A Canadian effort to address fractions teaching and learning challenges


Shelley Yearley
TLDSB – on assignment with Ontario Ministry of Education, Canada
shelley.yearley@tldsb.on.ca

Catherine D. Bruce
Trent University, Canada
cathybruce@trentu.ca

Teaching and learning fraction concepts provides challenges in primary schools all over the world. In this article, Shelley Yearley and Catherine Bruce describe a fractions-based research project conducted in Ontario, Canada.

Introduction

Fractions has proven to be an area of significant and consistent difficulty for both teaching and learning (see for example Gould, Outhred & Mitchelmore, 2006; Lesh et al., 1992; Moss & Case, 1999). The American National Mathematics Advisory Report (2008) states that "a major goal for K–8 mathematics education should be proficiency with fractions (including decimals, percents, and negative fractions), for such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). For example, Kamii and Clark (1995) found that only one third of 13-year-old students could correctly place a simple fraction on a number line, despite this being a learning objective for 11-year-olds.

To further investigate fractions teaching and learning in Canada, research partners from Trent University and the Ontario Ministry of Education embarked on a five-year research project with teachers in schools. This paper describes the initial stage and outcomes of the project which has also resulted in the development and launch of a digital research paper (www.edugains.ca/mathematics), video studies (www.tmerc.ca) and field-tested teacher resources.

Background

In the Ontario Mathematics Curriculum, by age 12, students are expected to be able to represent, compare, and order fractional amounts with unlike denominators using a variety of tools as well as understand the relationship among fractions, decimal numbers, and percents (Ontario Ministry of Education, 2005). However, annual large-scale testing for 8- and 11-year-olds in this same province by the Education Quality and Accountability Office has continually identified that fractions is an area of student weakness. Further, a Canadian college mathematics achievement project identified proficiency with fractions as an ongoing area of need for post-secondary students. In order to study learning challenges with fractions in more depth, our research team acquired funding and ethical approvals to engage in collaborative action research in three Ontario school districts beginning in 2011.

The teacher–researcher teams within each district were comprised of five to seven teachers working with children between the ages of 9 and 12 years. Each team was released for a total of five release days over a four-month period to meet together face-to-face for discussion and learning facilitated by university researchers and a provincial math consultant. Each of the three teams engaged in two parallel activities. The first was content learning about fractions. Some of the teacher participants expressed surprise about

the fact that the project was focused on fractions as typically little time was allocated to fractions instruction with the school year. Several of the teachers lacked comfort with this content area, resulting in fractions being left to the end of the school year.

The teacher–researcher teams engaged in tasks which were designed to build teacher content knowledge in the area of fractions. Additionally, as teachers engaged in unpacking student thinking, they increased their understanding of fractions. The second area of focus was to identify the problems students were facing with fractions learning and to develop a program of lesson interventions. The teacher–researcher teams co-planned and co-taught exploratory lessons grounded in current research and aligned with baseline student achievement data of precise fractions content understanding. The teams identified the concepts of representing, comparing, and ordering fractions to be fundamental problem areas for students.

The exploratory lessons were organised into bundles that attempted to coherently tackle a particular area of need. For example, one lesson in the representation bundle asked students to represent two different distances that two students had walked between home and school, and to determine which of the two students was closer to school (comparing the fractional distances from home to school). During this lesson, as with all other lessons, the teacher–researcher team observed students closely and documented their thinking through anecdotal notes, photos, and video. Individual students were selected and interviewed at the end of lessons to gather greater insights and data on what aspects of the learning tasks were helpful in building conceptual understanding and which activities reinforced existing misconceptions. As the teachers debriefed and developed a richer understanding of the fractions content, they gained more insight into what their students knew and could do. This in turn supported teachers in the refinement of instructional resources and pedagogical practices to maximise student learning. Three key learnings for the teacher–researcher teams were:

1. Sustained improvement in understanding fractions occurred most significantly when learning opportunities shifted from a more traditional unit of study approach to learning opportunities punctuated throughout the school year and based upon co-planned lessons.
2. An emphasis on addressing misconceptions and common difficulties with fractions equipped students for subsequent mathematics learning beyond fractions.
3. A sustained professional learning focus on fractions enabled a deep exploration of instructional and pedagogical decisions to support student learning.

Each of these outcomes will be discussed further.

Teacher professional learning outcomes

Punctuated instruction throughout the school year

As teachers became more aware of the depth and breadth of learning represented in a single learning outcome, lesson planning shifted from a focus of trying to get the students to demonstrate a specific skill towards providing opportunities to build from their current understanding to the desired learning. Teachers focused on creating learning opportunities which exposed and addressed a broader range of fraction representations and meanings. For example, there was a shift from a more traditional approach of asking students to represent a given fraction one way to asking students to represent the given fraction in as many ways as possible. One illustrative example asked students to select either $\frac{2}{5}$ or $\frac{4}{10}$ and represent it in a variety of ways. This small shift allowed students to demonstrate what they understood about fractions and surfaced misconceptions, providing teachers with more precise information for next steps in instruction.

The richness of student responses is reflected in the student sample (see Figure 1).

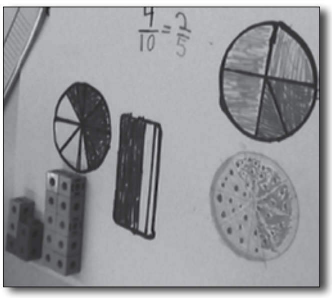


Figure 1. Student representations of four-tenths and two-fifths.

Note that these students have represented fractions as part–whole with continuous models (circles and rectangles) as well as with discrete, or set, models (linking cubes). The circle model responses demonstrate the consistent difficulty observed with partitioning of circle models. Students focussed on the number of partitions created and shaded rather than on the relationship between the shaded area and the whole. This reinforced the misunderstanding of a fraction actually being about two separate numbers rather than the relationship between the two numbers.

Additionally, although the students correctly stated at the top of the page that $\frac{4}{10} = \frac{2}{5}$ none of their representations demonstrate this equivalency. The lack of attention to a common whole (same shape and size) as well as the failure to partition equally prohibits the students from modeling the equivalency with pictures. The students appear to possess a procedural understanding which is conflicted by the pictorial representations, perhaps a reflection of a privileging of symbolic manipulation. The teachers were intrigued by the diversity and range of student understanding which could be ascertained through examination of the responses to this simple prompt.

For an online video study related to this lesson, please visit www.tmerc.ca/video-studies/.

Students used multiple meanings (e.g., part–whole, part–part, quotient) and representations (e.g., set, area) simultaneously as they shared their understandings. This meant that a more traditional introduction of meanings in isolation would not honour what the students knew and could do. Teachers adjusted their thinking to include broad learning outcomes over a period of two or three lessons and multiple opportunities to achieve the outcomes. In order to meet the timeline demands of the research project, teachers were inserting the fractions lessons into their pre-established program plans for mathematics. The lessons occurred in three day sequences, with a new sequence being delivered every two or three weeks. It became quickly evident that although students did not all progress through the learning in the same time, this punctuated instruction combined with the implementation of co-planned lessons increased their overall achievement by the end of the unit. Although representing only a small sample, this graphic (see Figure 2) shows the promising results as measured by pre and post-assessment scores (as percent) at one site.

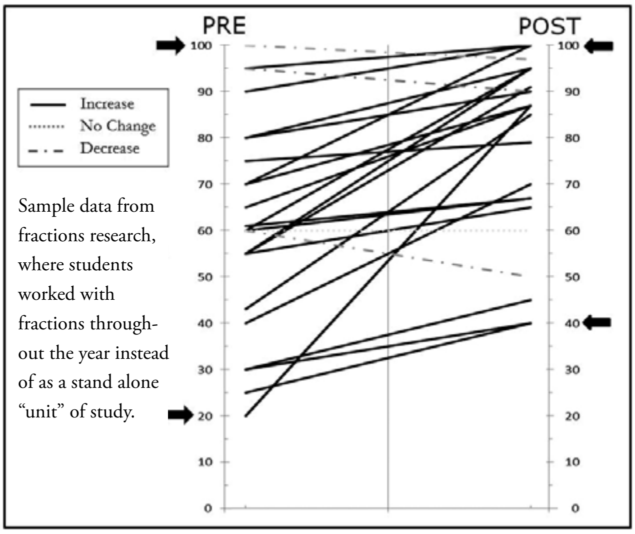


Figure 2. Student pre- post-assessment result.

Addressing misconceptions and common difficulties

Throughout the project, teacher content knowledge was deepened through examination of student thinking from the co-developed lessons as well as through precise opportunities to reflect upon co-observed lessons and to collaboratively plan subsequent learning tasks for the students. To demonstrate this teacher learning, we will further discuss the $\frac{2}{5}$ and $\frac{4}{10}$ lesson, which was implemented in five Grade 4 through 7 classrooms. After analysing the student responses, the teachers identified a number of common misconceptions and challenges (Table 1).

Teachers focused their efforts on common misunderstandings to identify appropriate instructional strategies that would support students in refining their understanding (table 2). These lesson interventions were informed by current research and affirmed through implementation in the teachers' classrooms.

It is important to note that teachers on all three teams identified a tension between wanting to immediately resolve student misconceptions and allowing misconceptions to be resolved by the student through subsequent learning. Teachers began to recognise that allowing students to resolve these misconceptions resulted in robust and flexible understanding.

Table 1. Teacher identified student challenges and misconceptions.

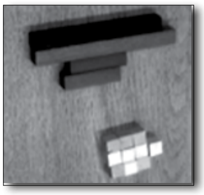
Challenges with content	Some common misconceptions about fractions
• Use of imprecise fraction language conflates student understanding of fractions (e.g., is "four tenths" but reading it as "four over ten" leads some students to understand it as "four tens", represented by four sets of ten, or alternately as literally "four over ten" represented as a ratio – see picture below). These multiple naming strategies appear to add to student difficulty in constructing a meaningful understanding of a fraction as a number.  Student representation of 'four over ten' • Over-reliance on circle representations for teaching and learning leads to limited success in representing fraction amounts that do not easily lend themselves to partitioning in a circle (such as $\frac{2}{5}$ and $\frac{4}{10}$; consider the difficulty of partitioning a circle into tenths or fifths compared to fourths or eighths). • Students' limited understanding of the meaning of fractions results in inappropriate strategies for comparing fractions (e.g., $\frac{2}{5}$ is equal to $\frac{1}{10}$ because 2 fives are 10 and 1 ten is 10). • Students who understand that a part-whole area model requires the partitioned areas to be of equal size, struggle with how much precision is required in representations. When comparing fractions, sometimes approximate drawings are sufficient, but other times exact drawings are required, depending upon the fractions' relative equivalency. • There appears to be a tendency in instruction to move quickly to symbolic notation of fractions and procedures related to fractions rather than to make informed choices about the best representation for sense-making.	• Size of the partitioned areas does not matter even when using an area model. • The numerator and denominator in a fraction are not related. • Fractions cannot represent 'parts-of-a-set' relationships. • All representations of fractions must always show the 'parts' as attached or touching, including set representations. • Equivalent fractions always involve doubling. • Number lines are a non-changing whole (where 1 is always the whole) as well as a flexible whole (where the entire length is the whole, regardless of the whole number labels which extend beyond 1). Some students used both definitions simultaneously when ordering fractions (absolute value of $\frac{1}{2}$ on a number line alongside a relative value of $\frac{1}{2}$ of the number line length).

Table 2. Teacher responses to common difficulties.

Common difficulties	Lesson implications
Fragile understanding of the meaning of, as well as the relationship between, the numerator and denominator.	Encourage students to select their own tools and representations in order to develop a sense of the whole as well as to consider the role of partitioning to explore the relationship between the numerator and denominator.
Limited procedural understanding for generating equivalent fractions.	Engage students in constructing equivalent fractions through their own reasoning using manipulatives rather than learning a single algorithm such as doubling both the numerator and denominator.
Conflation of characteristics of 'parts of a set' and area models (e.g., parts of a set must always be the same size).	Engage students in lessons that expose them to i) area models partitioned in non-congruent yet equal sized segments; ii) sets (collections of objects) of varied sizes.

Teachers also planned subsequent lessons to encourage the students to explore and resolve some complex questions raised in class. In one Grade 6 classroom where students were organizing fraction and decimal numbers on a number line for example, students engaged in a discussion where they debated about the equivalent fractions for the decimals 0.4 and 0.06. The teacher decided to begin the next day's lesson with the question "Is $0.4 = \frac{4}{100}$?" to identify and then clarify some of the incorrect conjectures students had previously made. In another classroom students were exploring the characteristics of discrete (set) and continuous (area) representations using pattern blocks. Students generated a wide range of fractions using both part-whole and part-part relationships. During the consolidation one student shared that she had generated the fraction of $\frac{1}{6}$ to represent her illustration of a single vertex circled on one hexagon, or 'one vertex of six is circled'. After detecting that the class was unsure whether this was an example of a discrete or continuous model, the teacher team decided that it would be a great starting point for the next lesson. The next day students were asked to share their thinking about the characteristics of the fraction represented. The teacher recorded student comments on the board. In addition to answering the specific question (that this was indeed a discreet/set model where the attribute in focus was 'circled angles'), this discussion allowed students to continue their thinking about fractions, gain clarification of the distinction between the models and meanings, and connect their fractions knowledge to other areas of study, such as geometry.

By punctuating the fractions instruction throughout the term, teachers were more able to provide remedial support to students as needed without negatively impacting overall student participation in the mathematics learning. For example, students who demonstrated a fragile understanding of the meaning of the numerator and denominator in the first set of lessons were provided with opportunities to develop their understanding prior to the second set of lessons being implemented. These opportunities were provided in the mathematics block within the context of the mathematics being studied or were explicitly supported through additional learning opportunities at school based on co-planned lessons. As well, teachers had opportunities to co-develop lesson sequences that built directly upon students' knowledge and needs, including those established in the lesson implications chart above.

Content-focused professional learning

Distinct from typical Canadian professional learning opportunities with a focus on a process such as communication or on a specific resource such as a text, this mathematics professional learning was built around the need to deepen understanding of the teaching and learning of the precise content area of fractions. The collaborative action research model encouraged ongoing co-planning of lessons and debriefing of student work, in combination with the extended timeline, which allowed teachers to make thoughtful connections between the research and their practice. Teachers found it very valuable to think about the mathematics content within the context of their students' thinking as well as for making explicit connections to instructional strategies that were most appropriate for the desired learning outcome. One teacher described the extended learning time for teachers and students as "slowing the math right down". Having a deep understanding of the content area supported the teachers in making precise instructional decisions both in-the-moment and upon reflection.

The teacher team also benefitted from collecting and examining student responses to the same activity across multiple grades. Teachers were able to better understand the developmental nature of fractions learning as well as the subtle nuances of the learning at different grades and stages (e.g., use of models and symbols predominately by 9-year-old students compared to inclusion of multiple number systems by 12-year-old students). The team was then able to identify a learning continuum across the grade band in a holistic manner rather than as a list of separate and discrete objectives or pieces of knowledge which developed in a singular moment. Further, planning shifted from a focus on acquiring mathematics answers to providing intentional opportunities for students to connect their prior knowledge and intuitions to new learning. The extended timeline of several months allowed the teachers to access additional information about

both the content and instructional considerations as well as engage in informal communications about student learning. Teachers reported feeling more confident in their ability to identify student assets with respect to fractions and to respond more precisely to student need.

Conclusions and implications

Data from student assessments as well as field notes from classroom observations suggest that the strategy teachers used of punctuating fractions instruction and learning over time in chunks, rather than condensing the fractions content into one single unit of study, had benefits for both students and teachers. However, this punctuated schedule had implications in terms of how texts and other teaching resources were used, and in terms of planning and reporting structures that required a more flexible organisation. A second component to the teacher–researcher work—that of co-planning and co-teaching fractions tasks—was also described by teachers in focus groups as being highly beneficial to improving the quality of student exposure to and experience with fractions. This combination of co-planning a more responsive fractions program with the punctuation of fractions learning over an extended timeframe resulted in three outcomes as observed in our study to date:

1. It encouraged heightened attention on listening to and noticing student thinking (Mason, 2002).
2. It increased opportunities to understand and address fragile or incomplete student conceptions of fractions (Gould, Outhred & Mitchelmore, 2006).
3. It led to increased student understanding of this complex content area as evidenced in the results of reliable pre-post tests.

On a broader scale, this first phase of the study has led the researchers to consider the potential value of choosing to focus on precise content and then persisting with learning about this same content for an extended action research cycle. We hypothesise that this content focus as a central feature of the professional learning helps to not only improve student learning but also builds teacher content knowledge. In future phases of the study we intend to test this hypothesis by engaging teachers in generating content maps and pathways that document their learning and teaching at the onset of the work, and over the course of the action research cycle. Next steps for our collaborative action research initiative also include the development and field-testing of teaching and learning fractions pathways, and moving forward to examining the content, models, and pedagogy of operations with fractions. Data collection from a larger group of students and teachers is continuing as part of this multi-year research study.

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