Pythagoras triples explained via central squares

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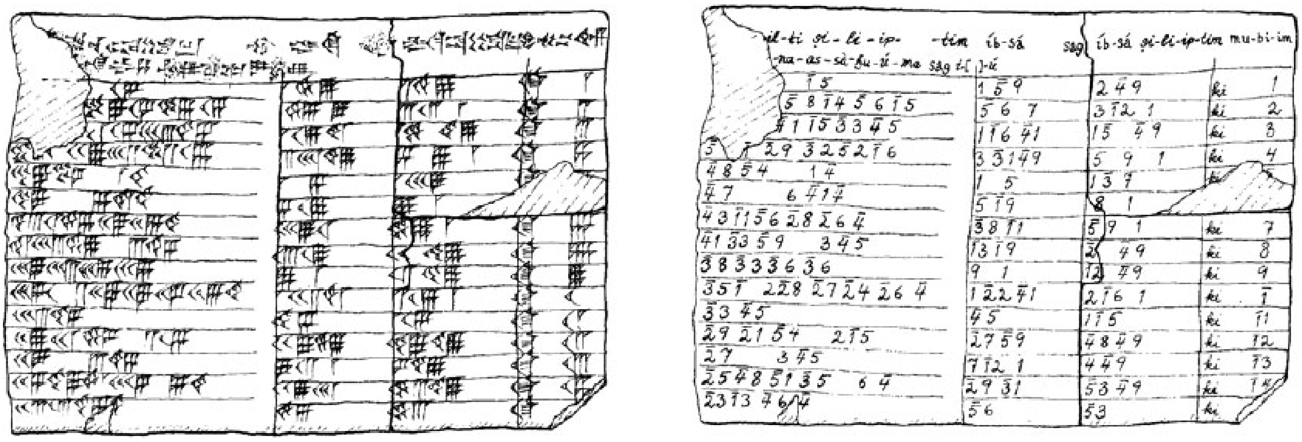
Very much like today, the Old Babylonians (20th to 16th centuries BC) had the need to understand and use what is now called the Pythagoras' theorem $x^2 + y^2 = z^2$. They applied it in very practical problems such as to determine how the height of a cane leaning against a wall changes with its inclination. This sounds trivial, but it was one of the most important problems studied at the time. A remarkable Old Babylonian clay tablet, commonly referred to as Plimpton 322 (Figure 1), was found to store combinations of three positive integers x, y, z that satisfy Pythagoras' theorem. Today we call them primitive Pythagorean triples where the term primitive implies that the side lengths share no common divisor.



(a) (b)

Figure 1. Old Babylonian clay tablet (known as Plimpton 322) stores combinations of primitive Pythagorean triples: (a) drawing of original and (b) translated (Friberg, 1981).

Why was the tablet built? Unlike what one may imagine, the reason was not an interest in the number-theoretical question, but rather the need to find data for a 'solvable' mathematical problem. It is even believed that this tablet was a 'teacher's aid' for setting up and solving problems involving right triangles (Friberg, 1981). This sounds like an environment not so different from our classrooms today. As human beings we share the same nature as the Old Babylonians to solve problems to live and evolve. The problems nowadays are more exotic and elaborated than a cane against a wall, but they share the same legacy. Right-angles are everywhere, whether it be a building, a table, a graph with axes, or the atomic structure of a crystal. While these are our contemporary challenges, we, like the Babylonians, strive to deepen our understanding of the Pythagoras' theorem, and on the various triples that generate these useful right-angles for our everyday practical applications.

Mack and Czernezkyj (2010) and Bernhart and Price (2012) have given an account on how to geometrically describe primitive Pythagorean triples using *equicircles*, that is, circles that are tangent to all three sides of a triangle. The interrelation between the triples (3, 4, 5) and (5, 12, 13) using equicircles is shown in Figure 2, and is then briefly described.

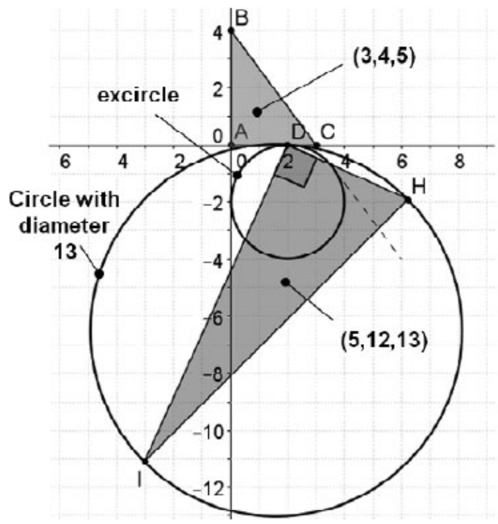


Figure 2. Geometrical interpretation of triples using the equicircles approach (Mack & Czernezkyj, 2010).

Begin with the triangle representing the first triple (3, 4, 5) located on the Cartesian plane, first quadrant, with the right angle at the origin and shorter side 3 on the *x*-axis as shown. A circle is than drawn: (1) tangent to the shorter side of the triple in line with *x*-axis; (2) tangent to the longer side in line with *y*-axis; and (3) tangent to the extension of the hypotenuse shown as a dashed line. The tangency to all sides of the triangle makes this circle an equicircle. Since this equicircle lays outside the triangle, it is named an escribed circle, or excircle.

Point D is where the excircle touches the shorter side of the triangle. A second larger circle is then drawn, passing through this same point D, and tangent to the *x*-axis. Its diameter is the length of the expected hypotenuse 13 of the second triple (5, 12, 13). The shorter side 5 of the triangle is formed by drawing a straight line from the tangent point D to the point H along the circle with diameter 13 such that its length becomes 5. The hypotenuse is the diameter 13, constructed from point H to point I.

Connecting point D and point I completes the triangle. Thus, the right-angle triangle defined by the triple (5, 12, 13) is formed. Other Pythagorean triples are generated by changing the diameter of the tangent circle, and applying the same procedure. An interactive tool to construct more triples using such approach is available online (see Boot, n.d.).

This paper now presents an alternative method that uses squares rather than circles to geometrically describe the Pythagorean triples, and how they are interconnected (Figure 3). Pythagoras is included in secondary education around the world including in Australian Curriculum (ACARA, n.d.), and hence this paper will be of interest for all. It was discovered by the author of this paper that each triple revolves around a unique central square—the cornerstone that allowed this geometrical interpretation of Pythagoras' and Plato's families of triples. As with the equicircle method, the central square method offers a new visualisation tool to interpret studies involving Pythagorean triples.

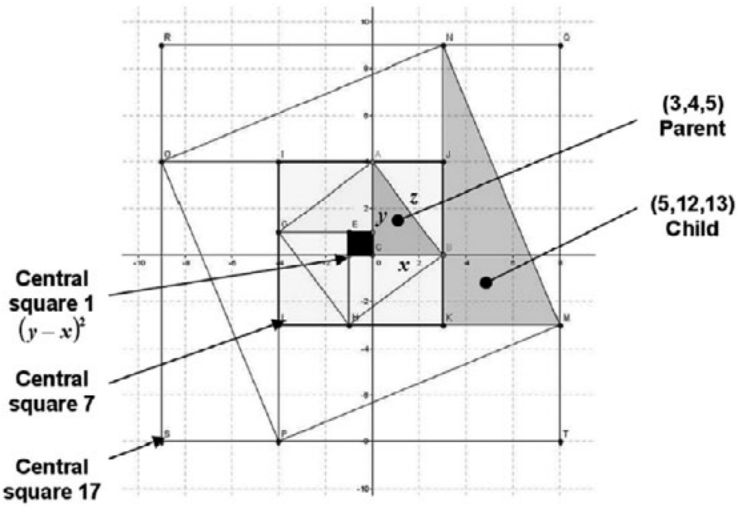


Figure 3. Geometrical interpretation of triples using central square approach.

Central square theory

The *central square theory* states that the right side of the equation z^2 is composed geometrically of four congruent right-angled triangles rotated around a central square $(y - x)^2$, which in turn when enclosed form a new square about which other Pythagorean triples revolve. Figure 3 shows how the central square theory interconnects parent–child triples. The main hypothesis assumed is that all triangles of triples relate to each other via intermediate squares.

This process is now briefly explained. Imagine the parent triangles revolve around a specific square. In this case the parent triple is the first triple (3, 4, 5), and the right-angled triangles revolve around a central square of side length 1 (see Figure 3). When enclosed, they form a new square side length 7 (=

3 + 4), about which triangles of a child triple revolves, that is, in this example (5, 12, 13). The central square theory implies that the process is iterative, and the new enclosing central square of side length 17 (= 5 + 12) is the foundation about which triangle-rectangles of a subsequent child triple will revolve.

Pythagoras' family

According to Euclid (300 BC), Pythagoras defined the following sequence of odd triples

$$(3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41)\ldots$$

where one side increases in steps of 2 between triples, that is, 3, 5, 7, 9. This incremental pattern when applied iteratively in conjunction with the central square theory generates the geometrical representation of the Pythagoras family. This is now explained step-by-step.

Start with the first triangle $(x_1, y_1, z_1) = (3, 4, 5)$ rotating around the unit side square (as shown in Figure 4a). The parameter n determines the position of the triple within the family, where for example $n = 1$ corresponds to the first triple $(x_1, y_1, z_1) = (3, 4, 5)$. Enclose it within a new square (Figure 4b) and extend each side radially by $x_2 = 5$ (Figure 4c). Unite the extensions to get the next triangle in the sequence of odd triples $(x_2, y_2, z_2) = (5, 12, 13)$ (Figure 4d).

The process is applied again. Enclose Figure 4d within a new square (Figure 4e), extend each side by $x_3 = 7$ (Figure 4f), and uniting the ends generates the next triple $(x_3, y_3, z_3) = (7, 24, 25)$ (Figure 4g).

The process is applied again. Enclose Figure 4g, extend sides by $x_4 = 9$ (Figure 4h) and uniting them gives $(x_4, y_4, z_4) = (9, 40, 41)$ (Figure 4i). The process continues for $n = 5, 6\ldots$ Note that the pattern of central squares evolves depending only on the increasing values of x. The sides y and z appear automatically as a consequence.

Plato's family

Remarkably, the geometrical representation of Plato's family of triples is constructed in the same manner. An important difference between the Pythagoras' and Plato's families of triples is that the first has odd sides, (i.e., 3, 5, 7, 9…) while the second has even sides, (i.e., 4, 8, 12, 16…). According to Euclid (300 BC), Plato defined the following sequence of even triples

$$(3, 4, 5), (8, 15, 17), (12, 35, 37), (16, 63, 65)$$

where one side (beginning with 4) increases in steps of 4 between triples, i.e., 4, 8, 12, 16… As before, this increment associated with central square theory gives the geometrical representation of Plato's family, here shown in Figure 5. Note that only by changing the circled terms in Figure 4i from 3, 5, 7, 9 to 4, 8, 12, 16, the geometrical pattern transforms the Pythagoras' triples into Platos' triples.

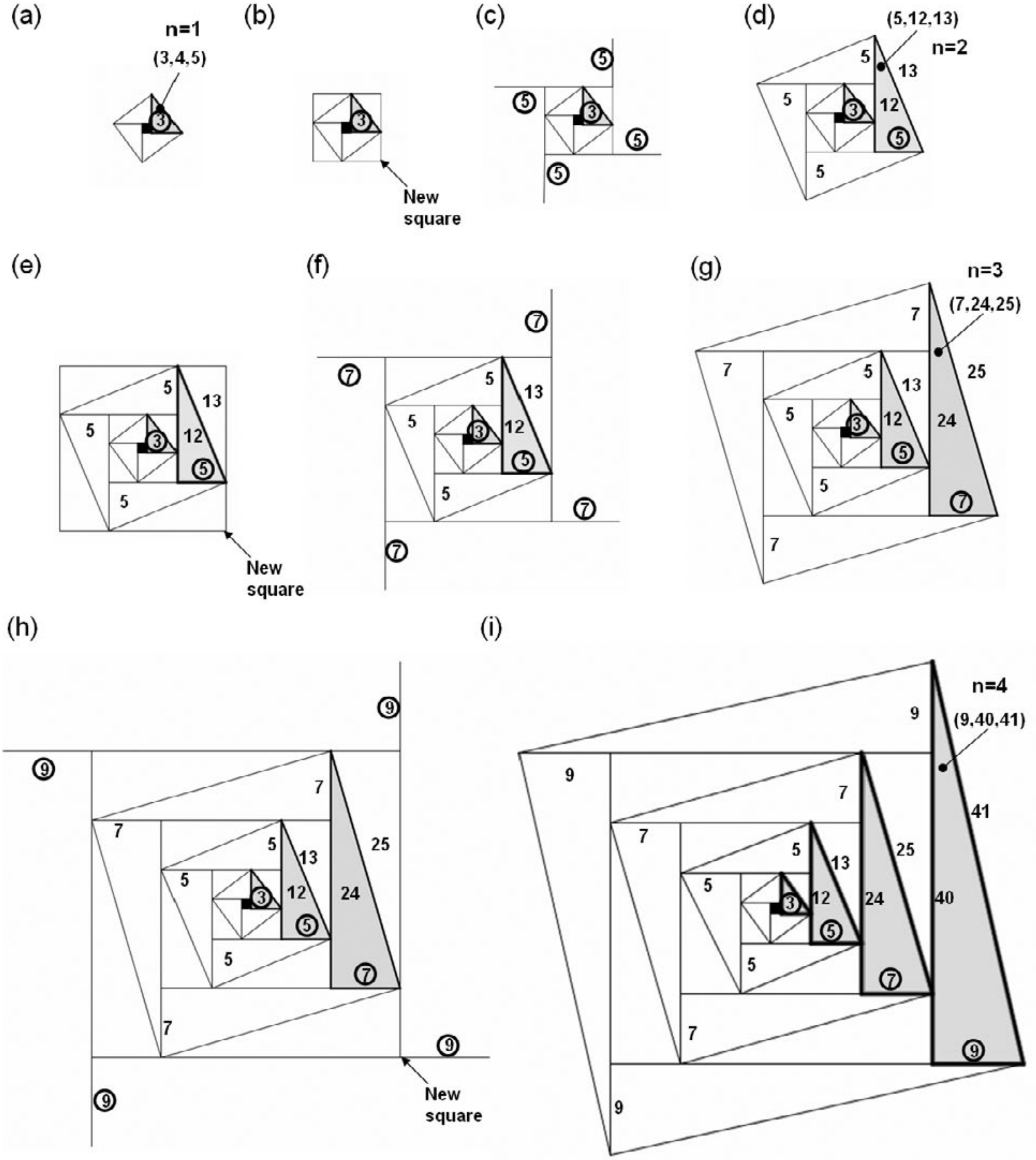


Figure 4. Step-by-step explanation on how to interconnect the triples in the Pythagoras' family using the central square approach.

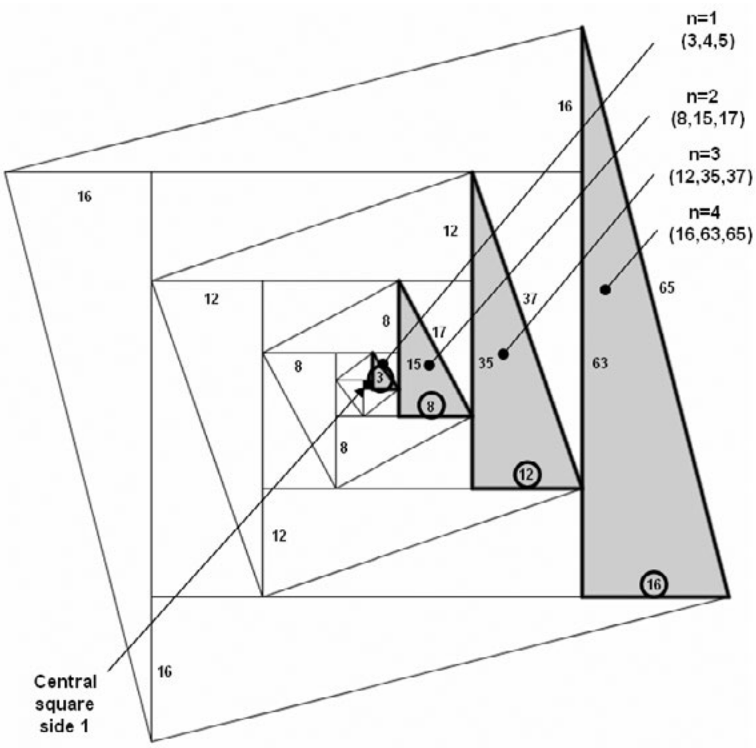


Figure 5. The geometrical representation of Plato's family via the central square approach.

Families of squares

The central square theory implies that each family of triples/triangles has an underlying family of squares interconnecting them. Figure 6a highlights the squares of the Pythagoras' family while Figure 6b those of Plato's family. Within this context, the description of the families expands to a sequence of triples (x, y, z) revolving around specific squares side(s) that connects them.

From the perspective of central square theory, the Pythagoras' family of triples is rewritten as

(Pythagoras)

(3, 4, 5)	(5, 12, 13)	(7, 24, 25)	(9, 40, 41)
[1]	[2]	[3]	[4]

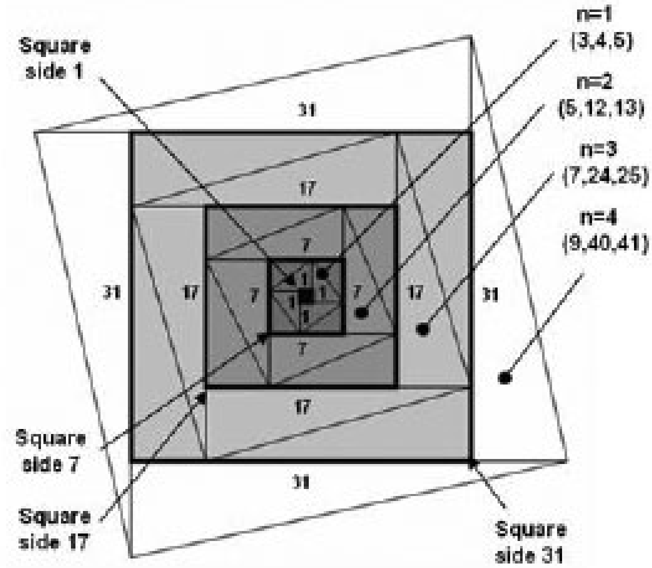
where [1], [7], [17], [31] represent the side squares about which the triples revolve.

Below is described in steps this method by which triples generate new squares that lead to new triples and subsequently new squares (Figure 7).

Step A: Start with the first triple (**3**, **4**, 5) revolving around a unit square [**1**]. When enclosed it gives a new square of side length 3 + 4 = [**7**]

Step B: The sides of the squares fan out by the next increment (3) + 2 = (**5**) —the shorter side of the new triple. The perpendicular side of the

(a)



(b)

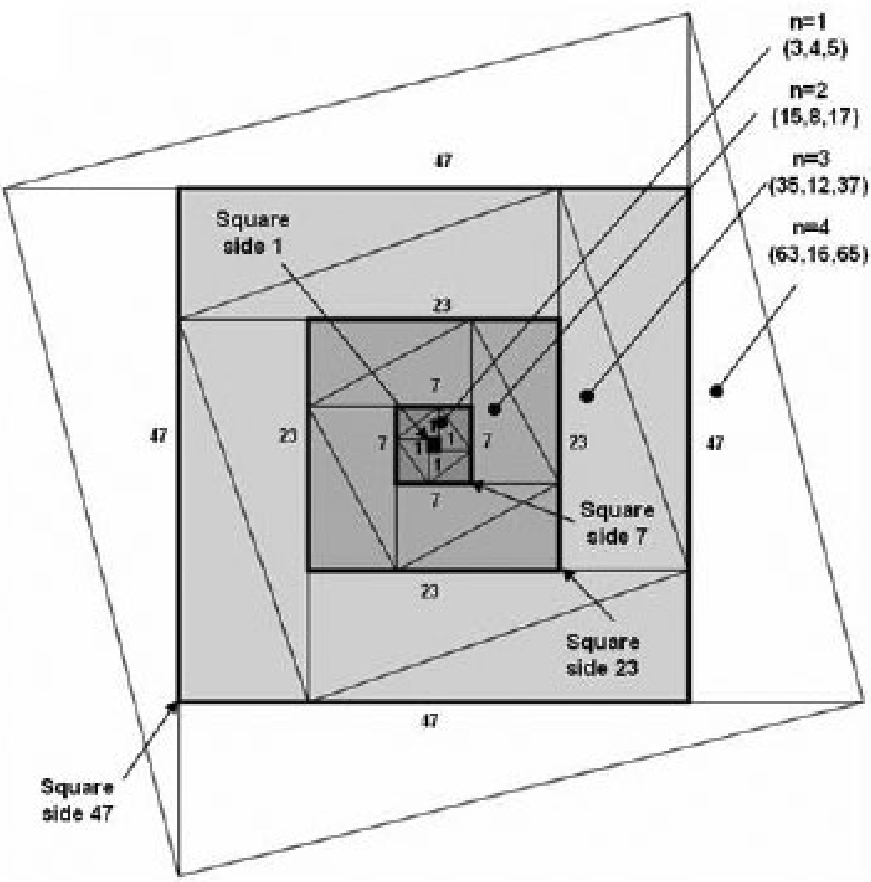


Figure 6. The squares of (a) Pythagoras' family and (b) Plato's family of triples.

new triple is the side length of the new square plus its shorter side [7] + (5) = (**12**).

Step C: Connecting the sides gives the next triple (**5**, **12**, 13). Enclosed gives the square of side (5) + (12) = [**17**] about which the following triple revolves.

Step D: The shorter side of the subsequent triple is found as before, that is, an incrementation of 2, or (5) + 2 = (**7**). The perpendicular side of the new triple is the square side plus the shorter side [17] + (7) = (**24**).

Step E: Connecting the sides gives the next triple (7, 24, 25). Enclosed gives a new square of side (7) + (24) = [**31**].

Step F: A new increment gives a new shorter side (7) + 2 = (**9**). Adding this with the side length of the corresponding square gives the longer side [31] + (9) = (40).

The result is the triple (9, 40, 41). This is the algebraic explanation of the geometrical steps shown in Figure 4, whose squares are highlighted in Figure 6a.

(Pythagoras)

				Step
(**3**, **4**, 5)				
[1]	[**7**]			A
(3, 4, 5)	(**5**, **12**, 13)			
[1]	[**7**]			B
(3, 4, 5)	(**5**, **12**, 13)			
[1]	[7]	[**17**]		C
(3, 4, 5)	(5, 12, 13)	(**7**, **24**, 25)		
[1]	[7]	[**17**]		D
(3, 4, 5)	(5, 12, 13)	(**7**, **24**, 25)		
[1]	[7]	[17]	[**31**]	E
(3, 4, 5)	(5, 12, 13)	(7, 24, 25)	(**9**, **40**, 41)	
[1]	[7]	[17]	[**31**]	F

Figure 7. Pythagoras' family created using the central square method.

Similarly, Plato's family is rewritten from the perspective of central square theory as

(Plato)

(3, 4, 5)	(8, 15, 17)	(12, 35, 37)	(16, 63, 65)
[1]	[7]	[23]	[47]

It is interesting to note that both Pythagoras' triple (5, 12, 13) and Plato's triple (8, 15, 17) revolve around the same central square side length [7]. Comparing Figure 6a and Figure 6b shows that this is because both series share the same first triple (3, 4, 5). Beyond the square [7], each family evolves with its unique sequence of squares.

Conclusion

The triangles formed by the triples in Pythagoras' or Plato's families can be geometrically interconnected via intermediate central squares—this forms the basis of the central square theory. This pattern of parent–child triple relationship allowed the geometric construction of both sequences, which seem to behave in a similar manner. Governed solely by specific increments in the smaller side of the triple and the identified geometrical pattern, it has been shown that both sequences start with the first triple (3, 4, 5), revolving around a square of side length unit, and evolve outwards into infinity. From the perspective of central square theory, the Pythagoras' or Plato's families are expressed not only as a sequence of triples, but also by their connecting sequence of squares.

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