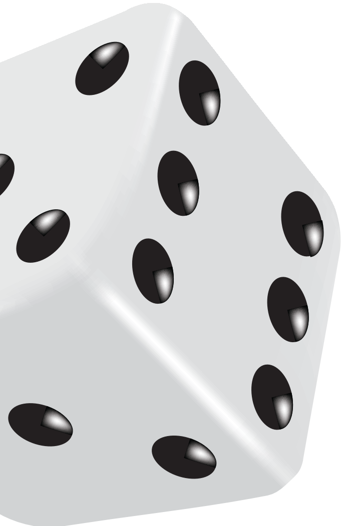

REFractions

The Representing Equivalent Fractions Game

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Stephen Tucker presents a fractions game that addresses a range of fraction concepts including equivalence and computation. The REFractions game also improves students' fluency with representing, comparing and adding fractions.

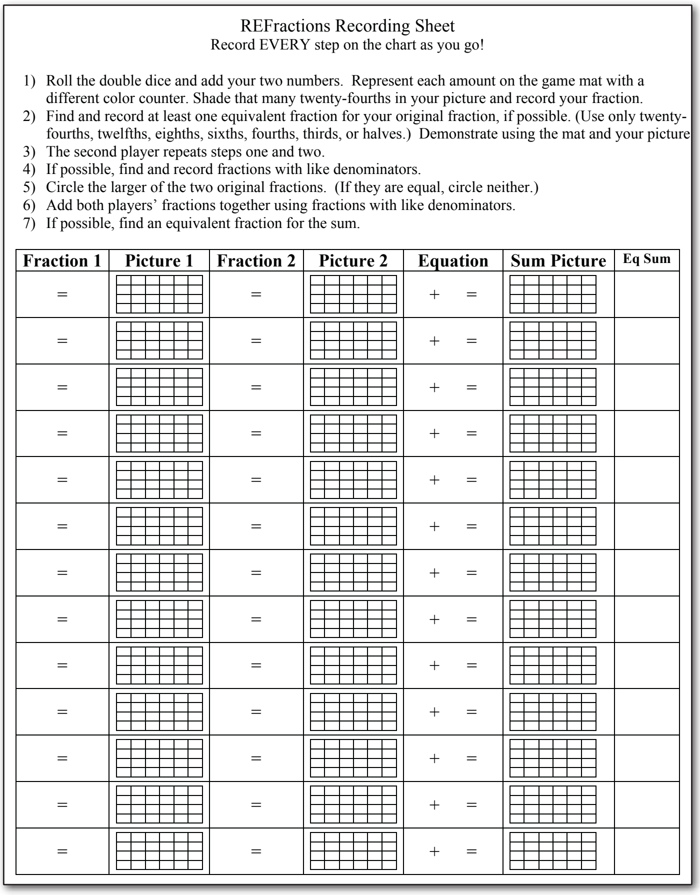
Students frequently encounter concepts that are difficult to grasp without significant interactions with concrete manipulatives. Although students often explore fraction concepts using manipulatives and activities, games usually lack effective recording components to transfer concepts and outcomes from concrete to symbolic forms. *REFractions: The Representing Equivalent Fractions Game* sheds light on concrete manipulation of fractional representation, comparison, and addition, connects them to pictorial representations, and extends them to symbolic representations. This supports the *Australian Curriculum* (Australian Curriculum, Assessment and Reporting Authority, 2012) goal that that students must "make connections between related concepts and progressively apply the familiar to develop new ideas. They develop an understanding of the relationship between the 'why' and the 'how' of mathematics" (p. 5). The game also connects to the *Standards for Mathematical Practice of The Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) in the United States, which indicates that fraction concepts are part of making "mathematically proficient students who can apply what they know [and] are comfortable making assumptions and approximations to simplify a complicated situation, realising that these may need revision later" (p. 7).

REFractions covers concepts including representing, comparing, adding, and finding equivalencies between fractions, all of which require reasoning skills and flexibility with various models for greater understanding. Throughout the game, students form what Clements and McMillen (1996) call "integrated-concrete knowledge" where the objects, their interactions with the objects, and the generalisations they create form useful mental structures (p. 271). The discourse and multiple representations involved in *REFractions* also parallel parts of Van de Walle, Karp and Bay-Williams' (2009) assertion that one should use pictures, written symbols, oral language, manipulative models, and real-world situations to model mathematics, and that the interactions between these representations helps develop new concepts (p. 27).

This article describes the necessary materials for gameplay, how to play the game, and a group of students' first experience with the game. It then discusses evidence of fraction strands and standards from gameplay, as well as possible variations or modifications to the game. Students need background knowledge of representing and illustrating fractions, an awareness that one can compare fractions and find equivalencies between fractions, and an understanding that they can add fractions. During gameplay over multiple sessions, students will develop each of these skills, as well as their mathematical discourse abilities.

Preparation

The purpose of this game is to increase students' fluency with representing, comparing and adding fractions in concrete and symbolic forms. Students use dice, representation mats, tiles and their recording sheets to develop and transfer their conceptual understanding of the concrete forms of fractions to the symbolic representations. To play *REFractions* each pair of students will need a recording sheet (Figure 1) and representation mat (Figure 2), a pencil, an eraser, tiles, and one double dice, which are translucent, coloured dice, each with a second, smaller opaque white die contained within. If double dice are not available, substitute two standard dice.



REFractions Recording Sheet
Record EVERY step on the chart as you go!

1) Roll the double dice and add your two numbers. Represent each amount on the game mat with a different color counter. Shade that many twenty-fourths in your picture and record your fraction.
2) Find and record at least one equivalent fraction for your original fraction, if possible. (Use only twenty-fourths, twelfths, eighths, sixths, fourths, thirds, or halves.) Demonstrate using the mat and your picture
3) The second player repeats steps one and two.
4) If possible, find and record fractions with like denominators.
5) Circle the larger of the two original fractions. (If they are equal, circle neither.)
6) Add both players' fractions together using fractions with like denominators.
7) If possible, find an equivalent fraction for the sum.

Fraction 1	Picture 1	Fraction 2	Picture 2	Equation	Sum Picture	Eq Sum
=		=		+ =		
=		=		+ =		
=		=		+ =		
=		=		+ =		
=		=		+ =		
=		=		+ =		
=		=		+ =		
=		=		+ =		
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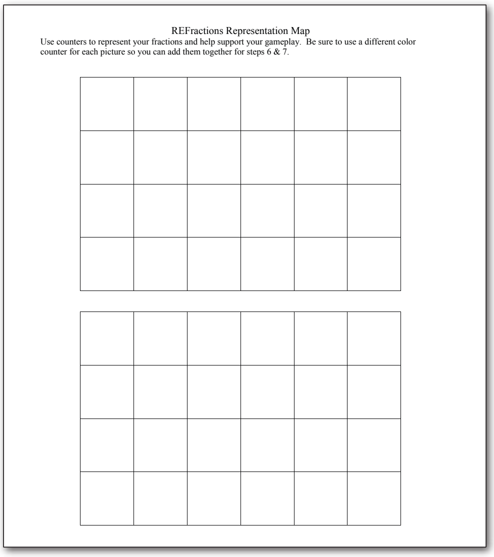
Figure 1. Blank recording sheet.



REFractions Representation Map
Use counters to represent your fractions and help support your gameplay. Be sure to use a different color counter for each picture so you can add them together for steps 6 & 7.

Figure 2. Blank representation mat.

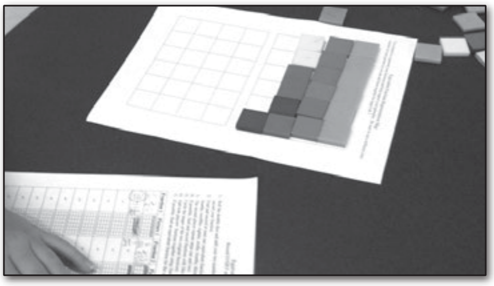


Figure 3. Representing steps with a different tile colour.

Different colours of tiles are important so students can use a different colour for each fraction they represent, compare and add, as shown in Figure 3. This intentional use of manipulatives aligns with Sowell's (1989) and Clements' (1999) claims that manipulatives strengthen the sequence of concrete-to-abstract understanding but are most effective when specifically purposed. The teacher will also need a set of the materials for the demonstration.

Before beginning the game, the teacher should review concepts and vocabulary that students will use during the game, which may include defining what a fraction represents, naming and representing fractions, comparing fractions with common (or uncommon) denominators, finding equivalent fractions, and adding fractions. Students need not be fluent in the latter skills before playing, but knowing they will encounter these ideas increases their awareness of the concepts. A demonstration version of the game featuring key vocabulary usage, discussion overtly connecting back to the target objectives, and careful following of instructions is a key component in insuring students participate and discuss productively.

How to play

The first step is for the first player to roll the double dice and add the two numbers, representing this fraction on the representation mat using one colour of tiles for each of the numbers (refer to Figures 1 and 2). Then the player shades that many twenty-fourths in the Picture 1 space and records the fraction in the Fraction 1 space (Figures 1 and 4). The first player then attempts to find equivalent fractions using only twenty-fourths, twelfths, eights, sixths, fourths, thirds, or halves. Students manipulate the illustration and tiles to find equivalent fractions while discussing the process. Often, students list multiple equivalent fractions and find many ways to record the illustration.

After the first player finishes, the second player repeats the first player's steps, but in the Fraction 2 and Picture 2 columns along the same row. During this process, the players should use key vocabulary words and refer to concepts reviewed or introduced in the opening discussion. To facilitate discussion, the teacher should focus students on the concepts they most need to develop. This may include questions regarding how students know their models are equivalent, why the addition sentence matches the concrete and pictorial examples, how they found equivalencies, why one fraction is greater or less than another, and how to find other models of equivalent fractions. The teacher should model and encourage specific mathematical language, such as numerator instead of 'top number' and denominator instead of 'bottom number'. If students have found multiple equivalent fractions, the fourth step, finding and recording fractions with like denominators, is often easier. The representation mat allows each player to show a fraction simultaneously and compare models when finding equivalencies. Then, students circle the greater original fraction, if they are unequal.

Next, students write the fractions in the equation space, find and illustrate the sum, and record it in the equation box. Students may use any symbolic representation they found that involves common denominators. For example, if students found that eight twenty-fourths and six twenty-fourths equal four twelfths and three twelfths, respectively, they may use the twenty-fourths or the twelfths in their equation. Students should use the representation mats to identify the sum, combining the amounts into one representation. When combining the tiles, students see each part and the whole at the same time as a visual representation of the addition process (Figure 3). Finally, students find and record an equivalent fraction for the sum, if possible, using the sum picture and representation mat to find any equivalencies.

Play continues until all space is used. At the end of the game, the teacher reconvenes the entire class to discuss key findings from the session, using exact mathematical vocabulary while referring back to the goals outlined in the beginning discussion. Students use evidence from their recording sheets and partner discourse to justify their reasoning, and the teacher should clarify misconceptions that arose during gameplay.

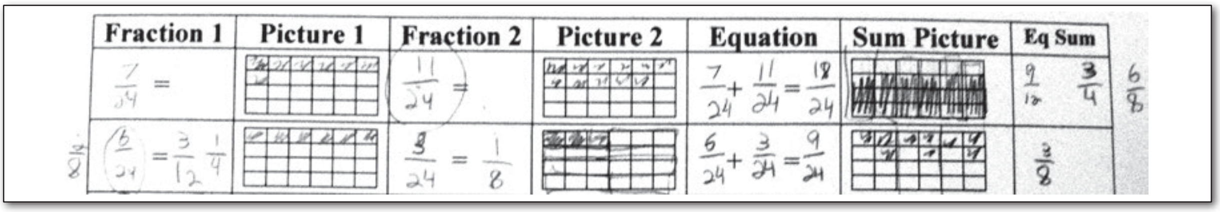


Figure 4. Demonstration rounds on recording sheet.

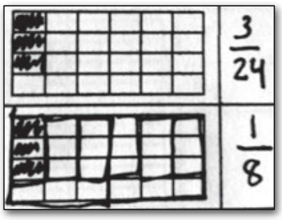
Evidence of fraction standards

This section of the article describes the implementation of the game in an American classroom of students who were between ten and eleven years old. The author began with a review of fraction concepts to gauge the background knowledge of the eight students who had just begun fourth grade in Mrs Moeller's class. The students had some prior exposure to representing, comparing and adding fractions, as well as finding equivalencies, but struggled to explain or demonstrate the concepts. This group was appropriate to test the game's efficacy, as it allowed ample room for exploration and growth.

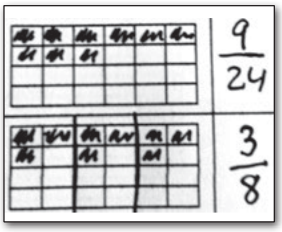
During the demonstration lesson, the author led two rounds of the game (Figure 4). In the first round, the group modelled the numbers that the teacher rolled on the double dice, using a different tile colour for each of the numbers. Students found no applicable equivalent fractions for eleven twenty-fourths or seven twenty-fourths, but determined that eleven twenty-fourths was greater because the pieces were all the "same size" but the eleven pieces were "more than the seven" pieces. This took significant coaching, as the students returned to "eleven is more" but had trouble with the "pieces in a whole", asserting that the denominator was "pieces in all" and not yet conceptually understanding why one did not add the denominators. For the sum of eighteen twenty-fourths, they found equivalent fractions by rearranging the tiles to show two partitions of twelve with nine tiles in each, six partitions of four with three tiles in each, and eight groups of three with six of the groups completely filled. Thus, the students were able to see different ways to connect equivalent fractions to the original sum of eighteen twenty-fourths.

For the second example, the author pre-selected six twenty-fourths and three twenty-fourths to find applicable equivalent fractions for each. Through demonstration on the mat and discussion, six twenty-fourths became one out of every four, three out of every twelve, and two out of every eight, as students preferred this model (Figure 5a). For three twenty-fourths, however, the author showed eight groups of three, one of which was completely coloured, in an effort to display multiple methods (Figure 5b). Together, the group added their original fractions after deciding they were not ready to use the equivalent fractions. Students illustrated three partitions of eight, each with three coloured, to demonstrate three eighths as an equivalent fraction for nine twenty-fourths, reverting to their more comfortable model (Figure 5c). To play the game, students chose their partners, collected their materials, and began working.

(a). Six twenty-fourths.



(b). Three twenty-fourths.



(c.) Nine twenty-fourths.

Figure 5. Reconstruction of contrasting methods for finding equivalent fractions.

Representing fractions

Students manipulated tiles on the representation mat and coloured corresponding sections of the picture cell on their recording sheets to explore the area/region model of fractions, representing fractions as equal coloured parts of on whole figure. This group of students was not yet ready to extend this context to placing fractions on a number line for linear representation or using the tiles to explore set models of fractions. Students developed their understanding of representing fractions using the area/region model, connecting the symbolic to visual representations with reasonable accuracy. When one student named a fraction as "fifteen ninths", his partner demonstrated that the fifteen pieces on the twenty-four mat equaled fifteen twenty-fourths, saying, "But we have fifteen pieces and it takes twenty-four to make a whole." This came about because the author previously reviewed that the denominator meant "the number of pieces it takes to make a whole", rather than "the number of pieces in all" to clarify confusion and allow the definition to include improper fractions.

Finding equivalent fractions

This is the crux of the activity and thus was the focus of most of the discussion. There were usually multiple ways to show any given fraction, and students were easily able to move between the twenty-fourths and twelfths, often splitting the even numerators into two halves (Figure 6). Sixths and fourths were also relatively easy to find, as one student pointed out that the mat is "a four by six array" and made the connection that she could find the number of fourths by counting how many rows of six she filled and find the number of sixths by how many columns of four she filled. Her partner, however, preferred to look at the models as four rows of "one out of every six" coloured and six columns of "one out of every four" coloured, respectively, and was able to expand this to "two out of every eight" and "three out of every twelve", visually demonstrating her expanding proportional reasoning skills. Figure 7 is a version of this thinking, as one group modelled "fifteen out of twenty-four is five out of every eight." With practice, students improved their ability to move fluently between the two models.

Another example of problem-solving for equivalent fractions came while a pair of boys tried to find an equivalent fraction for

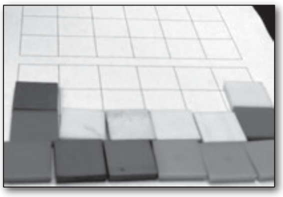


Figure 6. Fourteen twenty-fourths is equivalent to seven twelfths.

Figure 7. Fifteen twenty-fourths is equivalent to five eighths.

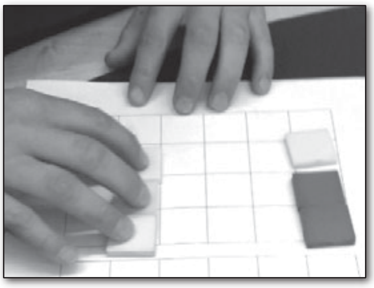


Figure 8. Finding equivalent fractions for six twenty-fourths.

six twenty-fourths (Figure 8). To explain, the boys placed their six tiles on the mat in two groups of three. They moved a group of three tiles across the mat, counting the groups of three it took to make the whole twenty-four, agreeing that there were eight groups of three with two groups completely coloured. Since it took eight equal groups to make a whole, they were using eighths, and with two coloured, their equivalent fraction was two eighths. The boys were elated, having taken ownership of their successful learning experience.

Comparing fractions with like denominators

This concept also appeared often, with discussions supporting conclusions shown on the mats and recording sheets. One student compared eight twenty-fourths and six twenty-fourths using the mat, illustration and symbolic representation, saying, "Eight twenty-fourths is larger than six twenty-fourths because eight is larger than six and all the pieces are the same size." Her partner added that the fractions equivalent to eight twenty-fourths would always be larger than the fractions equivalent to six twenty-fourths because of the original comparison.

Adding fractions with like denominators

Students expressed that they knew they could add fractions with like denominators, but were not sure why they could do so. One pair discussed adding eight twenty-fourths and six twenty-fourths with the author. The first student insisted on adding the numerators, and when questioned, said that, "If you add [the denominators] they will be less than they should be." Her partner demonstrated the addition on the mat, showing that there were now fourteen pieces. Next, the author asked if the number of pieces it took to make a whole changed, at which point both students excitedly exclaimed that it had not; they were still using twenty-fourths. The first student added that, "The size of the pieces is the same, too," thus, they had fourteen twenty-fourths. This exchange reveals many concepts one should develop over time, particularly how the pieces would get smaller if you added the denominators. This activity pushed students' boundaries for justifying answers while improving their conceptual understanding of fractions.

Extensions and variations

There are many options to adapt the game. Students may place their fractions on a number line or add their sums together to find combinations of fractions that equal a given target, including mixed numbers, using the materials to model the process. A further way to extend the activity is to request written explanations of justifications given during the game, such as why one fraction is larger than another, why two fractions add up to be a particular sum, or why two fractions are equivalent, given the illustration and how the students modified it to check their work. For other extensions, one can make students aware of adding or comparing fractions with unlike denominators, as they already have the fractions with unlike denominators listed as equivalent fractions. To simplify gameplay for remediation, the teacher can target specific fractions with equivalents and provide some or all of the fractions in advance. Where appropriate, reducing the game to using a twelve frame instead of a twenty-four frame and removing one of the dice allows students to start with more basic fractions.

Conclusion

REFractions: The Representing Equivalent Fractions Game encourages students to construct and connect various visual representations of fractions to find equivalencies, make comparisons, and add fractions while strengthening discourse skills. By purposefully maneuvering manipulatives and adjusting illustrations, while discussing these explorations with classmates and teachers, students may develop fluency with fraction concepts that readily map to *Common Core State Standards* and *Australian Curriculum* mathematics objectives. With a multitude of modifications to supplement the extensive standard gameplay, playing the game for multiple sessions with various partners is an insightful and enjoyable mathematics experience for teachers and students alike.

Acknowledgements

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Please contact the author for free full-page PDF versions of the recording sheet and game mat.