Some comments on the use of de Moivre's theorem to solve quadratic equations with real or complex coefficients

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Introduction

The roots of a quadratic equation with either real or complex coefficients can be found relatively easily from the 'standard' quadratic equation formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

A routine application of Equation (1) will furnish the desired roots, but it gives no indication of the location of these roots if the result contains complex numbers. This paper describes how a simple application of de Moivre's theorem may be used to not only find the roots of a quadratic equation with real or generally complex coefficients but also to pinpoint their location in the Argand plane. This approach is much simpler than the comprehensive analysis presented by Bardell (2012, 2014), but it does not make the full visual connection between the Cartesian plane and the Argand plane that Bardell's three dimensional surfaces illustrated so well.

de Moivre's method

Students studying a mathematics specialisation (ACARA, n.d., Unit 3, Topic 1 Complex Numbers) such as the Victorian VCE (Specialist Mathematics, 2010), HSC in NSW (Mathematics Extension in NSW, 1997) or Queensland QCE (Mathematics C, 2009) will be familiar with de Moivre's theorem and its applications to complex numbers, especially raising a complex number to a power. To recap, any complex number $\lambda + i\mu$ can be expressed in polar form as $r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{\lambda^2 + \mu^2} \text{ and } \theta = \tan^{-1}\left(\frac{\mu}{\lambda}\right)$$

with θ measured anticlockwise from the positive axis Ox. Then if p is a rational number, $[r(\cos\theta + i\sin\theta)]^p = r^p(\cos\theta + i\sin\theta)^p$ and de Moivre's theorem states:

$$r^p(\cos\theta + i\sin\theta)^p = r^p(\cos p\theta + i\sin p\theta) \qquad (2)$$

This result can be generalised to include fractional indices, as shown by Bostock and Chandler (1979, Chapter 12). If p and q be integers prime to each other, then *all* the values of

$$\left[r(\cos\theta + i\sin\theta)\right]^{\frac{p}{q}}$$

are given by

$$r^{\frac{p}{q}}\left[\cos\left((\theta+2\pi k)\frac{p}{q}\right) + i\sin\left((\theta+2\pi k)\frac{p}{q}\right)\right] \text{ where } k = 0, 1, 2, \ldots q-1 \qquad (3)$$

This latter form proves extremely useful for finding the (p/q)th root of a complex number.

Example 1

Suppose we are required to find the square roots of the complex number $z = 4 - 7.5i$. Firstly, z can be expressed in polar form thus:

$$r = \sqrt{4^2 + 7.5^2} = 8.5 \text{ and } \theta = \tan^{-1}\left\{\frac{-7.5}{4}\right\} = -61.93° \text{ or } 298.07°$$

Now apply de Moivre's theorem, as expressed in Equation (3) with fractional indices. Note also that because the argument must always be measured anticlockwise from the positive axis Ox, then in this case $\theta = 298.07°$.

$$\sqrt{z} = z^{\frac{1}{2}}$$

$$= (8.5)^{\frac{1}{2}}\left[\cos\left(\frac{298.07° + 360°k}{2}\right) + i\sin\left(\frac{298.07° + 360°k}{2}\right)\right], k = 0, 1 \text{ and } \frac{p}{q} = \frac{1}{2}$$

when $k = 0$, $\sqrt{z} = -2.5 + 1.5i$ and when $k = 1$, $\sqrt{z} = 2.5 - 1.5i$

These then are the two roots of the complex number $4 - 7.5i$. It is noted that de Moivre's theorem naturally anticipates results in the complex (Argand) plane. A standard graphical plot of the roots will always reveal q roots pitched equally around a circle of radius $r^{p/q}$ centred at the origin O. See Figure 1. In the current example, the roots are equi-pitched about the origin and lie diametrically opposite each other on a circle of radius $\sqrt{8.5}$ or 2.915; the diameter that links both roots is inclined at $\tan^{-1}(-1.5/2.5)$ or $-30.96°$ to the $\mathrm{Re}(x)$ axis and passes through the origin O.

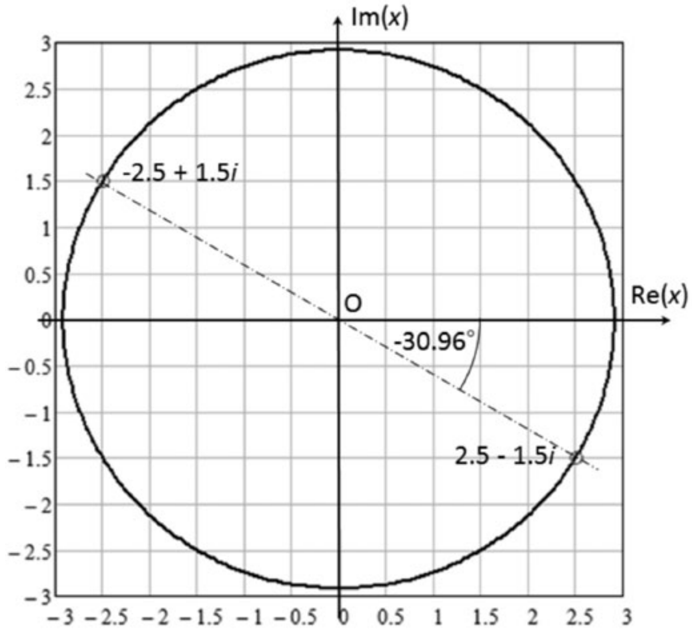

Figure 1. Plot in the Argand Plane showing both square roots of $z = 4 - 7.5i$.

The quadratic equation

Now consider the familiar quadratic equation $y = ax^2 + bx + c$ in which the coefficients a, b, c may be either real or generally complex. If the coefficients are purely real, then there are much simpler and faster ways of finding the roots. The efficacy of the method presented here is best appreciated for cases in which the quadratic equation has generally complex coefficients. Regardless of the nature of the coefficients, the roots follow from setting $y = 0$ and solving for the two values of x. By dividing the original quadratic equation through by the coefficient a and then completing the square, the following arrangement is obtained:

$$\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \tag{4a}$$

i.e.,

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \tag{4b}$$

Equation (4b) is sometimes referred to as the 'vertex' form of a quadratic equation. If the coefficients a, b, c are generally complex, then the terms $b/2a$ and $(b^2 - 4ac)/4a^2$ in Equation (4b) can both be expressed as single complex numbers, say $\alpha + i\beta$ and $\lambda + i\mu$. Equation (4b) hence becomes

$$\left[x + (\alpha + i\beta)\right]^2 = \lambda + i\mu \tag{5a}$$

This is akin to solving

$$z^2 = \lambda + i\mu \tag{5b}$$

Equation (5b) has been rendered in a standard form amenable to solution using de Moivre's theorem as expressed by Equation (3). This formulation is applicable to *any* quadratic equation with generally complex coefficients and represents a powerful technique for solving problems of this genre.

Armed with this information, if the complex number $\lambda + i\mu$ representing the RHS of Equation (5b) is rendered in polar form, it is easy to find the square roots using Equation (3) with $p = 1$ and $q = 2$. All that remains is to subtract the term $b/2a$ from both these square roots to give the required solutions for x. Whilst this procedure may seem a little involved, the results it delivers about the location of the roots more than justifies its use, as will shortly be explained, and some interesting solution strategies suggest themselves for finding the roots of cubic, quartic, and higher order polynomials. This will be discussed in more detail in the section entitled *Extension to higher order polynomials.*

Sample calculation

An example is now presented which will demonstrate the key features of this method. Consider the quadratic equation with the generally complex coefficients shown in Equation (6). It is desired to find its roots and plot them.

$$y = (1 + 2i)x^2 + (1 - 3i)x - 20 \tag{6}$$

From Equation (4b) the term $b/2a$ simplifies to $\alpha + i\beta = -0.5 - 0.5i$ and the term

$$\frac{\left(b^2 - 4ac\right)}{4a^2}$$

reduces to $\lambda + i\mu = 4 - 7.5i$. Hence Equation (6) can be expressed in the form shown in Equation (5a) as:

$$(x + (-0.5 - 0.5i))^2 = 4 - 7.5i \tag{7}$$

The RHS of Equation (7) can now be expressed in polar form and the square root found using de Moivre's theorem. This step has already been explained in Example 1. Then,

$$[x + (-0.5 - 0.5i)] = -2.5 + 1.5i \tag{8a}$$

and

$$[x + (-0.5 - 0.5i)] = 2.5 - 1.5i \tag{8b}$$

giving the final result for the two roots as:

$$x = (0.5 + 0.5i) - 2.5 + 1.5i = -2 + 2i \tag{9a}$$

and

$$x = (0.5 + 0.5i) + 2.5 - 1.5i = 3 - i \tag{9b}$$

It is noted that the square roots of $4 - 7.5i$ as plotted in Figure 1 are effectively translated by an amount $-b/2a$ in the Argand plane as the final part of the solution process; the point $-b/2a$ locates the new centre of the circle. The beauty of this method is that it always places the roots in the complex Argand plane, which is orthogonal to the Cartesian plane in which the quadratic equation was originally presented. Thus there is no doubt about the whereabouts of the roots.

If the techniques presented by Bardell (2014) are applied to the current problem, the roots are evident from the intersection of the hyperbolic paraboloid surfaces A and B, representing the real and imaginary parts of the quadratic equation respectively, with a plane at zero altitude (see Figure 2). A plan view of these surface intersections is shown in Figure 3, with de Moivre's solution from Equation (9) superimposed. The visual connection between the two methods should now be obvious.

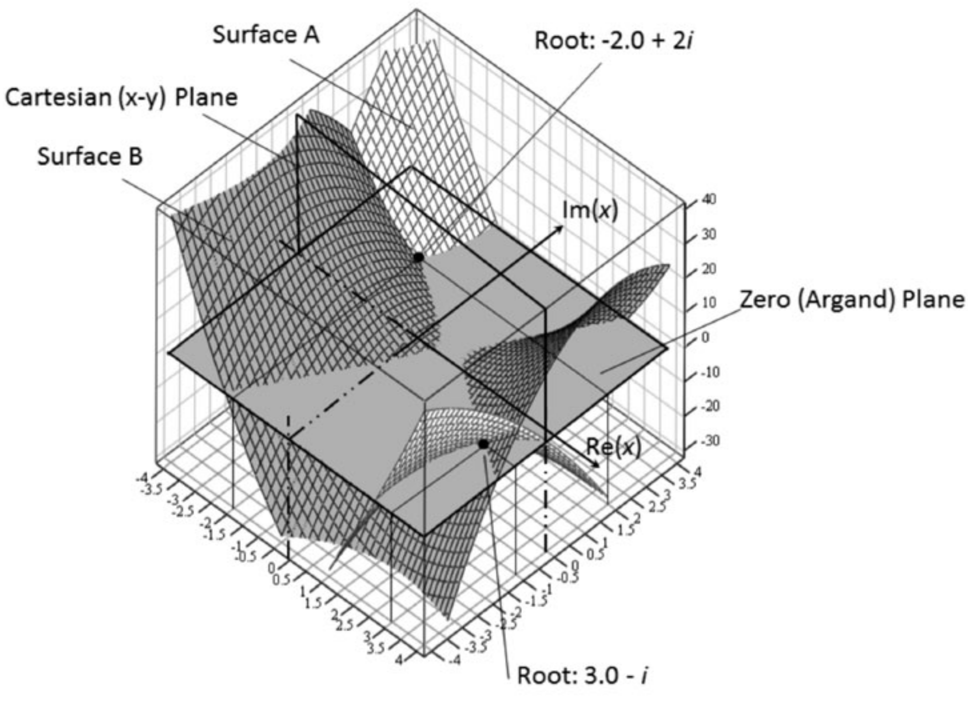

Figure 2. Equation (6) represented by the real and imaginary surfaces A and B.

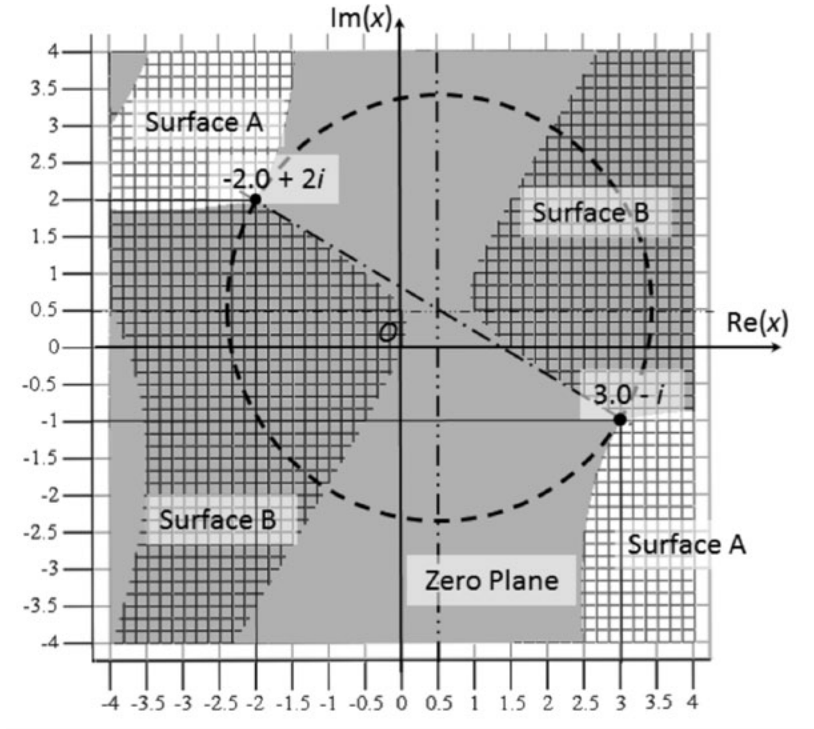

Figure 3. Plan view of Figure 2 showing the roots of Equation (6). The dashed line shows De Moivre's circle, translated now to an offset centre located at (0.5, 0.5).

Extension to higher order polynomials

We can use this technique to solve *any* polynomial with real or complex coefficients *provided* it can be rendered in the so-called 'vertex' form; i.e., $a(x + h)^p + k = 0$, where h and k are real or complex constants and p is the highest power of x present. In other words, if we can perform a similar action to completing the square on a higher order polynomial we can solve it immediately using de-Moivre's method.

Consider the general cubic polynomial $ax^3 + bx^2 + cx + d = 0$, in which the coefficients a, b, c, and d may be real or generally complex. This equation may be expressed in the vertex form $a(x + h)^3 + k = 0$ provided the coefficients satisfy the condition $b^2 - 3ac = 0$. Then, by setting

$$h = \frac{b}{3a} \quad \text{and} \quad k = d - ah^3 = \frac{\left(27da^2 - b^3\right)}{27a^2}$$

the cubic can be rewritten in vertex form, which is immediately amenable to solution using the techniques described in this paper. Note, the vertex form is easily found by expanding $a(x + h)^3 + k = 0$ and comparing the coefficients of each power of x with those of the general cubic polynomial. In this way, $a = a$, $b = 3ah$, $c = 3ah^2$ and $d = k + ah^3$. Eliminating h between the second and third equations gives the invariant condition $b^2 = 3ac$. Substituting $h = b/3a$ into the last equation gives

$$k = \frac{27da^2 - b^3}{27a^2}$$

A similar procedure can be used for quartics, quintics, etc. but the invariants thus generated become increasingly difficult to satisfy, thereby reducing the scope to render these equations in vertex form.

For example, $3x^3 - 18x^2 + 36x - 20 = 0$ can be expressed in vertex form since the condition $b^2 - 3ac = 0$ is satisfied with $(-18)^2 - 3 \times 3 \times 36 = 0$. Then, $h = -2$ and $k = 4$. The vertex form is thus $3(x-2)^3 + 4 = 0$. This can easily be checked by expanding and simplifying this vertex form to recover the original cubic equation. So, after rearrangement, $(x-2)^3 = -\frac{4}{3}$ and this problem reduces to finding the cube roots of $-\frac{4}{3}$, which by de Moivre's method are found to be -1.101, $0.55 - 0.953i$, and $0.55 + 0.953i$. Now add 2 to these results to find the three roots of the original equation, viz., 0.899, $2.55 - 0.953i$ and $2.55 + 0.953i$.

It is observed that a general cubic will have three roots which when plotted in the Argand plane will form an isosceles triangle. Only in those cases when the cubic reduces to the vertex form do the roots form an equilateral triangle and lie equally spaced around de Moivre's circle. This observation applies whether the coefficients are real or complex. The cube roots of unity, equally spaced at 120° intervals around the unit circle centred at the origin, is arguably the best known illustration of this fact.

It is understood that this technique will only be of use in a limited number of cases, since few cubic polynomials will lend themselves to expression in vertex form. The applicability to higher order polynomials becomes even more limited since the conditions the coefficients have to satisfy get ever more demanding. Nonetheless, on those few occasions when it does work, the method described herein will provide a very quick, neat and simple solution.

Conclusions

This paper has demonstrated how a simple application of de Moivre's theorem may be used to not only find the roots of a quadratic equation with real or generally complex coefficients but also to pinpoint their location in the Argand plane. It has further been shown that the roots are always spaced equidistant from their common local origin, which is located at the point $-b/2a$. For a quadratic equation defined by complex coefficients any combination of roots can be forthcoming, and these are captured by the current method. Also, for the case with real coefficients, if $b^2 - 4ac \geq 0$, then the roots will always lie on the $Re(x)$ axis and be equally spaced about the point $-b/2a$, whilst if $b^2 - 4ac < 0$, then the roots will always lie on a line parallel to the $Im(x)$ axis through the point $-b/2a$ and be equally spaced either side, i.e., in front and behind, of the $Re(x)$ axis as expected.

An attempt has been made to generalise the ideas developed for solving the quadratic equation to apply to higher order polynomials. However, this has met with limited success because it is only applicable to that class of polynomial that can be expressed in vertex form.

Finally, the approach adopted here should help students and teachers alike appreciate the rich interplay that exists between complex numbers and polynomial equations. In this context, it should amply satisfy the following ACARA (n.d., Rationale) stated aims: "developing students' understanding of concepts and techniques drawn from complex numbers, and developing students' ability to solve applied problems using concepts and techniques drawn from geometry, trigonometry, and complex numbers".

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