The modelling of reasoning and justification methods in the teaching of **fraction division** at Year 4 level in Vietnam



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Norton, Thao and Duy provide an interesting insight into the teaching of fraction division in Vietnam. The article highlights one of the many teaching strategies available to teachers for building fraction concepts.

Introduction

This paper is set against the reform agenda currently being implemented in Vietnam. Part of the rationale for this is the fact that in 2012 Vietnam joined the Programme for International Student Assessment (PISA). This international assessment of students was initially organised by the Organization for Economic Cooperation and Development (OECD) which assesses student knowledge at age 15 in specific areas. The domains assessed include: ability to apply knowledge and skills in specific areas; ability to analyse; reason; and capacity to explain and communicate. This has prompted a fresh look at Vietnamese educational processes. While the focus of this paper is on the teaching of reasoning and justification for the division algorithm to Vietnamese students, the topic area is relevant internationally and may add to teachers' repertoire of strategies to deepen students' understanding of fractions.

There are two strands to the paper that become interwoven. The first is the challenge of teaching higher-order thinking associated with problem solving and, in particular, the development of reasoning. The second strand involves the teaching of fractions, an area that students internationally find difficult. We use the content of fractions to model the development of a teaching approach coined reasoning and justification or RJ. RJ embraces many of the attributes of constructivism that holds that students ought to construct or create their own understandings in their minds as they attempt to solve meaningful problems (Davis, Maher & Noddings, 1990; James & Carpenter, 1992). Clearly all teachers want students to understand and internalise concepts, connecting new concepts to existing knowledge. But rather than encouraging students to invent their own methods, as suggested by some researchers (e.g., Gravemeijer, 2003; Heirdsfield & Cooper, 2004; Kamii & Dominick, 1997), the RJ approach has a clear intention of making explicit the process of reasoning and justification for the development of a particular algorithm. Thus, from a pedagogical perspective, the approach has more in common with those methods, recommended by Hattie (2009), of visible learning where teachers strive to make clear the underlying concepts to be learnt. Hattie (2009, p. 25) noted that most students learn most effectively when teachers behave as "deliberate change agents, and as directors of learning." Thus, we do not propose that students are to be left to their own devices to "discover" the reasoning for fraction algorithms; rather they will be carefully guided to make the connections between patterns and operations. Transparency of structure is considered important by cognitive load theorists (e.g., Cooper, 1990; Kirschner, Sweller & Clark, 2006) and those advocating direct instruction (e.g., Engelmann, 1992).

Literature review

Reform curriculums (e.g., Australian Curriculum, Assessment and Reporting Authority (ACARA), 2012) have content strands such as number and algebra; space and measurement; and probability and data. These set out what has to be taught in terms of the facts and procedures of mathematics. Usually there are also process or proficiency strands, and in the ACARA (2012) document there are four proficiency strands. The first is understanding or being able to make connections between related concepts, including appreciating the how and why of connected ideas of concepts that may be expressed in different ways. The second proficiency strand is fluency or being able to quickly and accurately carry out skills and procedures and recall facts. The third proficiency strand is problem solving, which refers to being able to make the right choices about which procedures to use after interpreting data and to formulate a plan to arrive at a meaningful solution. Finally there is reasoning. The National Council of Teachers of Mathematics (2011, p. 4) offers the following rationale for the study of reasoning and proof: "Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena." The teaching of reasoning has been associated with general academic success (e.g., Fowler & Watford, 2000). ACARA (2012, p. 5) defines *reasoning* as:

> Students develop an increasingly sophisticated capacity for logical thoughts and actions, proving, evaluating, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce or justify strategies used and conclusions reached, when they adapt the *known* from the *unknown*, when they transfer learning from one context to another, when they prove that something is true or false and when they compare and contrast related ideas and explain their choices. [emphasis not in original]

We can see in this description that the ACARA curriculum authors are intent on encouraging students to think logically and from what is known to the unknown. Students who engage in such processes can develop a sense of proof. Proof in mathematics has a unique certainty; it is absolute. Establishing proof is the process of establishing a certainty based upon other certainties.

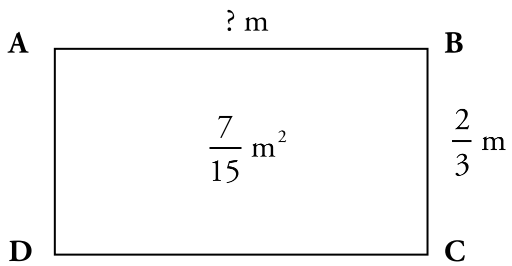
Fractions are a worthy topic of investigation for the teaching of reasoning, in part because they are important for further study and also because many students do not understand the logic and reasoning behind fraction operations. The US Department of Education (2008, p. xix) noted: "Difficulty with fractions (including decimals and percent) is persuasive and is a major obstacle to further progress in mathematics, including algebra." Much has been written about the difficulties students experience with fraction concepts (e.g., Brown & Quinn, 2007; Jigyel & Afamasaga-Fuata'i, 2007; Lamon, 1999). The research data indicates that a significant proportion of students find almost every aspect of fraction conceptualisation difficult. Brown and Quinn (2006; 2007) found that most Year 9 students had fragmented knowledge of fractions and chose to apply algorithms with little understanding of the meaning behind them. Many of the errors indicated that students misapplied short cuts. Students also applied the wrong algorithm for the context. Part of the reason for such failure was that students did not understand the reasoning underpinning the algorithms, and Booker, Bond, Sparrow and Swan (2010) suggested that formal algorithms be introduced after students had gained an understanding of the reasoning behind them.

Misunderstanding of fractions is not confined to school children. Indeed Norton and Nesbit (2011) and Norton (2010) found that a relatively high proportion of prospective primary school teachers had limited understanding of fractions. In an unpublished survey of primary school teachers in Haiphong Vietnam (n = 40) the second author found that few primary school teachers were able to construct an explanation for the algorithm for division of common fractions. Van de Walle (2007) noted that the "invert and multiply algorithm was one of the most poorly understood procedures in the K–8 curriculum" (p. 329). With this background in mind we set out a set of teaching steps for the teaching of division of fractions.

Using the RJ method for the teaching of the fraction division algorithm

In the sections below we set out how fraction division is usually taught in Vietnamese government schools and describe an alternative sequence designed to develop reasoning and understanding. The question below is given as an example of the type of question requiring fraction division and is typical of the type of activity set in Year 4 primary text books (e.g., Do Dình Hoan, 2004, p. 135). This text is the dominant guide to primary curriculum in Vietnam.

a) Example: The rectangle ABCD is given with the area and one side given.



The rectangle has an area $\frac{7}{15}$ m^2.
If the width is $\frac{2}{3}$ m what is the other length?

In order to find the length we use the division $\frac{7}{15} \div \frac{2}{3}$. (In Vietnam this would be expressed as $\frac{7}{15} : \frac{2}{3}$. Full colon is the symbol for division or ÷).

b) The algorithm: *Take the first fraction, multiply with the inverse of second fraction.*

The fraction $\frac{3}{2}$ is called the inverse of the fraction $\frac{2}{3}$, because it is inverted or turned upside down.

We have: $\frac{7}{15} \div \frac{2}{3} = \frac{7}{15} \times \frac{3}{2} = \frac{21}{30}$

The text book does not give any explanation as to how the algorithm is derived or why this procedure of multiplying by the reciprocal is necessary. The standard Vietnamese teaching sequence for teaching algorithms for fraction addition, subtraction and multiplication is as follows (Vu Quoc Chung, 2007, p. 168).

1. Put the problem in the real life context.
2. Use visual models to find a result.
3. Discuss students' attempt to derive the algorithm from the visual models.
4. The teacher confirms the algorithm.

Our RJ model adapts this process as follows:

1. Start with a problem structure that the students understand.
2. Build upon existing structures or knowledge with a varied structure.
3. Rationalise and generalise to form an algorithm.
4. Confirm that the algorithm works against known solutions.
5. Practise the algorithm in a variety of contextual settings.

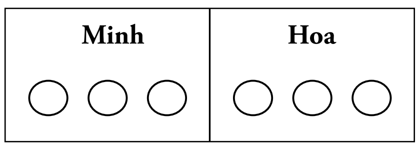
In mathematics education the sequence is critical. The prerequisite for the following activities is a thorough understanding of fraction naming, and renaming of common fractions, mixed numbers, as well as fraction multiplication. The teaching sequence below is based on the model for teaching fraction division presented by Norton (2011).

Step 1

Start with a problem with a structure that the students understand (in this case, whole-number division).

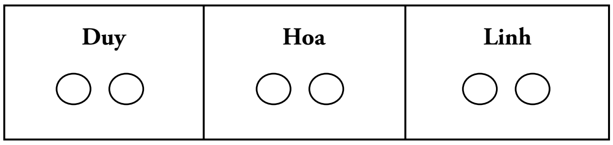
Problem 1

There are 6 marbles to be shared among 2 students (Minh and Hoa). How many marbles does each student get? This problem can be modelled with materials.



Answer: Each student receives 3 marbles. Symbolically this is expressed as:
6:2 (6 ÷ 2 = 3)

Problem 2



Answer: There are 3 students who receive a full share. Symbolically this is expressed as 6:2 ($6 \div 2 = 3$). Note that the problem structure has changed from sharing objects among students to stating the share and asking the number of students who receive a share. Students ought to understand whole-number division and follow the logic of this shift in context.

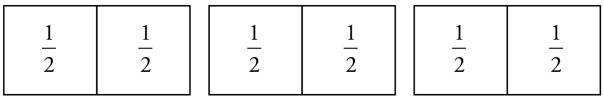
Step 2

Build upon existing structures or knowledge with a varied structure.

Problem 3

There are 3 bars of chocolate. If each student wants to have a half a bar of chocolate, how many students will receive a share? Materials or diagrams can be used to illustrate the solution.

Symbolically this is $3 \div \frac{1}{2}$ (3 divided by $\frac{1}{2}$)



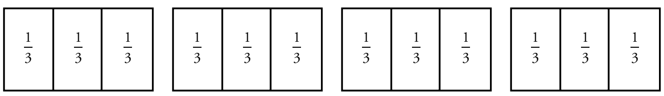
Answer: There are 6 students who receive a share of the chocolate.

Thus the symbolic summary is: $3 \div \frac{1}{2} = 6$. (3 divided by $\frac{1}{2} = 6$)

Problem 4

There are 4 chocolates. If each student wants $\frac{1}{3}$ of a piece of chocolate, how many students will receive a share?

Symbolically this is expressed as $4 \div \frac{1}{3}$. Again materials or models can be used to find the solution. Each of the four chocolate bars is divided into thirds and 12 can be counted.

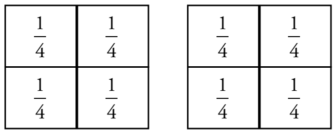


Answer: There are 12 students who receive a share of the chocolate. Thus the symbolic summary is: $4 \div \frac{1}{3} = 12$ (4 divided by $\frac{1}{3} = 12$).

Problem 5

There are 2 chocolates. If each student wants $\frac{1}{4}$ of a piece of chocolate, how many students will receive a share?

Symbolically this is expressed as $2 \div \frac{1}{4}$.



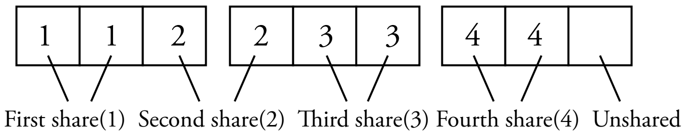
Answer: There are 8 students who each receive of a piece of chocolate.

Thus the symbolic summary is: $2 \div \frac{1}{4} = 8$ (2 divided by $\frac{1}{4} = 8$).

Problem 6

There are 3 chocolates. If each student wants $\frac{2}{3}$ of a piece of chocolate, how many students will receive a share? How much of a share of the chocolate will remain?

Symbolically this is expressed as $3 \div \frac{2}{3}$.



Answer: There are 4 students who receive a full share, and $\frac{1}{3}$ of a bar of chocolate remains. Since a full share is $\frac{2}{3}$ of a bar, the remaining unshared $\frac{1}{3}$ of the chocolate bar represents $\frac{1}{2}$ of a full share.

Symbolically this is expressed as: $3 \div \frac{2}{3} = 4$. This is a critical point and is counter intuitive since many students will consider that since $\frac{1}{3}$ of the chocolate bar remains unshared the solution should be $4\frac{1}{3}$.

The reasoning as to why the solution is not $4\frac{1}{3}$ but rather $4\frac{1}{2}$ needs to be discussed and understood. Using material models will help students to appreciate the solution.

Problem 7

There are 4 chocolates. If each student wants $\frac{3}{4}$ of a piece of chocolate, how many students will receive a share? How many chocolate pieces remain?

Symbolically this is expressed $4 \div \frac{3}{4}$. The shares, each of $\frac{3}{4}$, are numbered.

1	1	2	3	3	4	5	5
1	2	2	3	4	4	5	

Answer: There are 5 students who receive a full share of $\frac{3}{4}$. There remains $\frac{1}{4}$ of the chocolate bar unshared (or $\frac{1}{3}$ of a full share of the ration to be shared).

So, the symbolic summary of this problem is $4 \div \frac{3}{4} = 5\frac{1}{3}$.

Problem 8

There are $2\frac{1}{2}$ chocolates. If each student wants $\frac{3}{4}$ of a piece of chocolate, how many students will get a full share? How much of a share will remain?

Symbolically this is expressed $2\frac{1}{2} \div \frac{3}{4}$.

First bar	**Second bar**	**Half a bar**

Divide each chocolate into 4 parts or quarters. Each student receives 3 parts or three quarters of a chocolate bar. The number of full shares is modelled below. Person 1 receives three quarters and so on.

1	1	1	2	2	2	3	3	3	remains

Answer: There are 3 full shares and $\frac{1}{10}$ of chocolate bar remains (or $\frac{1}{3}$ of a full share)

In summary: $2\frac{1}{2} \div \frac{3}{4} = 3\frac{1}{3}$.

Step 3

Rationalise and generalise to form an algorithm. The next step is to have the students deduct from the patterned information. The symbolic summaries of the problems 3, 4 and 5 above are listed and students are asked to find a pattern or short cut. This is relatively simple for problems 3, 4 and 5; however, some students will benefit from being encouraged to set the finding out in a table such as that below. The degree of scaffolding provided in assisting the students to make meaning from the table depends on the teacher disposition and the students' abilities and problem-solving disposition. Teachers working with capable students and favouring an inquiry approach will likely use more questioning and less telling.

Problem	Solution from models	Fraction algorithm applied
$3 \div \frac{1}{2}$	6	$\frac{3}{1} \times \frac{2}{1} = \frac{6}{1} = 6$
$4 \div \frac{1}{3}$	12	$\frac{4}{1} \times \frac{3}{1} = \frac{12}{1} = 12$
$2 \div \frac{1}{4}$	8	$\frac{2}{1} \times \frac{4}{1} = \frac{8}{1} = 8$

Some students might guess that the following pattern has emerged from the data.

Answer: To divide a whole number by a fraction, we take the dividend and multiply it by the inverse of the divisor. Can such a generalisation be applied where there are remainders (e.g., in problems 6, 7 and 8)?

Again students are encouraged to set out their findings in a table to look for patterns.

Problem	Solution from models	Fraction algorithm applied
$3 \div \frac{2}{3}$	$4\frac{1}{2}$	$\frac{3}{1} \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$
$4 \div \frac{3}{4}$	$5\frac{1}{3}$	$\frac{4}{1} \times \frac{4}{3} = \frac{16}{3} = 5\frac{1}{3}$
$2\frac{1}{2} \div \frac{3}{4}$	$3\frac{1}{3}$	$\frac{5}{2} \times \frac{4}{3} = \frac{20}{6} = 3\frac{1}{3}$

Note that in the case of mixed numbers, these numbers need to be converted to improper fractions. Students are encouraged to develop their own summary of the 'rule' or algorithm developed, and a class discussion can ensue such that a definition that approximates the formal algorithm is achieved.

> **Answer:** To divide a fraction by a fraction, we take the dividend multiplied by the inverse of the divisor.

Step 4

Confirm that the algorithm works against known solutions. In effect this was achieved with the use of material or diagrammatic models.

Step 5

Practice the algorithm in a variety of contextual settings. Use the developed algorithm to find solutions to the following, for example:

1. Find the side length of a rectangle with area $\frac{5}{2}\text{m}^2$ if the width is $\frac{1}{4}\text{m}$ long.
2. 3 kilograms of rice was to be divided into rations of $\frac{3}{4}$ kg each. How many rations could be obtained, and what part of a ration remains?
3. There were $4\frac{3}{4}$ cakes to be shared. If each share was to be $\frac{2}{3}$ of a cake, how many complete shares could be obtained, and how much of a share remains?

Students can be encouraged to check their calculations with models to verify that the rule they developed was valid for these applications. Teachers are likely to find that although these problems can be modelled with contexts and materials, once students have confidence that the algorithm has been proved they will tend to use the most efficient methods available to them.

Discussion and conclusion

In the teaching sequence above, reasoning consistent with NCTM (2011) and ACARA (2012) recommendations was modelled. The teaching method termed RJ starts with what is known (whole-number division) and adapts materials or visual models in contextual settings in a logical sequence to find something that is unknown (fraction divisions). The steps were set out in a sequence as follows:

1. Start with a problem structure that the students understand.
2. Build upon existing structures or knowledge with a varied structure.
3. Rationalise and generalise to form an algorithm.
4. Confirm that the algorithm works against known solutions.
5. Practice the algorithm in a variety of contextual settings.

These steps are essentially inductive in that the conclusion or generalisation was reached from specific examples. The questioning sequence is designed to help students to construct or create understanding as they attempt to solve meaningful problems, and in this way is consistent with early constructivist pedagogy (Davis, Maher & Noddings, 1990). However, the method is highly teacher-centred in that the teacher explicitly leads the students on a logical journey to construct the algorithm for fraction division. This aspect of the approach reflects the principles recommended by Hattie (2009) in regard to visible learning with an emphasis on transparency of structure, features of teaching considered important by cognitive load theorists (e.g., Cooper, 1990; Kirschner, Sweller & Clark, 2006) and those advocating direct instruction (e.g., Engelmann, 1992).

Acknowledgement

It is rare in mathematics education to find a completely novel approach. The authors are aware that the use of models to prove the fraction division algorithm has history. We, like most learners, build on the insights of those who have gone before us.

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