# Graphical solution of the monic quadratic equation with complex coefficients

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## Introduction

There are many geometrical approaches to the solution of the quadratic equation with real coefficients. Here it is shown that the monic quadratic equation with complex coefficients can also be solved graphically, by the intersection of two hyperbolas; one hyperbola being derived from the real part of the quadratic equation and one from the imaginary part. Both hyperbolas are of relatively simple form. Special solutions correspond to one or both of the hyperbolas being degenerate.

Considering the importance of the quadratic equation with complex coefficients as a bridge between the quadratic equation with real coefficients and more abstract algebraic approaches, rather little has been written about it. One well known work, Hardy (2008) partially examines relations amongst the coefficients of the monic quadratic with complex coefficients and an online source, McCarthy (n.d.) investigates the general quadratic in greater detail. Two recent articles in this journal by the same author address, in detail, the visualisation of the roots of the quadratic with real, Bardell (2012) and complex, Bardell (2014) coefficients by examining three dimensional plots.

This article is of potential interest to secondary school students with some exposure to complex numbers and first year university students. For example, in the Queensland Senior Mathematics Syllabus C, one suggested learning experience is "solve polynomial equations with real and complex coefficients".

## Theory

The monic quadratic with complex coefficients can be written in the form $z^2 + (b + Bi)z + (c + Ci) = 0$, where $b$, $B$, $c$ and $C$ are all real. Replacing $z$ by $x + iy$, we find $(x + \mathrm{i}y)^2 + (b + Bi)(x + iy) + (c + Ci) = 0$. Expanding this yields $x^2 + 2xyi - y^2 + bx - By + Bxi + byi + c + Ci = 0$. The real part gives

$x^2 - y^2 + bx - By + c = 0$ (1) and the imaginary part gives $2xy + Bx + by + C = 0$ (2). Both (1) and (2) represent rectangular hyperbolas with centre

$$\left(\frac{-b}{2}, \frac{-B}{2}\right)$$

Equation (1) can be rewritten as

$$\left(x + \frac{b}{2}\right)^2 - \left(y + \frac{B}{2}\right)^2 - \frac{b^2}{4} + \frac{B^2}{4} + c = 0$$

There are three general cases depending on the value of $4c + B^2 - b^2$. If $4c + B^2 - b^2$ is negative the foci of the hyperbola lie on a horizontal line, if positive, the foci of the hyperbola lie on a vertical line and if zero the hyperbola is degenerate, consisting of two intersecting lines of gradient ±1. All of this is guaranteed by the absence of a term in $xy$ in equation (1).

Equation (2) can be rewritten as

$$2\left\{\left(x + \frac{b}{2}\right)\left(y + \frac{B}{2}\right) - \frac{bB}{4} + \frac{C}{2}\right\} = 0$$

Again there are three general cases depending this time on the value of $2C - bB$. If $2C - bB$ is negative, the foci lie on a line of gradient +1; if $2C - bB$ is positive, the foci lie on a line of gradient –1; and if zero, the hyperbola is degenerate consisting of a horizontal and a vertical line. All of this is guaranteed by the absence of terms in $x^2$ and $y^2$ in (2).

For convenience, $4c + B^2 - b^2$ will be referred to as $\delta_1$ and $2C - bB$ will be referred to as $\delta_2$.

## Number of solutions

In general, two hyperbolas can have up to four points of intersection. In the case of two rectangular hyperbolas with the same centre and rotated 45° with respect to each other, there will be just two points of intersection and so the quadratic $z^2 + (b + Bi)z + (c + Ci) = 0$ will have two solutions. In the case that both of the conics are degenerate, there will be just one point of intersection, corresponding to repeated roots.

## The non-degenerate cases

There are four different non-degenerate cases, corresponding to the four non-degenerate combinations of (1) and (2), i.e.,

- $\delta_1 > 0, \delta_2 > 0$
- $\delta_1 > 0, \delta_2 < 0$
- $\delta_1 < 0, \delta_2 > 0$
- $\delta_1 < 0, \delta_2 < 0$.

***Case 1:*** $\delta_1 > 0$, $\delta_2 > 0$

The equation $z^2 + (-2 + 2i)z + 5 + 10i = 0$ becomes $x^2 - y^2 - 2x - 2y + 5 = 0$ and $2xy + 2x - 2y + 10 = 0$. These are plotted in Figure 1, where (1) refers to the hyperbola from equation (1) and similarly for (2). The points of intersection can be seen to be $(-1, 2)$ and $(3, -4)$, so the roots are $z = -1 + 2i$ and $z = 3 - 4i$. In this case $b = -2$, $B = 2$, $c = 5$, $C = 10$, so $\delta_1 = 4c + B^2 - b^2 = 20 + 4 - 4 = 20 > 0$ and $\delta_2 = 2C - Bb = 20 - 4 = 16 > 0$ .

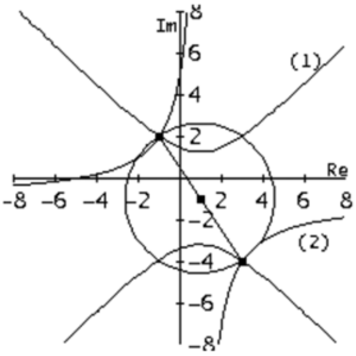



*Figure 1. Case 1.*

***Case 2:*** $\delta_1 > 0$, $\delta_2 < 0$

The equation $z^2 + (-3 - 3i)z + 8 - i = 0$ becomes $x^2 - y^2 - 3x + y + 8 = 0$ and $2xy - x - 3y + 1 = 0$. These are plotted in Figure 2. The points of intersection are $(2, 3)$ and $(1, -2)$, so the roots are $z = 2 + 3i$ and $z = 1 - 2i$.

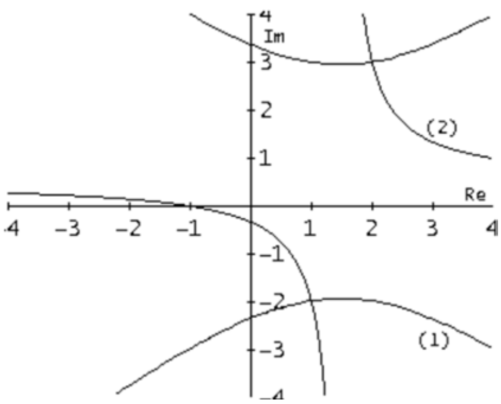



*Figure 2. Case 2.*

***Case 3:*** $\delta_1 < 0$, $\delta_2 > 0$

The equation $z^2 + (4 - 5i)z - 3 - 9i = 0$ becomes $x^2 - y^2 + 4x + 5y - 3 = 0$ and $2xy - 5x + 4y - 9 = 0$. These are plotted in Figure 3. The points of intersection are $(-3, 3)$ and $(-1, 2)$ so the roots are $z = 1 + i$ and $z = -1 + 2i$.

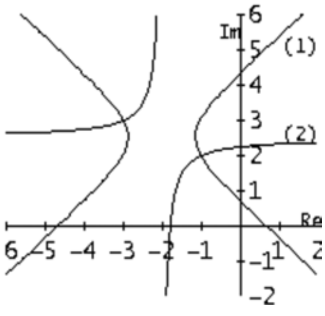



*Figure 3. Case 3.*

***Case 4:*** $\delta_1 < 0$, $\delta_2 < 0$

The equation $z^2 + (-8 - 3\text{i})z + 5 + 9i = 0$ becomes $x^2 - y^2 - 8x + 3y + 5 = 0$ and $2xy - 3x - 8y + 9 = 0$. The points of intersection are (7, 2) and (1, 1) so the roots are $z = 7 + 2i$ and $z = 1 + i$ as shown in Figure 4.

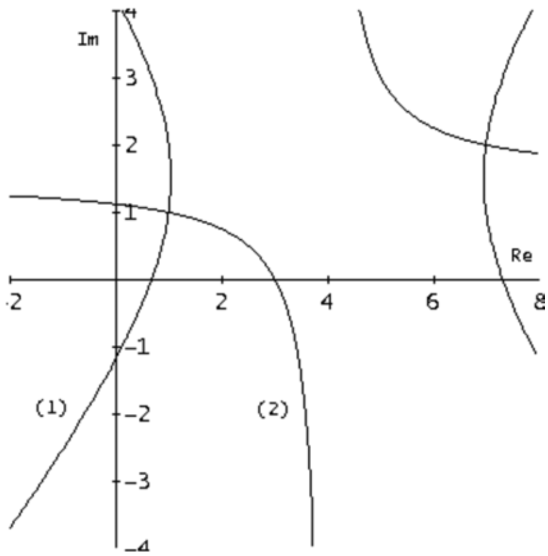



*Figure 4. Case 4.*

Note that each pair of solutions lie at the opposite ends of a diameter of the circle centred at

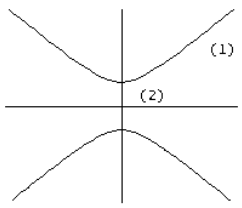
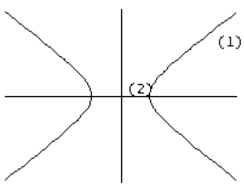
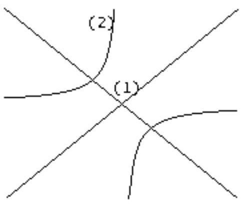
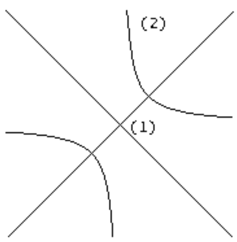
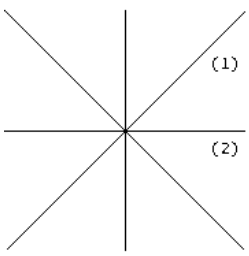
$$\left(\frac{-b}{2}, \frac{-B}{2}\right)$$

with radius given by half the modulus of the discriminant of the original equation, as shown in the first figure. Note also that although each of the points of intersection is found by considering points of intersection of equations (1) and (2) in the Cartesian plane, these points correspond directly to the roots in the complex plane. The axes labels in the figures are those appropriate to the complex plane.

## Special cases

Special cases correspond to one or both of the hyperbolas (1) and (2) being degenerate, which amounts to one or both of them being a cross shape; (1) leads to an × shaped cross and (2) leads to a + shaped cross. There are five possible combinations of $\delta_1$ and $\delta_2$; $\delta_1 > 0$, $\delta_2 = 0$; $\delta_1 < 0$, $\delta_2 = 0$; $\delta_1 = 0$, $\delta_2 > 0$; $\delta_1 = 0$, $\delta_2 < 0$; and $\delta_1 = 0$, $\delta_2 = 0$. The centre of each of these can be located in general position, on the *x*-axis, on the *y*-axis or at the origin. A detailed list is given in Table 1.

*Table 1*

| **Shape** | **Centre** | **Nature of roots** | **Roots** |
|---|---|---|---|
|   | general | Same real, distinct imaginary | $\alpha + i\beta_1, \alpha + i\beta_2$ |
| | on *x* axis | Complex conjugate | $\alpha + i\beta, \alpha - i\beta$ |
| | on *y* axis | Distinct imaginary | $i\beta_1, i\beta_2$ |
| | origin | Equal and opposite imaginary | $i\beta, -i\beta$ |
|   | general | Distinct real, same imaginary | $\alpha_1 + i\beta, \alpha_2 + i\beta$ |
| | on *x* axis | Distinct real | $\alpha_1, \alpha_2$ |
| | on *y* axis | Equal and opposite real, same imaginary | $\alpha + i\beta, -\alpha + i\beta$ |
| | origin | Equal and opposite real | $+\alpha, -\alpha$ |
|   | general | Two distinct roots | $\alpha_1 + i\beta_1, \alpha_2 + i\beta_2$ |
| | on *x* axis | Distinct real, equal and opposite imaginary | $\alpha_1 + i\beta, \alpha_2 + i\beta$ |
| | on *y* axis | Equal and opposite real, distinct imaginary | $\alpha + i\beta_1, -\alpha + i\beta_2$ |
| | origin | Equal and opposite roots | $\alpha + i\beta, -\alpha - i\beta$ |
|   | general | Two distinct roots | $\alpha_1 + i\beta_1, \alpha_2 + i\beta_2$ |
| | on *x* axis | Distinct real, equal and opposite imaginary | $\alpha_1 + i\beta, \alpha_2 - i\beta$ |
| | on *y* axis | Equal and opposite real, distinct imaginary | $\alpha + i\beta_1, -\alpha + i\beta_2$ |
| | origin | Equal and opposite roots | $\alpha + i\beta, -\alpha - i\beta$ |
|   | general | Repeated complex roots | $\alpha + i\beta$ |
| | on *x* axis | Repeated real roots | $\alpha$ |
| | on *y* axis | Repeated imaginary roots | $i\beta$ |
| | origin | Repeated zero | $0$ |

Note: In each case (1) refers to the hyperbola from equation (1) and similarly for (2). The real axis is horizontal and the imaginary axis is vertical.

***Two of the better known special cases***

***Case 5***

$z^2 + (-6 - 4i)z + 5 + 12i = 0$ becomes $x^2 - y^2 - 6x + 4y + 5 = 0$ and $2xy - 4x - 6y + 12 = 0$. The unique point of intersection is (3, 2), so the repeated root is $z = 3 + 2i$. Here $\delta_1 = 0$, $\delta_2 = 0$ and the centre is in general position. This is shown in Figure 5.

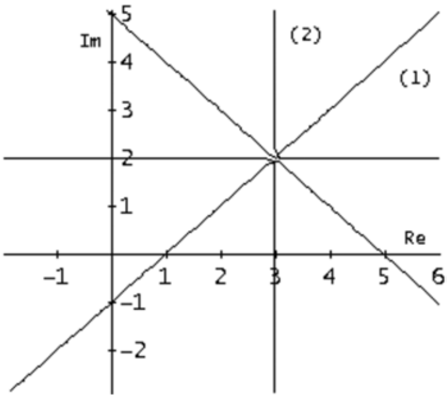



*Figure 5. Special case* $\delta_1 = 0$, $\delta_2 = 0$.

***Case 6***

$z^2 + 4z + 5 = 0$ becomes $x^2 - y^2 + 4x + 5 = 0$ and $2xy + 4y = 0$. The points of intersection are (–2, 1) and (–2, –1), so the complex conjugate roots are $z = -2 \pm i$. Here $\delta_1 > 0$, $\delta_2 = 0$ and the centre lies on the $x$ axis. This is shown in Figure 6.

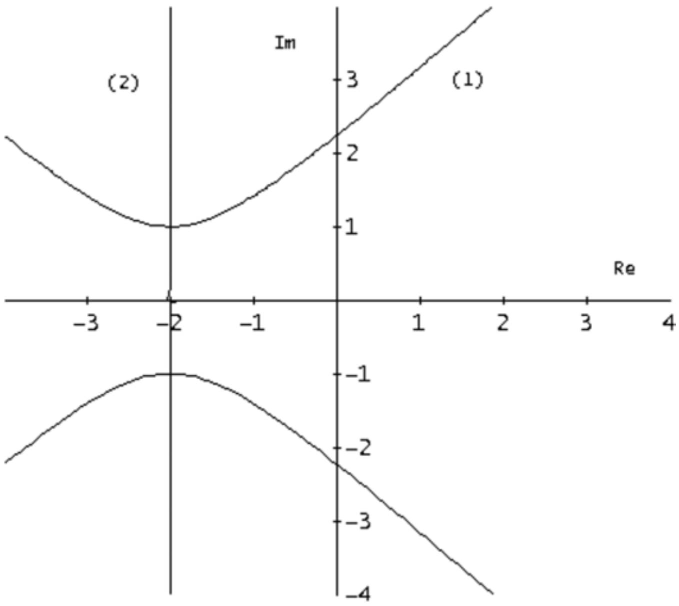



*Figure 6. Special case* $\delta_1 > 0$, $\delta_2 = 0$.

## Summary

The quadratic equation with complex coefficients can readily be solved by considering the intersection of two hyperbolas in the Cartesian plane. The solutions lie in the complex plane. There is a surprising amount of variety in the solutions.

### References

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