (using appropriate geometry definition, postulates and theorems), and students who use DGS cannot be allowed to ignore learning and presenting formal proofs for the problems.

Three types of geometric constructions

There are three types of geometric constructions, based on the tools that are used to construct them: compass-and-straightedge (with or without relating to unit values), compass only, and straightedge only. The first type is the original, classic construction method using only a compass and straightedge, and without using any measurements, and is the construction method implied when one refers to building the basic structures.

All compass-and-straightedge constructions consist of repeated application of five basic construction methods using points, lines, and circles that have already been constructed. These methods are:

- creating a line through two existing points;
- creating a circle through one point with centre another point;
- creating a point which is the intersection of two existing, non-parallel lines;

- creating one or two points at the intersection of a line and a circle (if they intersect); and
- creating one or two points at the intersection of two circles (if they intersect).

Another approach to basic constructions is by relating to the 'numbers' that the ancient Greeks were able to construct with compass and straightedge while actually doing arithmetic geometrically by lengths of segments. Knowing how to construct a parallel line to a given line, they were capable of making arithmetic constructions for two given segments, one of length x and the other length y, and a unit length of 1. Through basic geometry and algebra, other related lengths could be constructed. Five arithmetic constructions are possible:

$$x+y,\ x-y,\ x\cdot y,\ \frac{x}{y} \text{ and } \sqrt{x}.$$

The second type of geometric construction uses compass alone. According to the Mohr–Mascheroni theorem (Eves, 1968), any geometric construction that can be performed with a compass and straightedge can be done with a compass alone. In this case, a straight line is defined by any pair of points.

The third type of construction is with straightedge alone. In this case, the Poncelet–Steiner theorem (Eves, 1995) states that whatever can be constructed by compass and straightedge together can be constructed by straightedge alone provided one is given a single circle and its centre. Actually, this means one action with the compass to provide the circle and its centre. Steiner proved this theorem in the second volume of his writings: "Die geometrischen konstruktionen ausgeführt mittels der geraden linie, und eines festen kreises".

Geometric constructions with straightedge alone

This article focuses on the third type of geometric construction: that with straightedge alone, and most of the constructions presented are conducted without being given the centre of the circle. In particular, we shall focus on how to construct a perpendicular (and occasionally a parallel) to a straight line from a point either distant to or on a line using only a straightedge, and given a circle and a line passing through its centre. In other words, using a circle and a diameter, but without knowing the precise location of its centre.

Geometric constructions using only a straightedge is a topic in plane geometry that is not directly expressed in the program of studies (which is unfortunate, given that it encourages thought, creativity and originality in the methods of solution, and presents opportunities for integrating solutions while developing unique strategies). Often, when dealing with a construction problem, especially with only a straightedge (or only a compass), interesting solutions are revealed that emphasise the beauty of mathematics.

Solving construction tasks using only a straightedge requires a wider and more profound knowledge of plane geometry than that required to solve problems using a compass and straightedge. Such constructions allow one to apply the important properties of known geometric shapes in the Euclidean or projective plane, and thus to diversify the process of studying geometry.

Presentation of the task

A circle with diameter AB is drawn on a plane, as well as some random point denoted by the letter P. A straight line passing through point P and perpendicular to diameter AB is to be constructed, using a straightedge only.

There are seven possible locations of point P (see Figure 2), and the complexity of the problem, and its solution, depends on the location of point P relative to line AB and to the given circle, and the full solution of the task requires solutions for all seven cases.

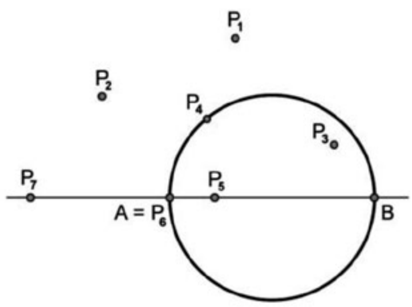

Figure 2. The seven possible locations for point P.

The seven possible locations are as follows:

- *Case 1:* Point $P = P_1$ is located outside the circle, so that its projection on the line AB lies on the diameter AB.
- *Case 2:* Point $P = P_2$ is located outside the circle, and its projection on the continuation of AB lies outside the circle.
- *Case 3:* Point $P = P_3$ is an interior point of the circle, which does not lie on the diameter.
- *Case 4:* Point $P = P_4$ is located on the circular arc, and it does not coincide with the ends of the diameter A and B.
- *Case 5:* Point $P = P_5$ is an interior point on the diameter.
- *Case 6:* Point $P = \mathrm{A} = P_6$ is located at the end A of the diameter.
- *Case 7:* Point $P = P_7$ is located on the continuation of the diameter AB, outside the circle.

Because the solution of the task requires some knowledge of the material learned in plane geometry as part of the high-school program of studies (including the ones presented in the Australian schools), when this problem is presented to students, it should be presented only after they have studied the circle and special segments in the triangle (altitudes, angle bisectors, etc.). In addition, the solution requires some knowledge of the following special properties of trapeziums (of which the students are usually not aware), such as the Steiner theorem for a trapezium, and which should be introduced to them during presentation of the problem.

Special properties of the trapezium

Property A: The Steiner theorem for the trapezium. (Note: The Steiner theorem is stated in the introduction, above. While many other mathematicians have proven this theorem, in this article there is reference only to Steiner's proof.)

Given that: $AM = MB$, $DN = NC$, it will be proven that the points E, M, F and N lie on the same straight line, as shown on Figure 1.

Steiner's proof of the trapezium theorem: To prove the theorem, it will be proven that the line EF intersects the bases of the trapezium at the points M and N (Figure 3).

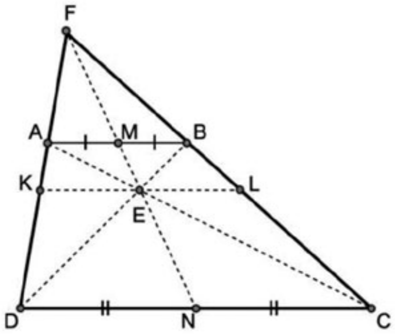

Figure 3. Line EF intersects the bases of the trapezium at the points M and N.

Draw segment KL through point E parallel to base AB. From the theorem "the point of intersection of the diagonals of a trapezium bisects the segment parallel to the base whose ends are on the sides of the trapezium and which passes through the point of intersection" (can be proven using Thales' theorem), it is concluded that $KE = EL$, that is, EF is a median to the side KL in triangle ΔFKL.

Another theorem is also used: "In any triangle ΔFKL, the median to the side KL bisects any segment parallel to KL, whose ends are located on the sides of the angle $\angle KFL$". (This theorem can also be proven using Thales' theorem).

This theorem is applied for the two triangles ΔFKL and ΔFDC, and then obtaining that FE intersects the segment DC at the point N and the segment AB at the point M.

We use Steiner's theorem to solve our task. At the end of the article is another example for a geometrical problem that can be solved simply using Steiner's theorem for the trapezium, as well as an application of the presented constructions.

Property B: Is a result of the Steiner theorem for the trapezium.

Let M and N denote the middles of the bases of trapezium $ABCD$.

The continuations of the sides of two trapeziums with same height, a common large base, and a small base with a similar length, intersect at points which are located on one line parallel to the bases of the trapeziums.

Figure 4 shows two trapeziums: *MBND* and *AMND*, which have a common large base and equal small bases. Points *G* and *F* are on one straight line parallel to the bases of the trapeziums.

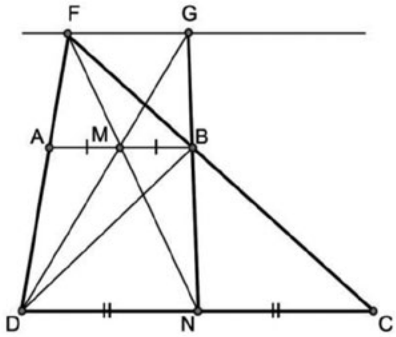

Figure 4. Two trapeziums: MBND and AMND, which have a common large base and equal small bases.

The proof employs the similarity of the triangles: $\Delta FNC \sim \Delta FMB$ and the triangles: $\Delta GDN \sim \Delta GMB$, and the inverse of Thales' theorem.

Constructing an application of the Steiner theorem for the trapezium (will be used below)

The application: To construct a straight line parallel to a given segment that passes through a given point, when the midpoint of the segment is given; i.e., given a point *P* and a segment *AB* with midpoint *C*, construct a straight line that is parallel to *AB* and that passes through point *P*.

Description of the construction: Draw a straight line through points *A* and *P*. On the continuation of the line, choose some point X_1. Connect point X_1 with points *B* and *C* (Figure 5). Connect point *B* with point *P*. The segments *BP* and CX_1 intersect at point X_2. Draw a straight line connecting points *A* and X_2, whose continuation intersects BX_1 at point X_3. The line PX_3 is the sought line.

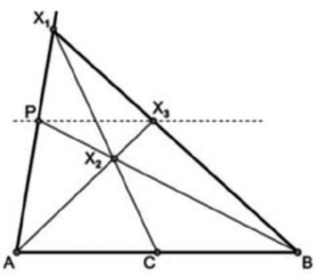

Figure 5. Description of constructing a parallel line through a point P to a given segment AB and its midpoint C.

Proof of the construction: An indirect proof is given. Suppose that line PX_3 is not parallel to *AB*. Draw through the point *P* the straight line *PQ*, which is parallel to *AB* (Figure 6).

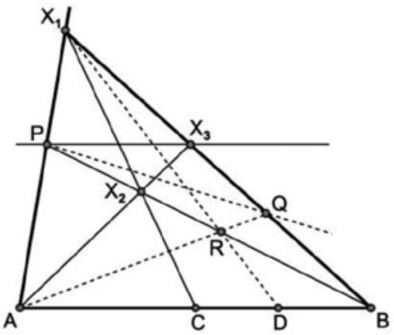

Figure 6. Description of indirect proof to a construction of a parallel line through a point P to a given segment AB and its midpoint C.

Since points X_3 and Q do not coincide, lines AX_3 and AQ are different lines, and they intersect segment BP at different points X_2 and R, respectively. Therefore, the lines X_1X_2 and X_1R are also different lines, and they intersect segment AB at different points C and D, respectively, which means that the point D is not the middle of segment AB. On the other hand, from Steiner's theorem, in trapezium $ABPQ$, the line X_1R intersects base AB, which means that D must be the midpoint of AB. Hence, we have a contradiction! Therefore, the assumption is not correct, and $PX_3 \parallel AB$.

The description of this construction is the general process for constructing a line parallel to a given segment (with its midpoint). It is important to note that according to the Steiner theorem for the trapezium, the following two constructions with a straightedge alone are equivalent:

- given a segment and its middle point, construct a parallel to the segment through any point; and
- given a segment and a parallel line to the segment, bisect the segment.

A list of other geometrical properties which are relevant for the solution of the task:

The properties of angles in a circle:

- An inscribed angle that rests on the diameter is a right angle (ACARA: An angle in a semicircle is a right angle).
- Inscribed angles that rest on the same arc (or on equal arcs) are equal (ACARA: Two angles at the circumference subtended by the same arc are equal).

The properties of a diameter and a chord:

- A diameter that passes though the midpoint of a chord (or through the midpoint of the arc that corresponds to the chord), is perpendicular to the chord.
- A diameter that is perpendicular to a chord bisects it.

The properties of the heights in a triangle:

- The three heights (altitudes) of a triangle intersect at one point (are concurrent at a point), called the *orthocentre* of the triangle.
- In a triangle ΔABC, the three heights (altitudes) AA_1, BB_1, CC_1 are also the bisectors of the angles in the triangle $\Delta A_1B_1C_1$, as shown in Figure 7.

Triangle $\Delta A_1B_1C_1$ is called the *orthic triangle* or *altitude triangle* of triangle ΔABC.

It is interesting to indicate that the incentre (that is the centre for the inscribed circle) of the orthic triangle is the orthocentre of the original triangle (see Figure 8). Also, the orthic triangle provides the solution to Fagnano's problem, posed in 1775, of finding for the minimum perimeter triangle inscribed in a given acute-angle triangle (Holand, 2007; Rademacher & Toeplitz, 1957). To prove the property of the orthic triangle, refer to Figure 9.

Inscribe ΔBCB_1 with a circle. Obviously, BC is the diameter of this circle and hence point C_1 is on the circle. Therefore, quadrilateral BCB_1C_1 is inscribed in the circle to obtain, according to the sums of the opposite angles, $\angle ABC = \angle AB_1C_1 = \beta$. Similarly, circumscribe quadrilateral AB_1A_1B with a circle, to obtain $\angle BAC = \angle CA_1B_1 = \alpha$. Finally, isolate A_1C_1AC to obtain $\angle ACB = \angle AC_1B_1 = \gamma$.

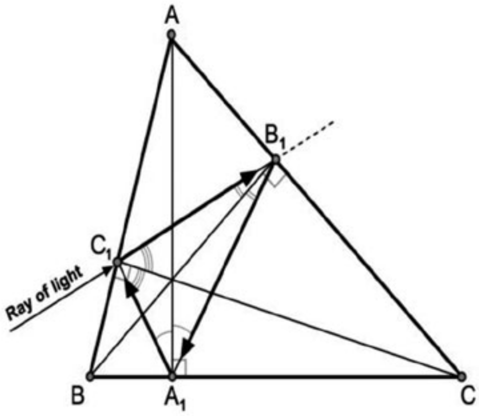

Figure 7. The three heights AA_1, BB_1, CC_1 of triangle ABC are also the bisectors of the angles in the triangle $\Delta A_1B_1C_1$.

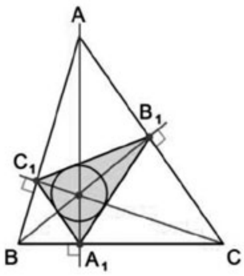

Figure 8. The centre of the inscribed circle of the orthic triangle $\Delta A_1B_1C_1$, is the orthocentre of the original triangle.

The result shows that the three triangles, ΔA_1BC_1, ΔAB_1C_1 and ΔA_1B_1C, are similar to each other and to the original triangle, ΔABC. At each vertex of triangle $\Delta A_1B_1C_1$ are two equal angles of $90° - \alpha$, $90° - \beta$ and $90° - \gamma$, and therefore the heights of the original triangle ΔABC are the bisectors of the orthic triangle $\Delta A_1B_1C_1$.

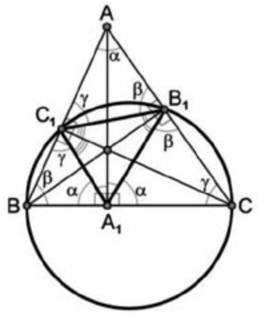

Figure 9. Proof of the property of the orthic triangle $\Delta A_1B_1C_1$.

This property of the orthic triangle (see Figure 7) also has a physical significance: any ray of light that penetrates through point C_1 in direction C_1B_1 continues to travel indefinitely along route $C_1B_1A_1$, since the angle of impact of the light on each side (assuming it is coated with a mirror surface from the inside), is equal to the angle of return. The sides of the orthic triangle form an 'optical' or 'billiard' path reflecting off the sides of ΔABC.

It is interesting to note that in high school geometry studies, students learn about the special points related to the points of intersection of the three mid-perpendiculars, the three angle bisectors, the three medians, and the three heights in a triangle. They learn that the first intersection point is the centre of the circle circumscribed around the triangle, the second is the centre of the inscribed circle, and the third is the centre of gravity. However, they are not given any special property for the fourth one, the orthocentre. From here, though, one can see that the point of intersection of the heights is the centre of the inscribed circle in the orthic triangle.

Now that we have presented all the geometric statements required for the completion of the task, that is, to find the perpendicular to diameter AB passing through given point P using a straightedge only (see Figure 2), it is possible to proceed.

Solution of the task

There are seven different solutions, depending on the location of point P (see Figure 2), plus another option using the Steiner theorem in a more straightforward manner.

Case 1: P is located at point P_1.

Description of the construction: As shown in Figure 10, point P_1 is connected to points A and B (the ends of the diameter). These lines intersect the circumference of the circle at points X_1 and X_2. Connect point A with point X_2, and connect point B with point X_1. The two segments AX_2 and BX_1 intersect at point X_3. The line P_1X_3, whose continuation intersects the diameter at point M, is the requested perpendicular.

Proof of the construction: Segments AX_2 and BX_1 are heights in the triangle ΔAP_1B (inscribed angles resting on the diameter), and X_3 is the point of intersection of the heights in the triangle, therefore P_1M is the third height in the triangle.

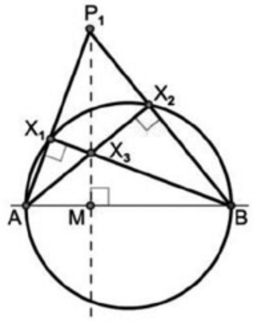

Figure 10. Construction of a perpendicular from point P_1 to the diameter AB.

Case 2: P is located at point P_2.

Description of the construction: The stages of the construction are as in the first case, and as shown in Figure 11.

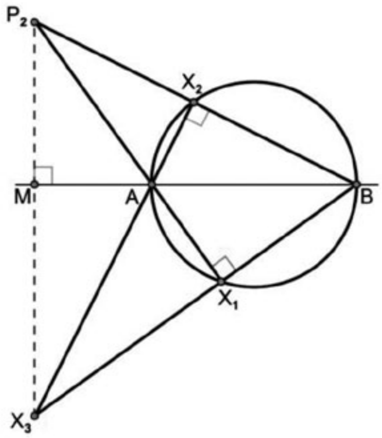

Figure 11. Construction of a perpendicular from point P_2 to the diameter AB or its continuation.

Proof of the construction: In this case, triangle ΔAP_2B is obtuse-angled. Therefore, the point of intersection of the heights is X_3, where BX_1 is the height to the side AP_2 and AX_2 is the height to the side BP_2. Point X_3 is outside the triangle ΔAP_2B. The segment P_2X_3 is the third height to the side AB.

Case 3: P is located at point P_3.

Description of the construction: The stages of the construction are as in the first case, and as shown in Figure 12.

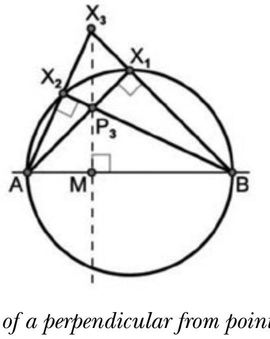

Figure 12. Construction of a perpendicular from point P_3 to the diameter AB.

Proof of the construction: AX_1 and BX_2 are heights in the triangle ΔAX_3B, and therefore X_3M is the third height that intersects the diameter at point M.

Case 4: P is located at the point P_4.

Description of the construction: Point A is connected with point P_4, and a point, X_1, is selected on the continuation of line AP_4. Point B is connected with point X_1. The segment BX_1 intersects the circle at point X_2 (Figure 13). Point A is connected with X_2, and point B with P_4, resulting in the point of intersection X_3. A straight line is drawn through point X_1 and point X_3, and the continuation of the line intersects diameter AB at point X_4. Point X_2 is connected with point X_4, and the continuation of the line intersects the circle at the point X_5. The line P_4X_5 is the perpendicular to the diameter.

Proof of the construction: As in the previous cases, segments AX_2, BP_4 and X_1X_4 are three heights in the triangle ΔABX_1. Therefore, from property 3b, we also obtain that they are the bisectors of the angles in the orthic triangle $\Delta P_4X_2X_4$. Ray AX_2 bisects angle $P_4X_2X_4$. This means that point A is the middle of arc P_4X_5, and therefore, from property 2a, chord P_4X_5 is perpendicular to diameter AB.

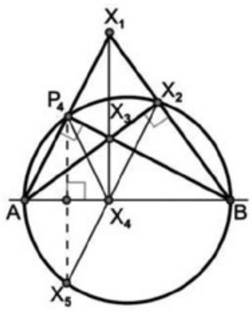

Figure 13. Construction of a perpendicular from point P4 to the diameter AB.

Cases 5, 6, and 7: P is located at point P_5, P_6 or P_7.

Description of the constructions: (The description is similar for all three cases): Draw perpendiculars X_1X_2 and X_3X_4 to diameter AB in the following manner: Choose a point, X_1, on the circle (as the location of P_4), and draw from it a perpendicular to the diameter to point X_2 (the construction is carried out as in case 4). In the same manner, choose a point, X_3, and draw a perpendicular to point X_4. Points M and N are, respectively, the points of intersection of perpendiculars $X_1\ X_2$ and $X_3\ X_4$ with diameter AB (Figure 14).

Proofs of the constructions: Point P_5 is connected with point X_1 and point X_2. The segments intersect X_3X_4 at points X_5 and X_6, so that trapezium $X_1X_2X_6X_5$ is obtained, where points M and N are the midpoints of its bases. Point X_2 is connected with point N, and point M with point X_5. The continuations of the straight lines intersect at point X_7. From property B, which stems from the Steiner theorem for a trapezium, line P_5X_7 is perpendicular to diameter AB.

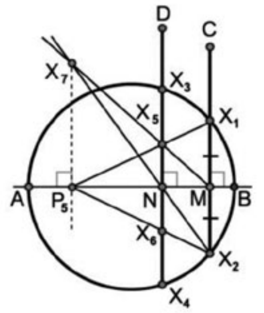

Figure 14. Construction of a perpendicular to the diameter AB at point P_5.

Straightforward application of the Steiner theorem: Another option is to apply the Steiner theorem for trapezium (see Figure 15). With respect to case 1, from an arbitrary point C, a perpendicular to AB is constructed. It intersects the given diameter at point M, and the circle at points X_1 and X_2. Then, construct a parallel line to X_1X_2 that passes through point P_5 according to the assignment described earlier in this paper.

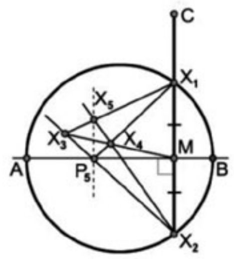

Figure 15. Straightforward application of the Steiner theorem for constructing a perpendicular to the diameter AB at point P_5.

It is interesting to note that for the proofs of cases 1–4 (to construct a perpendicular to a line through an external point), one needs only to apply the theorem about the intersection of the three altitudes of a triangle at a single point. However, for cases 5–7 (to construct a perpendicular at a point on the segment AB or its continuation), one must apply the Steiner theorem.

Extension of the task

An interesting question is: What extra given condition is needed to be able to locate the centre of the circle with only a straightedge?

First, given a segment and its centre point, it is possible to construct a diameter of a given circle (see Figure 16).

Through points H_1 and H_2 on the circle, it is possible to construct parallel lines to the given segment CD (shown in section 5.3 above). Then, using the Steiner theorem for trapezium $H_1K_1K_2H_2$, it is possible to bisect each of the chords H_1K_1 and H_2K_2 (points L and N). The chord through L and N is a diameter.

Equivalently, given a circle and two parallel lines, it is possible to construct a diameter of the circle (see Figure 17).

Choose two points on l and two points on k and draw the trapezium $F_1F_2G_2G_1$, then, using the Steiner theorem, find the midpoints of the bases of the trapezium (one midpoint is sufficient) and apply the previous construction of a diameter of the circle.

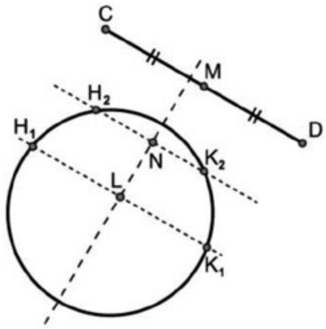

Figure 16. Constructing a diameter of a given circle by given a segment CD and its midpoint M.

Therefore, to construct the centre of a given circle, we need two nonparallel segments with their midpoints, or equivalently a parallelogram, or a triangle and its centre of gravity (the intersection point of its medians). In this case, we are able to construct two diameters whose intersection is the centre of the circle, which means that it is possible to construct with straightedge alone every construction that can be constructed with compass and straightedge. Obviously, it is possible to create many more conditions for finding the centre of a given circle by construction with a straightedge alone.

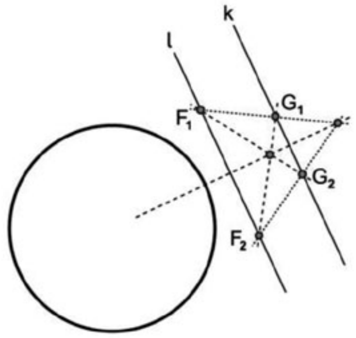

Figure 17. Constructing a diameter of a given circle by given two parallel lines

Application of the task

In order to demonstrate the applicability of the Steiner theorem for the trapezium in problem solving, we present another problem that can be easily proven using this theorem.

Problem: Given is a trapezium with lengths of bases a and b, and where the sum of the angles of the lower base is 90°. That is, the following is given: $AB = a$; $DC = b$; $\alpha + \beta = 90°$; $AE = EB$, $DF = FC$. Using a and b, express the length of the segment connecting the midpoints of the bases (Figure 18).

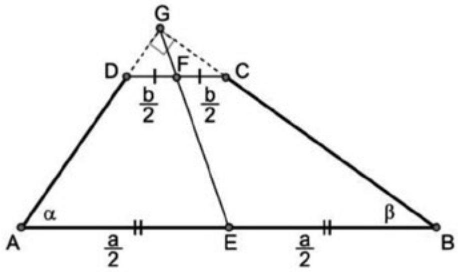

Figure 18. Application of Steiner's theorem to express the length of FE.

Proof using Steiner's theorem: Extend the sides up to their point of intersection, and obtain a right-angled triangle ΔGAB. From Steiner's theorem for a trapezium, points G, E, and F are on one straight line.

GF and GE are medians to the hypotenuse in right-angled triangles, which are equal to half the hypotenuse, and therefore:

$$FE = GE - GF = \frac{a-b}{2}$$

Note: Other proofs of this problem can be given, but they are longer and either requires auxiliary constructions and the use of algebraic formulas for the trapezium, or the use of trigonometric tools.

Additional applications

Another application of the constructions presented above is to create some shapes that can be used either in art or in tessellation. For example, given a rhombus, construct using straightedge alone a rectangle with its sides equal to the diagonals of the given rhombus (see Figure 19).

Now, it is obvious that, given the segment DB and its midpoint, it is possible to construct a parallel to it through point A, as well as through point C and the same is applied for segment AC and points D and B. Repeating this shape again and again creates a nice tessellation pattern. Another possibility is to start with a parallelogram rather than a rhombus.

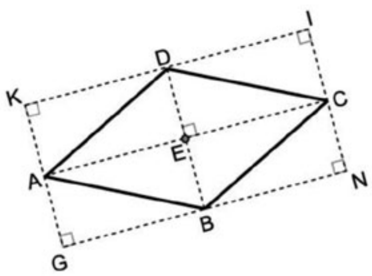

Figure 19. Constructing a rectangle with its sides equal to the diagonals of a given rhombus

Another example of application is the following task: Given parallelogram $ABCD$ with sides $AB = a$ and $BC = b$, construct lengths $m \cdot a$ and $n \cdot b$, m, $n \in N$ using straightedge alone (see Figure 20).

Here too, use the method of constructing a parallel line to a given segment with its midpoint (the diagonals of the given parallelogram) to obtain another parallelogram, and then repeat this procedure as needed. In this case too, one can adapt this work to art and tessellations.

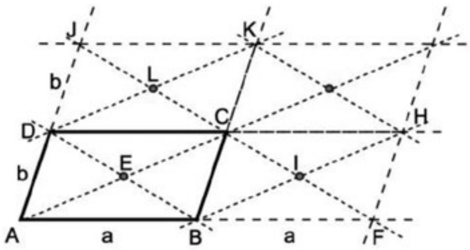

Figure 20. Constructing lengths of m and n times the corresponding sides a and b of a given parallelogram.

Conclusion

Inspiring students to love and be excited by mathematics is one of the goals of the good teacher. The geometric constructions topic is remarkable and can be used for implementing knowledge, theorems, and properties of geometrical shapes, as well as challenging thought. An important branch in this field is geometric construction with restrictions on the types of tools used. Such geometric construction requires the ability to differentiate between possible and impossible constructions based on rudimentary construction methods that, with their application, allow the performance of more complicated ones.

In addition, several studies have pointed out that introducing historical background to mathematical topics plays a vital role in making mathematics more interesting and accessible (see for example Heiedi, 1996) and "it demythologises mathematics by showing that it is the creation of human beings" (Marshal & Rich, 2000, p. 706). Hence, investigating the development of geometric construction through the history of mathematics enriches it significantly.

To this purpose, this article presents examples of geometric constructions to construct a perpendicular on the diameter of a circle using only a straightedge. (The centre was not given, and in some cases, the centre of the circle was also sought.) Steiner's theorem, a mathematical pearl that indicates the minimal conditions required for classical geometric constructions, is presented as a tool for the constructions.

This article is enhanced, the authors believe, by a brief history of geometric constructions, and by focusing specifically on constructions with aid of a straightedge (without markings) alone. Indeed, Steiner claimed that every construction that can be conducted using a compass and straightedge can be done using only a straightedge, given a circle and its centre (equivalent to one use of a compass). In addition, several mathematicians who lived in different ages are mentioned, with a description of how they coped with geometric construction problems, some of which involved proving that some specific constructions were actually impossible, and which required the creation of new mathematical theories and proofs to finally prove them.

Mention is made of the use of the dynamic geometry software as a helpful, powerful tool for teaching geometry, especially for enhancing the technical part of geometric constructions.

The authors believe that the geometric constructions presented in this article can make a significant contribution to the education of both pre- and in-service mathematics teacher as well as to the high school geometry curriculum. Through such geometric constructions, including emphasising their place in the history of mathematics, the learner can expand their conception of mathematics and, hopefully, appreciate the fascination that is in mathematics.

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