Learning to like algebra through looking

Developing upper primary students' functional thinking with visualisations of growing patterns

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Karina Wilkie discusses functional thinking in the primary classroom. She provides a useful learning progression with sample responses to a growing pattern task.

For many of us, algebra conjures up unpleasant memories of a difficult abstract topic in secondary school involving manipulation of letters and numbers according to confusing rules and procedures. In recent years we have seen algebra explicitly introduced into the Australian Curriculum right from the early years of schooling, with research showing that learning algebra is possible even for very young children (Blanton & Kaput, 2004; Warren, 2000). But how does this type of learning connect to the more formal algebra study students are introduced to at secondary school?

A foundational ability that effectively supports students' later learning is that of being able to use 'functional thinking'. Functional thinking is defined as thinking that "focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual instances) to generalisations of that relationship across instances" (Smith, 2008, p. 143). It helps students understand the notion of change and of how varying quantities (or 'variables') relate to each other. It underpins students' learning about how pronumerals (letters in algebraic equations) are used to represent these variables and are a powerful way of expressing relationships in the world around us.

The study of functions and calculus is an important aspect of students' senior secondary mathematics learning and the conceptual understandings that students can develop in primary school about algebra paves the way for later learning in the secondary years (e.g., Cai & Moyer, 2008; Carraher, Schliemann, Brizuela & Earnest, 2006).

What does functional thinking look like in the upper primary years of schooling? The Australian Curriculum: Mathematics refers to describing, continuing and creating patterns and sequences, and to describing the rule that creates a sequence (Australian Curriculum and Assessment and Reporting Authority [ACARA], 2009). This rule is a generalisation of a relationship between two varying quantities or numbers (variables). But teaching students to find the rule for a number sequence often hides this relationship between two amounts. For example, the following is a number sequence in which three is added to the previous term to obtain the next term in the sequence:

$$2 \underset{+3}{\rightarrow} 5 \underset{+3}{\rightarrow} 8 \underset{+3}{\rightarrow} 11 \quad 14 \quad 17 \ldots$$

Upper primary school students can often recognise that each number (or item) is three more than the previous number; this is known as recursive or near generalisation. "Add 3 to an item to get the next item in the sequence" is an example of a recursive rule. Yet many students would struggle if they were asked to find the 100th item in the sequence as this would involve finding an explicit rule or far generalisation which involves finding a corresponding

relationship between the position number of an item and the item itself — the number in the sequence:

2	5	8	11	14	17...
1st	2nd	3rd	4th	5th	6th

× 3 then −1

In this case, "multiply the position number of an item by 3 and then subtract 1 to get the item itself in the sequence" is an example of an explicit rule. (The 100th term would be 299.)

Research in recent years has highlighted the benefit to students' conceptual understanding of algebra if they learn to relate varying quantities in a geometric (or figural) growing pattern rather than looking at one-dimensional number sequences (e.g., Kaput, 2008; Markworth, 2010). Being able to find a rule that shows how these varying quantities (or variables) are related to each other is a key part of functional thinking. This article describes a useful teaching approach to giving upper primary students experiences that promote the development of their functional thinking. It is based on a research project in which I worked with 10 upper primary school teachers and their classes for a sequence of five lessons on geometric growing patterns throughout one year. The teachers were involved in cycles of collaborative planning, implementing, evaluating, and revising the sequenced lessons with the researcher and their teaching teams. I outline a learning progression which is helpful for structuring such a lesson sequence and for assessing students' responses to tasks. I will also share two key teaching strategies that can support students' conceptual understanding of functions and variables.

Learning about functions and variables by generalising geometric growing patterns

There are two ways of describing the relationship between two varying amounts or variables. As previously illustrated with a number sequence, a generalisation can either be recursive (near) or explicit (far). Recursive generalisation looks at how both amounts vary step by step and is also known as co-variation. It is useful for exploring the concept of slope or gradient of a function when represented graphically (Kuchemann, 2010). Explicit generalisation looks at how one quantity can be used to calculate a related quantity and is also known as correspondence (Confrey & Smith, 1994). Both types of generalisation are important for functional thinking. Students tend to be able to learn recursive generalisation (or co-variation) but often need targeted learning experiences that help them progress to the second type—explicit generalisation or correspondence (Stacey, 1989; Warren & Cooper, 2008). Explicit generalisation helps them pay attention to the correspondence between two quantities or amounts, no matter which item they are looking at in a growing pattern or sequence. For example, consider the growing pattern in Figure 1.

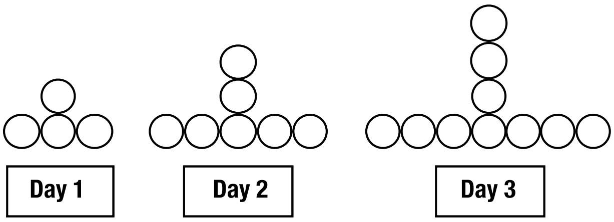
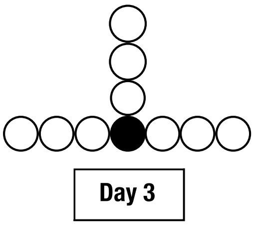


Figure 1. The 'upside-down T plant' growing pattern.

Two variables in this growing pattern could be the total number of leaves on the plant and the day number.

A recursive rule for this 'upside-down T plant' growing pattern (as nicknamed by the students in the project) would be: "Each day the plant grows 3 more leaves and so the total number of leaves increases by 3 every day". An example of an explicit rule (often harder for students to find) would be: "Each branch of the plant has the same number of leaves as the day number plus there is an extra one in the middle." So if you multiply the day number by 3 and add 1 more, you will be able to find the total number of leaves for the plant on any day. Helping students focus on how the day number can be used to find the number of leaves on the plant supports their ability to generalise explicitly. In the research project, the 222 students were found to visualise the structure of the upside-down T plant in four different ways: recursively with extra leaves being added on the ends, a central leaf with 3 branches, a horizontal line with an extra branch on top, and a vertical line with two branches on either side.

The second visualisation, which led most easily to explicit generalisation (Wilkie & Clarke, 2014) is shown in Figure 2.



Day number x 3 + 1 = Total number of leaves
or, $3n + 1 = t$

n = day number
t = total number of leaves

Figure 2. A useful way to visualise the structure of the upside-down T plant for explicit generalisation.

The students were encouraged first to describe their own visualisation of the structure of the growing pattern and how it changes, then try to continue the pattern (day #4 and day #5). They were then asked to find items later in the pattern and create a rule to find the total number of leaves for any day number. Having a partner with whom to share their rule verbally helped many students to put it in writing. Students often chose to write a number sentence containing both words and numbers as shown in the box in Figure 2. This often led to their interest in symbols to replace the variables (in this case n for the 'day number' and t for the 'total number of leaves'). Although the use of pronumerals (letters) to represent variables is not explicitly introduced in the Australian Curriculum: Mathematics until Year 7 (ACARA, 2009), several upper primary students were keen to explore and experiment with them. This tended to flow naturally from students' own initial attempts to create a number sentence for their rule which often contained a mixture of numbers and words, for example: "The answer is the day × 3 + 1". They were shown that "the answer" was the total number of leaves and could be represented with the letter t and that 'the day' could be represented with n for day number. The important concept that letters stand for quantities or numbers rather than the objects themselves is one that takes time and experience to develop; generalising growing patterns provides a conceptually appropriate opportunity for this development. (Suggesting n instead of d for the day number and t instead of l for the total number of leaves was a deliberate strategy to help students focus on a letter as representing a number rather than the shorthand for a word or phrase.) A conceptual understanding of what variables actually are and how they are represented symbolically in equations is a foundational aspect of learning algebra and helps students when grappling with algebraic equations and linear functions in early secondary school.

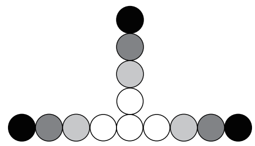
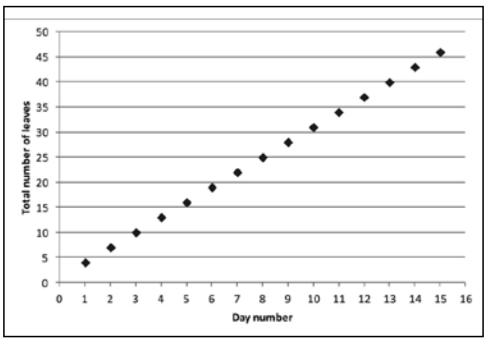
In summary, the key conceptual understandings for upper primary students to develop functional thinking include:

- growing patterns can be described and extended by identifying the consistent change between one item and the next in the pattern (recursive generalisation);
- different quantifiable aspects of a growing pattern correspond to each other and their relationship can be described with a rule for finding one quantity given the other quantity (explicit generalisation);
- explicit rules for a growing pattern can be described in words or with number sentences (equations) containing letters or symbols to represent the varying quantities;
- varying quantities in growing patterns are called "variables" and because they can be different numbers (depending on the item in the pattern), letters or symbols are used to represent them in equations;
- the relationship between two varying quantities in a growing pattern can be represented with an equation, a table, and a graph.

Teaching and assessing functional thinking

It is helpful to know how to plan a sequence of lessons that each give students repeated experiences with generalising geometric growing patterns and that help them progress from describing and extending patterns, to finding a recursive rule, to finding an explicit rule. Markworth (2010) in her research with Year 6 students developed a learning trajectory and it was adapted to create a learning progression for the research project described in this article. It is presented in the first column of Table 1. This progression can also be used as an assessment rubric when assessing the types and levels of students' generalising ability. The second column of Table 1 shows examples of students' typical responses at each level of the rubric.

Table 1. Learning progression for functional thinking in upper primary years with sample student responses (Wilkie, 2013, adapted from Markworth, 2010, p. 253).

<table>
<tr><th>Learning progression level</th><th>Sample responses using the ‘upside-down T plant’ growing pattern</th></tr>
<tr><td>1. Extend a geometric growing pattern by identifying its physical structure, features that change, and features that remain the same (figural reasoning).</td><td>”The plant is an upside-down T shape with 3 stems that keep growing each day. I can continue the pattern and colour the plant’s structure to show what I see.”
</td></tr>
<tr><td>2. Identify quantifiable aspects of items that vary in a geometric growing pattern.</td><td>“The number of leaves of the plant increases every day and the day numbers increase too. The number of leaves on each stem increases every day.”</td></tr>
<tr><td>3. Articulate the linear functional relationship between quantifiable aspects of a geometric growing pattern by identifying the change between successive items in the sequence (co-variation or recursion).</td><td>“Each day, the plant has 3 more leaves than the previous day. Each day, the plant looks the same as the plant the day before but with 3 extra leaves, one on each stem.”
“To find out how many leaves there are on the 10th day, I can keep adding on 3 for each day.”
“4, 7, 10, 13, 16, 19, 22, …”</td></tr>
<tr><td>4. Generalise the linear functional relationship between aspects of a geometric growing pattern by:
4.1 describing the relationship between a quantifiable aspect of an item and its position in the sequence (correspondence);
4.2 using symbols or letters to represent variables; or
4.3 representing the generalisation of a linear function in a full, symbolic equation.</td><td>4.1 “Each stem of the plant has the same number of leaves as the day number, for example, on Day 2, each stem has 2 leaves:”
4.1 “The total number of leaves is 3 times the day number (for the leaves on the 3 stems) plus the 1 leaf that stays the same in the middle. So for Day 2, there are 3 times 2 (6) leaves plus 1 more, 7 leaves altogether.”
4.1 “For Day 20, there are 3 times 20 (60) leaves plus 1 more, 61 leaves altogether.”
4.2 “‘n’ is the number of the day. The total number of leaves on the nth day equals 3 times n plus 1 more.”
4.3 “‘t’ is the total number of leaves on the nth day. The rule for the ‘upside-down T’ plant is t = 3n + 1.”</td></tr>
<tr><td>5. Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations.</td><td>“An ‘upside-down T’ plant with 100 leaves would occur on Day 33 (when n = 33) since 3 x 33 + 1 = 100.”
“This table of plant leaves and days is not of an ‘upside-down T plant’ since here t = 4n:”
<table><tr><th>n th day</th><th>t leaves</th></tr><tr><td>2</td><td>8</td></tr><tr><td>5</td><td>20</td></tr><tr><td>9</td><td>36</td></tr><tr><td>20</td><td>80</td></tr></table>
“Here is a scatter plot showing the total number of leaves on each day for the ‘upside-down T’ plant. The dots go up 3 each time. The dots form a straight line (which shows a ‘linear’ relationship between the variables).”
</td></tr>
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For example, a score of 3 means that a student can describe a recursive rule ("+ 3 each day" or something similar). A score of 4.2 means that a student can write an explicit rule using some sort of expression (for example: "3 × n plus 1"). A score of 5 means that the student is able to apply their functional thinking in flexible ways and to represent a relationship between variables in different ways, for example with a table of values, with a rearranged equation (to make the other variable the subject), or graphically.

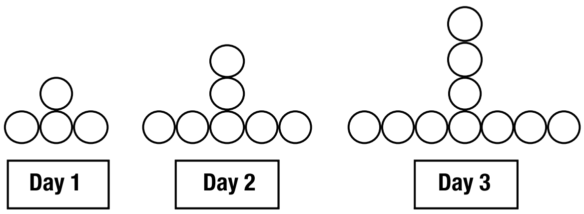
Teachers in the project used the rubric to score each student's initial level of generalisation, to group or pair students for future lessons, and to make decisions about the level of difficulty of the next growing pattern to introduce to students. Table 2 outlines a sequence of questions asked of students about each growing pattern, using the upside-down T plant as an example. These questions were found to help students use their visualisation of a particular growing pattern to progress to finding an explicit rule for the relationship between two quantities in the growing pattern.

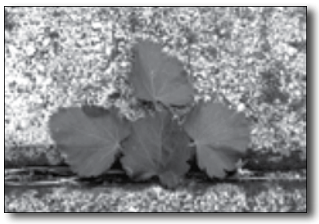
Helping students visualise the structure of the same growing pattern in different ways

A key finding from the research project was that the students were keenly interested in each other's different ways of visualising (and generalising) the same growing pattern. Sharing times at the end of each lesson became a forum for students to explain and demonstrate their different rules to the class. The students grappled with how different expressions for an explicit rule nevertheless produced the same answer. An important aspect of secondary school algebra is learning to evaluate and re-arrange algebraic equations. Sharing times for upper primary students to compare their different rules for the same pattern provided an engaging opportunity for them to explore these skills conceptually, as is demonstrated with the following example of students' different responses to the daisy chain pattern (Figures 3–8). (This growing pattern is more complex than the upside-down T plant and was presented to students towards the end of their lesson sequence on functional thinking).

Table 2. An example of useful questions to help students generalise a growing pattern, using the upside-down T plant as an example.

Questions to help students generalise the upside-down T plant growing pattern





a) Add pictures for what the upside-down T plant will look like on each of the next 2 days (Day #4 and Day #5).
b) What do you notice about the structure of the plant and the way it grows each day? If you can, colour the leaves of the pictures in different colours to show what you see, and explain your thinking below.
c) How many leaves will the plant have on Day #7? Explain / show how you obtained your answer.
d) How many leaves will the plant have on Day #17? Explain / show how you obtained your answer.
e) If someone gives you any day number, how do you find the number of leaves the plant will have on that day? Explain/show how you obtained your answer.
f) On what day number would the plant have 100 leaves? Explain/show how you obtained your answer.

Enrichment questions

g) If you take a plant and double its day number, will you also double the total number of leaves of the matching plant? Why/why not?
h) Can you draw a scatterplot for the total number of leaves on each of the first 10 days of the plant's life? What do you notice?
i) If someone gives you the total number of leaves for the growing plant, how can you find its day number?

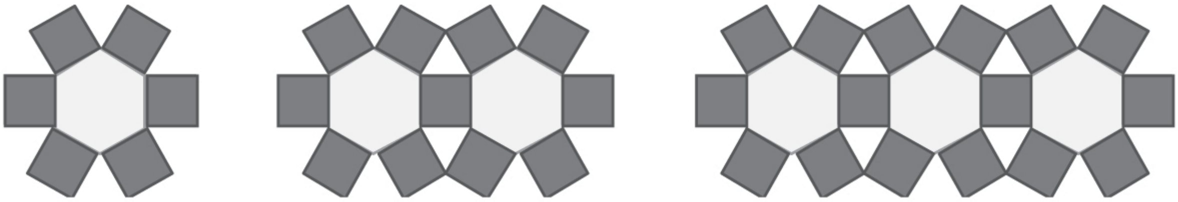


Figure 3. The daisy chain growing pattern.

The students began by creating the daisy chain growing pattern with pattern blocks. By building it themselves, they could pay attention to the changing aspects of the pattern and notice that a petal (square block) was 'dropped' each time a new flower was added to the chain (it is this aspect that makes this pattern more complex and therefore more challenging to generalise). Figure 4 presents a typical recursive rule for this daisy chain expressed by one of the students who added six blocks on each time to the original daisy with seven blocks.

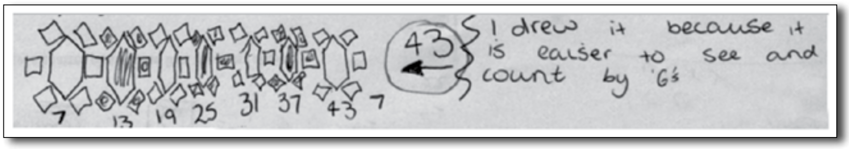
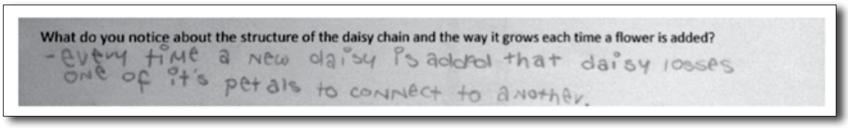
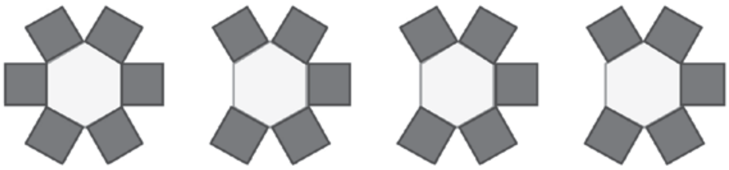


Figure 4. A student's recursive rule for the daisy chain growing pattern.

A few different ways of visualising the daisy chain pattern emerged from students' responses in each class. This paved the way for comparing different explicit rules for their equivalence, and for evaluating which rules were easy to use when calculating the total number of blocks for long daisy chains (e.g., 100 flowers). One way students visualised the daisy chain was as one 'whole' flower with 'broken' flowers added each time (one petal pulled off each time a flower is added). This is presented in Figure 5.





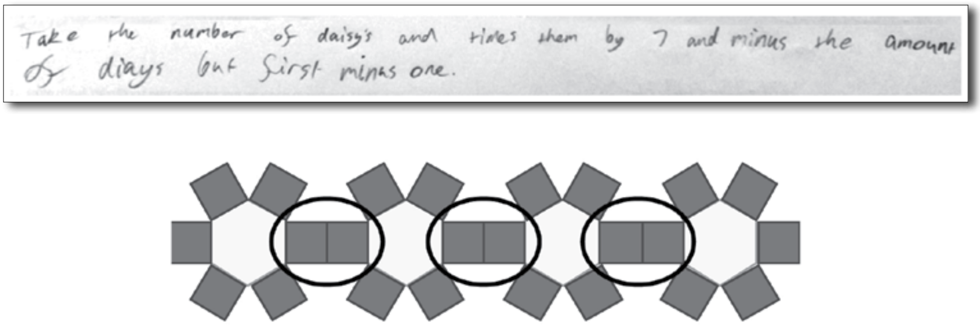
7 + 6 + 6 + 6

$$\mathbf{t = 7 + (n - 1) \times 6}$$

Number of blocks for 1st daisy chain	1 less than the number of daisies since the 1st daisy is already accounted for.	6 blocks needed for each additional daisy in chain	***n*** = number of daisies in a chain ***t*** = total number of blocks

Figure 5. Visualising and generalising the daisy chain growing pattern as made up of one whole flower and then broken flowers.

A second way of visualising the daisy chain pattern emerged from the students' sharing time and is presented in Figure 6. Some students saw all whole flowers joined in a chain but then had to subtract the petals that had been 'pulled off' in between.



$$7 + 7 + 7 + 7 - 3$$

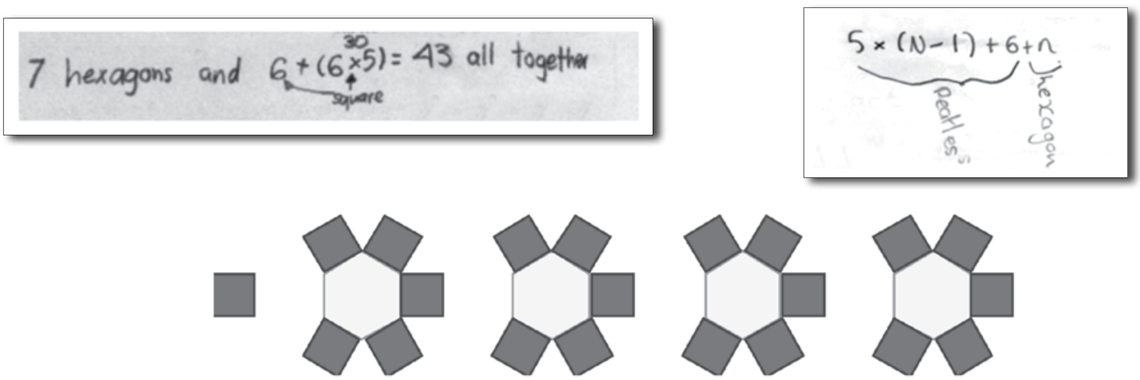
$$t = n \times 7 - (n - 1)$$

7 blocks needed for each daisy in chain.	1 less than the number of daisies equals the number of "pulled off" petals.	n = number of daisies in a chain t = total number of blocks

Figure 6. Visualising and generalising the daisy chain growing pattern as whole flowers with missing petals in between.

Providing students with pattern blocks and encouraging them to make chains of different lengths helped them find the relationship between the number of flowers in the chain and the number of petals that had been 'pulled off' to connect the flowers.

A third way students visualised the daisy chain pattern involved seeing the hexagons and squares separately, and then adding their different amounts together, as shown in Figure 5. Although their explicit rule 'worked', students realised that when they compared it to other expressions of the rule, it involved a lot of calculation and was less efficient than other rules for long daisy chains.



$$4 + 6 + 5 + 5 + 5$$

$$t = n + 6 + (n - 1) \times 5$$

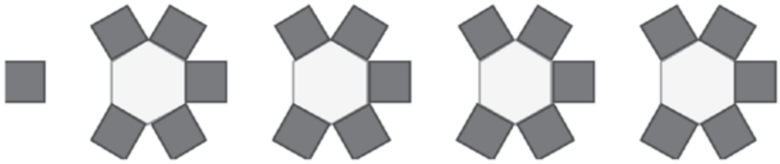
Number of hexagons.	Number of squares around 1st daisy.	Number of square blocks around rest of daisies (n daisies subtract 1st daisy).

Figure 7. Visualising and generalising the daisy chain growing pattern as made up of hexagons and squares.

A few students from different classes actually came up with a fourth and unanticipated way of visualising the daisy chain pattern, which actually proved to be the most elegant and simple way of all and elicited expressions of admiration from the other students. This type of visualisation is presented in Figure 8.

Think of them all as a broken daisy (6) and plus one

You think they are all 6, but then the starting point must have 1. So you keep the one aside and add it when you multiply the 6 by how many daisies there are.



1 + 6 + 6 + 6

$$t = 1 + n \times 6 \text{ (or } 6n + 1)$$

1 petal (square) at front of pattern.	Number of blocks for "broken daisies" joined on.	***n*** = number of daisies in a chain ***t*** = total number of blocks

Figure 8. Visualising and generalising the daisy chain growing pattern as made up of hexagons and squares.

During sharing times, the students were encouraged to compare each other's rules and to test them out to see if the different rules produced the same answer (total number of blocks). Discussions about the use of brackets and order of operations emerged naturally in this context. The daisy chain growing pattern, because of its complexity, also provided a great scenario in which students could grapple with how to express their explicit rules mathematically. For example, some students who visualised the chain as one whole with broken daisies (Figure 5) knew that the number of broken daisies was "one less" than the total number in the chain and wanted to know how to write this in their number sentence (– 1). An example is presented in Figure 9.

It would be (7×N)-(the number just before N)=T

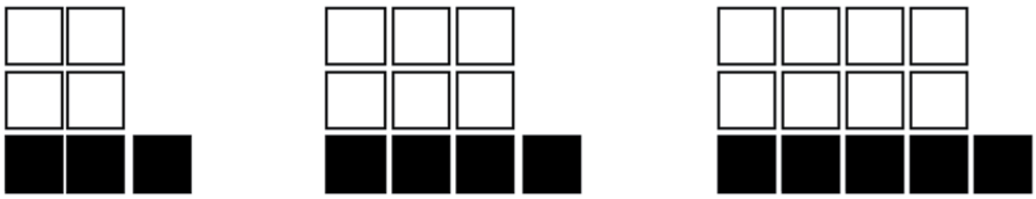
Figure 9. A student attempting to express the operation of subtracting the 'pulled-off' petals from the whole daisies.

Because of their prior experience with simpler growing patterns and with visualising different geometric structures to generalise explicitly, the students were more likely to persevere with a more challenging pattern such as the daisy chain.

Helping students explore variables by using non-sequential items from a growing pattern

As previously mentioned, research has found that students can struggle to move from finding recursive rules to finding explicit rules. Kuchemann (2010) recommended including tasks that focus on the structure of an individual item of a pattern—treating it as a prototype—to help shift students' recursive strategies (describing changes in each successive stage of a pattern) and to focus their attention on the correspondence between the two variables. In this research project some non-sequential items from a growing pattern, presented in Figure 10, were offered as the challenge to some of the students (in the latter half of the lesson sequence). Unlike typical generalisations involving the position number and the total number of blocks or counters as the variables, this Tile trucks task involved looking for a relationship between the numbers of white blocks and black blocks in the pattern.

If you are given a certain number of black blocks for a tile truck, how do you find the matching number of white blocks?



One possible rule: Double the number of black blocks (b) then subtract 2 to get the number of white blocks (w), i.e., 2b – 2 = w.
An alternative rule: Take away one from the number of black blocks and then double the result to get the number of white blocks, i.e., 2(b – 1) = w.

Figure 10. The tile trucks task and finding the relationship between two variables in a non-sequential growing pattern.

A key finding of this research project was that a small number of students (from three different classes) were able to find more than one way of visualising and generalising the structure of a growing pattern themselves (presented in Figure 11) since previous research had found that upper primary students had not been able to demonstrate this (e.g., Rivera & Becker, 2006). It seems that repeated experiences with discussing and comparing visualisations during sharing time paved the way for some students being able to apply their functional thinking in this sophisticated way.

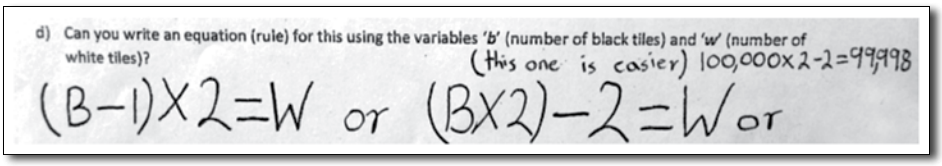


Figure 11. A student's different ways to visualise the relationship between two variables in a non-sequential growing pattern.

These students were additionally challenged to evaluate which rule was the easier to use with large numbers of blocks. One response using 100 000 black blocks can be seen in Figure 11.

Concluding comments

Although students may find the generalisation of growing patterns quite challenging initially, this research project found that a sequence of lessons starting with simple growing patterns and progressing to more complex patterns supported students' gradual development of functional thinking. Generalising and even creating their own geometric patterns with hands-on materials additionally engaged the students' interest. It was as if they relished the challenge of finding the rule and then enjoyed sharing their discoveries with the rest of the class (one class did not want to leave for morning recess as they had not finished sharing yet!). A handful of students even progressed to generalising and representing non-linear (quadratic) growing patterns (involving matchstick grids). One student reflected, "I loved figuring out the rule."

These experiences of visualising and generalising geometric growing patterns gave students a novel context in which to develop their conceptual understandings of what a functional relationship actually is and what it can look like in mathematics (e.g., a worded description, a symbolic equation with pronumerals representing the variables, a table of values, and a graph). This provided a great foundation for upper primary school students both to engage effectively with and to enjoy learning algebra. Another student reflected, "We did not muck around a lot. I found it very fun." Our intent is that these experiences will provide students (and their teachers) with a positive perspective on algebra and pave the way for success in their secondary years of mathematics.

Acknowledgement

The author would like to acknowledge with appreciation the teacher participants and School Mathematics Leaders from the Contemporary Teaching and Learning of Mathematics project, funded by the Catholic Education Office, Melbourne, who contributed to the study on which this article is based.

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