Where is the square? Activities to stimulate spatial reasoning

Samuel Obara
Texas State University, USA
<so16@txstate.edu>

The National Council of Teachers of Mathematics (NCTM, 1989, 2000) and the new *Australian Curriculum: Mathematics* for senior secondary (ACARA, 2010) highlight the importance of teaching spatial reasoning as early as preschool when mathematics is introduced. Studies have shown that there is a relationship between spatial abilities and mathematical achievement (Burnett, Lane, & Dratt, 1979; Casey, Nuttall, Pezaris, & Benbow, 1995; Geary, Saults, Liu, & Hoard, 2000). Activities that enhance spatial reasoning skills are invaluable to, and should be encouraged in, classroom instruction. Casey, Andrews, Schindler, Kersh, Samper, and Copley (2008) define spatial skills as "the ability to think and reason through the comparison, manipulation, and transformation of mental pictures" (p. 270). In this article I present an activity (Aichele & Wolfe, 2007, p. 11) to pre-service high school teachers. Though the activity seems simple, it can be challenging to students who have not been exposed to spatial tasks. The activity goal is to create a square from a given polygon by making one straight cut so that two pieces, when put together without overlap or gaps, form a square as shown in Figure 1.

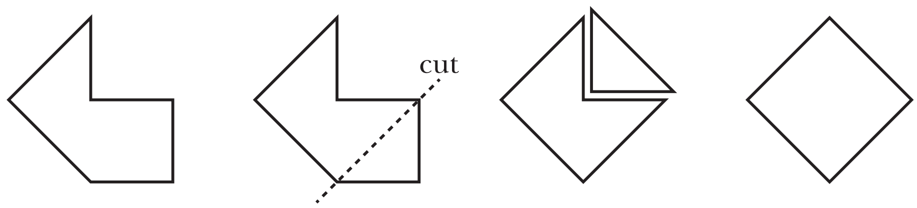


Figure 1. Steps in forming a square (Aichele & Wolfe, 2007, p. 9).

The students received a handout (Figure 2) to make a straight cut so that the two pieces can be configured to make a square. A few examples from the activities along with students' commentaries are reported here. These are intended to illustrate the process involved not only in solving the problems but in learning spatial reasoning.

During the brainstorming exercise on the strategy for the activity, one of the students described the characteristics of a square:

> Squares have four right angles and four congruent sides and can be oriented in any direction. Therefore, if an angle is not a right angle, then it will have to become one. The sides of a square are congruent; consequently, longer sides may have to be cut into smaller segments.

After brainstorming, I discussed the example (Figure 1) by asking students why I did what I did, or if there was any other way to do it. The aforementioned student noted that it worked because the cutting was done in such a way that the two pieces could form a square. Each student was given a multiple copies of handouts in case they made a mistake, a pencil, and a pair of scissors and a protractor to use.

Readers are encouraged work through the polygons (Figure 2) and find out which ones are more challenging and why; what can students learn from the activity.

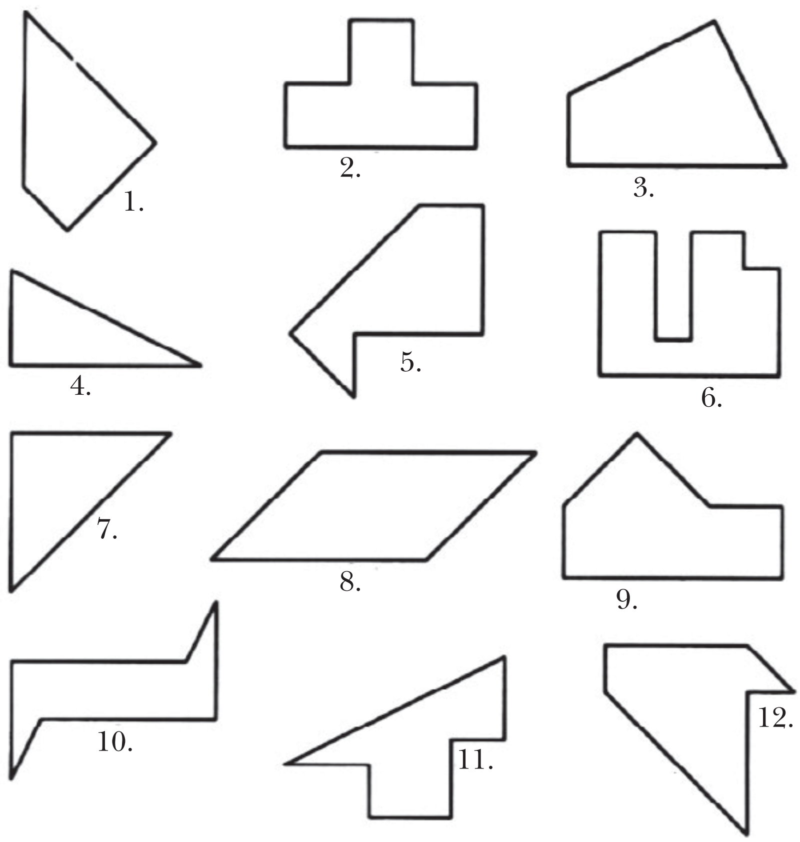


Figure 2. Two congruent halves activity (Aichele & Wolfe, 2007, p. 9).

Students' engagement with the activity

The class was divided into small groups of three, but each member of the groups was expected to do the activity independently as well. The students were free to start with any polygon on the handout and work through. As students were working on it, one student noted: "After 'cutting' a few polygons, I noticed that in all cases the longest side was cut to form a square. I also noticed that most of the time, the cut created at least one right angle."

In a follow-up question why this 'cut' be an apparent strategy in this activity, the student noted that the longest side must always be cut in some way to reduce it so that the length of larger part (if it is not cut equally) is the length of one side of the square. Most of the time the cut created at least one right angle as right angles must be created to be the final angle in the square.

Out of the 12 polygons investigated, a few were selected (5, 11 and 12) for classroom discussion since they provided a unique and meaningful learning experience to the students.

Polygon 5

Polygon 5 was interesting, noted one student: "When I came to polygon 5, it was difficult, so I skipped the problem and came back to it at the end." After working on the problem for a few more minutes, the answer appeared obvious to her and she shared her views on the challenges she had faced. She tried two strategies discussed.

First scenario (first cut)

Student A examined the polygon and was convinced that cutting off the piece (Figure 3a) was the right way to construct a square. She shared her idea with her classmate who at the same time was struggling with the same problem, and they shouted out: "We got it—polygon five." They said that one should cut the marked piece and rotate to the indicated section, and they were confident in their answer. The two were very excited, only their excitement ended abruptly when I challenged them to cut off the piece using a pair of scissors and test their conjecture. The group cut off the piece and noticed that the cut-off triangle was too big and did not make a square with the second (Figure 3).

(a) (b)

Figure 3. First cut, polygon 5.

Second scenario (second cut)

Student A tried to look at the polygon from different perspectives to figure out how else it could be cut in such a way without compromising the characteristics of a square. When asked to explain her thinking process, she noted: "I noticed that there were 'ear' and a 'mouth' that could fit together, so I tried to place them in two separate pieces. I used my earlier guidelines for cutting the longest side in such a way that it creates a right angle when extended to the opposite corner," (see Figure 4b).

She then noted, "When making the cut, the objective was to make right angles for the corners of the square, so that when the cut part is rotated, it can form a right angle corner for the square. This created the right angle as a result of the cut, limited the number of cuts and that helped me find the correct cut."

Still, other students could not visualise how the cut could form a square until it was cut and fitted, as shown in Figure 4b.

(a) (b)

Figure 4. Second cut, polygon 5.

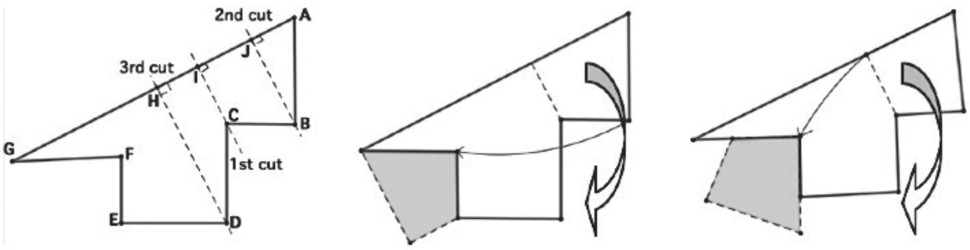
One should note that the longest line may be the line that can be cut to solve the problem. On the other hand, students should look for right angles in the polygons. Finally, student should be given multiple copies of the polygon(s) so that they have the option of actually making several 'investigative cuts.'

Polygon 11

The second challenging polygon to cut was number 11. This was one of the polygons that some students had the most trouble with; in fact, some students almost abandoned the activity because they had no idea of what to do. As one student put it: "I spent much time looking at it. I didn't know where to start. I knew I would have to cut the longest side, but I was not sure in which way to go with it".

First scenario (first cut)
As the students were trying different options, one student noted: "I decided to 'try something' and drew the first cut (CI as shown in Figure 5). I put the two pieces together in my head, but they did not form a square."

The student was not very sure if that was the case, but nonetheless proceeded next to cutting off the piece and moving it by first matching point B to point F (Figure 5b). The student noticed that matching point B with F did not produce a square. The students did not give up at this point but tried another option, but this time matching point I with point F. Once again, the two pieces did not make a square, even after the students made sure that the cut created a right angle, perpendicular to segment AG. One student said, "That was the only option for the first cut because after the first cut, one right angle was created at point I and the existing right angle at point B. To form a square, its interior angles were to be right angles so you had to fit at point F and the other outside to avoid a concave polygon."



(a) *(b)* *(c)*

Figure 5. First cut.

The student goal at this point was to create at least one right angle as a result of the cut.

Second scenario (second cut)
So the student next chose point B as the start of the cut, but quickly realised that although she was creating a right angle triangle that could fit in point C and F, it would not result in a square. Therefore, she moved to the third cut.

Third scenario (third cut)
The student did not expect this cut to work, but the she continued anyway to the third cut (HD). As she put it, "This one seemed to form two congruent

polygons. That attracted my attention. Maybe it will work, but I have no idea how it could be put together to form a square."

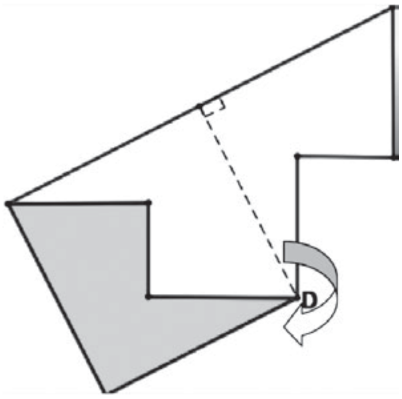


Figure 6. Third cut.

One of the students in another group had a hard time with this particular problem. He could not visualise mentally how the cut that formed two congruent polygons could be moved around so that the two pieces might form a square. He had to cut the pieces and physically move them around. This is one of the advantages of this activity—that students can cut and move the objects around rather than just visualising with the aim of developing spatial skills of students. One student noted, "I had trouble visualizing in my head how to move the piece around to form a square and decided to cut off the yellow polygon with a pair of scissors [third cut DH] and tried it around the white polygon."

After careful consideration, another student pointed out that the part is transformed through a 90° rotation at the center with rotation taking place at point D. The idea of cutting the polygon out and moving it, helped students to figure out why the two polygons formed the square. Students chose the three cuts because at minimum there must be at least two congruent segments in the three scenarios and at maximum the polygons can be congruent (the third cut).

Polygon 12

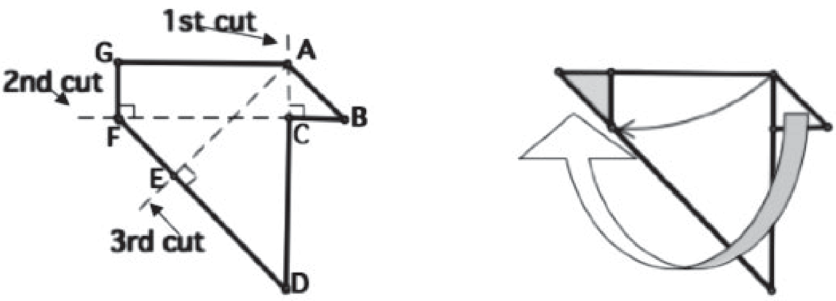
This polygon also challenged students in groups and when working individually. Three scenarios emerged as likely ways to cut it so as to form a square.

First scenario (first cut)

This was noted to be one option to use in that it created a right angle when cut out. One student said, "I first started looking at the part represented by the polygon as my first cut (Figure 7). I cut that portion, but the only place to put this was in the left hand corner, making it into a triangle."

Second scenario (second cut)

After the student noted that the first cut could not make a square, she moved to the next cut (2nd cut). Using what was learned from the first cut, it was obvious that she could not make a square. Frustrated about her situation, the student teamed up with a group to investigate any possibility. They pointed out that the main technique is to create at least one right angle as the result of the cut, which led to the third scenario.

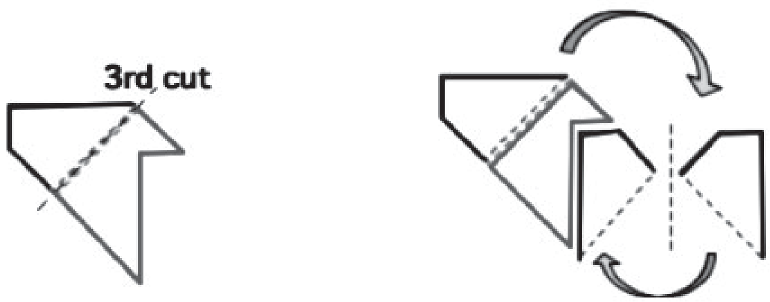


(a) *(b)*

Figure 7. First, second and third cut.

Third scenario (third cut)

The strategy was to have cuts that go through a point and make a right angle at the opposite side. The student had an option to explore, which was through point A and perpendicular to segment FD at point E. This resulted in the third cut. In the group, the students could not visualize how the two polygons ABCDE and EFGA could make a square. Some students did mentally figure out how that could generate a square, but it was a challenge for most. The group resolved to cut the two pieces apart and try to discover how that could generate a square. The polygon below shows how the students investigated this scenario. Sure enough, the two polygons generated a square.



(a) *(b)*

Figure 8. Third cut.

Conclusion

Spatial skills are vital skills that can be learned at any age. Some students begin the activity at different performance levels than others but skill can be improved across the ability levels. The activities presented may seem to be simple at a glance, but they can be a challenge, depending on the extent to which the student has developed spatial skills. These activities are appropriate to any student at any grade level and can challenge students to think critically. Spatial skills highlight fundamental characteristics of shapes like squares. For example: What is a square? How can it be constructed? These are simple but crucial concepts that students need to know, which the activity addresses. This lesson fits in the section of properties of polygons and their construction. Involving students in such activities, not only highlights properties of a square, but also figure out which pieces go where-which is a skill that most students lack and need developed and hence enhancing their spatial skills. This activity may seem easy to do but it avails the opportunity for students to share strategies in the learning process. These are hands-on activities enable students to first conjecture and then to test their ideas by cutting and fitting the pieces. Having a protractor is very useful tool for students in identifying complementary and supplementary angles in solving the problem. Students can work either individually or in small groups. As a teacher, try to facilitate meaningful conversation within groups and throughout the entire class. If administered well, these activities can provide opportunities for students to learn ideas that they may eventually take for granted.

References

Aichele, D. B. & Wolfe, J. (2007). *Geometric structures: An inquiry-based approach for prospective elementary and middle school teachers.* Upper Saddle, NJ: Prentice Hall.

Australian Curriculum Assessment and Reporting Authority [ACARA]. (2010, May 14). *ACARA releases senior secondary Australian curriculum for national consultation.* Retrieved from http://www.australiancurriculum.edu.au

Burnett, S. A., Lane, D. L. & Dratt, L. M. (1979). Spatial visualization and sex differences in quantitative ability. *Intelligence, 3*, 345–354.

Casey, M. B., Andrews, N., Schindler, H., Kersh, J. E., Samper, A. & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26* (3), 269–309.

Casey, M. B., Nuttall, R., Pezaris, E. & Benbow, C. P. (1995). The influence of spatial ability on gender differences in math college entrance. *Developmental Psychology, 33*, 697–705.

Geary, D. C., Saults, S. J., Liu, F. & Hoard, M. K. (2000). Sex differences in spatial cognition, computational fluency, and arithmatical reasoning. *Journal of Experimental Child Psychology, 77*, 337–353.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.