Promoting and assessing mathematical generalising



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The difficulties faced by young children when generalising are highlighted and ideas about how teachers can select and design tasks that promote generalisation are provided.

Helping students generalise mathematical ideas is an essential component of teaching and learning of mathematics (Lannin, Ellis, Elliott & Zbiek, 2011). However, it can be challenging for us as primary teachers to assess and promote generalisation. Because generalisation is an essential part of mathematics instruction, in this article we describe the types of difficulties young learners often experience with generalisation, and share the strategies we have used to assess and promote generalisation in young learners.

What does it mean to generalise?

Students naturally engage in generalising, both inside and outside of the mathematics classroom (Mason, 1996). For instance, outside of the mathematics classroom children reason that both A, B, and C in Figure 1 are chairs, but D is not.

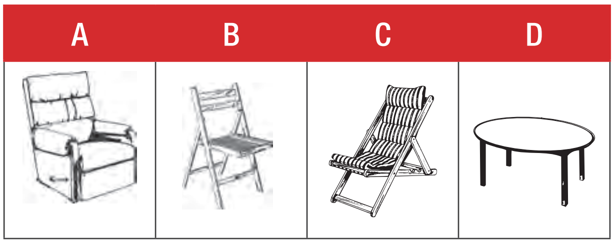

Figure 1. Four objects children see in everyday life.

As it relates to mathematics, students reason that E, F, and G in Figure 2 each contain three objects, while H does not. While it often seems that students arrive at both of these conclusions with little effort, such understanding is the product of sophisticated general reasoning.

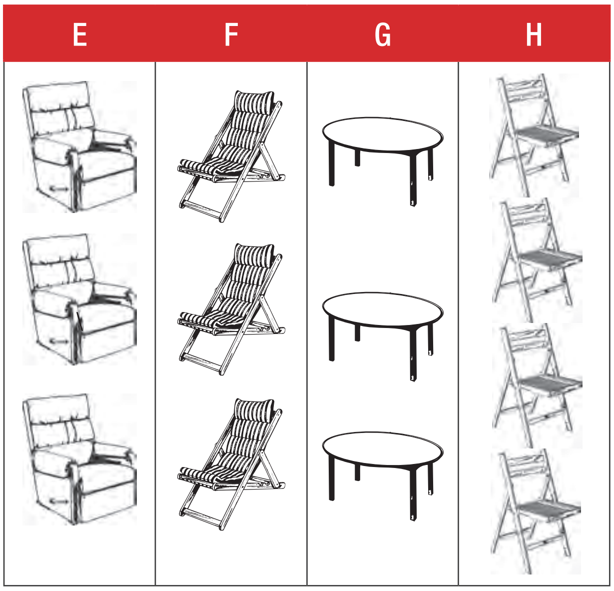

Figure 2. Diagrams comparing and contrasting what is meant by "three objects."

As a student generalises, he or she: (a) attends to common features across representations, problems, or situations and abstracts an idea from those commonalities; or (b) determines to which new representations, problems, and situations an idea applies (Lannin et al., 2011). For instance, the student who noted that there were three objects in sets E, F and G in Figure 2 attended to the number of objects in each set, while ignoring other features, such as the colour, shape, and intended use of the objects. Further, the student seems to understand the final count describes

the number of objects. The student attended to common features and abstracted an idea from it, so the student has generalised in the first sense of the term.

If the student can extend these ideas about how much 'three' is to new situations with different types of objects, then the student has generalised in the second sense of the term. For example, a student who notes there are three groups of ten when three ten-sticks are provided, or who notes there are three dollar coins in her pocket, has applied the idea of 'three' to new situations that involve objects that can represent different units.

In our work with students, we found that students do not always see the common features that we intend. Further, sometimes children do not extend their ideas to the various representations, problems, or situations that are appropriate, though students sometimes extend to situations beyond those that we view as appropriate. Therefore, we must be aware of how students generalise and the generalisations they develop in our classrooms. Moreover, we must design instruction that allows students to articulate their reasoning to help us understand how students generalise. We believe this is especially important because how students generalise is a critical barrier to learning for the struggling learners with which we worked.

From our work with struggling learners, we present multiple situations that highlight the difficulties students have in their efforts to generalise, and the instructional strategies that could help to address these difficulties. Further, we discuss how the nature of the mathematical tasks can reveal these difficulties and how these can be used to promote generalising.

Difficulties noticing common features across representations, problems and situations

To illustrate the difficulties students experience in attending to common features, we provide two transcripts that detail a conversation we had with Allison, a student in Year 2 who was struggling in mathematics. The two parts of the conversation occurred one after the other over a period of fewer than two minutes. As you read the transcript, consider the characteristics of the mathematical task we presented to Allison.

Transcript 1

Teacher: There are 27 cubes right here, right now. [Two sticks of ten and one stick of seven Unifix cubes are on the table. The teacher places another stick of ten cubes on the table.] Now how many are there?

Allison: Um, [Pauses for a few seconds and looks toward the ceiling.] 37.

Now, consider the similarities between the mathematical task described above and a second task we presented to Allison in which she was asked to determine 27 + 10 = __ when written for her on a whiteboard.

Transcript 2

Teacher: I was doing this in second grade [Year 2] the other day and I was having a hard time figuring this out. Somebody wrote this down. [Writes '27 + 10 = __' on whiteboard.] Do you know what [the answer to] this is?

Allison: Yeah.

Teacher: What is that? [Points to whiteboard.]

Allison: [Raises five fingers, and counts first two.] 28, 29. [Restarts counting, and quietly counts all ten fingers.] 28, 29, 30, 31, 32, 33, 34, 35, 36, 37.

Teacher: What did you get?

Allison: [Writes '37' on whiteboard.] Well, this is the smart way since I just have to change the 2 into a 3, and leave the 7 all by itself.

Notice that despite using two different representations for 27 and 10 more, one with manipulatives and the other in written form, the two tasks involve the same quantities (27 and 10) and the same operation (addition). However, Allison does not notice these similarities across the two situations, and approaches the second mathematical situation as if it were new and unrelated to the first. She counted (one-by-one) to determine her response to the second situation, even after she had just solved a similar mathematical task moments earlier, not counting by ones to do so.

For students who do notice these important similarities, the presentation of these two mathematical tasks one after the other seems odd. These students often state, "We just did this

problem," or "This is the same as the last one." However, Allison does not recognise and connect these two situations. As such, learning mathematics is more difficult for her as she uses different (and often inefficient) strategies for what are, in essence, the same situation. Further questioning is needed to determine what Allison does or does not notice about the features of these two problems, but if Allison does not notice common features, as it appears from these responses, we must design instruction to support Allison in making these important connections.

For these students, it is important that we employ targeted instructional strategies to encourage Allison and others to see similarities between the two mathematical tasks. For example, we could provide Allison with these two tasks (without asking her to actually compute the result) and ask Allison: "What do you notice about these two problems? How are the two problems similar? How are the two problems different?" These questions can help focus her attention on the deeper structure of the tasks, and help Allison see the connections that she did not recognise. In presenting new mathematical tasks, and when encouraging students to connect mathematical representations of similar (or different) situations, prompting students to consider how one mathematical task can help them to solve a second mathematical task can also help students like Allison to look for these important connections.

Difficulties applying mathematical ideas to new representations, problems, and situations

To further illustrate the difficulties students have generalising, consider the challenges students face in determining the appropriate application of mathematical ideas to new representations, problems, and situations. To demonstrate these challenges, we provide three transcripts of a conversation with Meghan, a student in second grade/Year 2 who was also struggling in mathematics. Similar to the first set of transcripts, these occurred one after the other, over a period of fewer than three minutes. As you read the first two transcripts (Transcripts 3 and 4), consider the similarities and differences between the first and second sets of mathematical tasks, and the mathematical idea that Meghan appears to understand based on her response to these first two problems.

Transcript 3

Teacher: How many counters are there right there? [Ten counters, no particular arrangement.]
Meghan: [Pause.] Ten.
Teacher: Ten. If I pull this one away [removes one counter], how many are left?
Meghan: Nine.
Teacher: I am going to put this one back in [puts removed counter back to make set of ten]. I am going to pull this one away [Removes one counter, different from counter removed the first time.] How many are left?
Meghan: Nine.
Teacher: [Puts removed counter back to make set of ten.] I am going to pull this one away. [Removes one counter, different from counters removed before.] How many are left?
Meghan: Nine.
Teacher: [Puts removed counter back to make set of ten.] I am going to pull this one away [Removes one counter, different from the counters removed before]. How many are left?
Meghan: Nine. [Laughs.] That's fun.

Transcript 4

Teacher: [Two sets of ten counters on table.] How many counters all together there?
Meghan: Twenty.
Teacher: I'll pull this one away. [Removes one counter.] How many are left?
Meghan: I don't know. [Pause.] 19, because 10 [points to 10] and then... No, one more and that makes 20, and take one out and that makes 19.
Teacher: That's good thinking. I am going to put this one back [puts removed counter back to make two sets of 10], and now there are 20 again. I am going to pull this one away.
[Removes one counter, different from counter removed before.] Now how many are left?

Meghan: Nineteen.
Teacher: Still 19? How do you know?
Meghan: Because my brain told me.
Teacher: Really? What if I put this one back ? [Puts removed counter back to make set of two sets of 10.] I am going to pull this one away. [Removes one counter, different from counters removed before.] How many are left?
Meghan: Nineteen.
Teacher: Still 19? I am going to put this one back. [Puts removed counter back to make set of two sets of 10]. Pull this one out. [Removes one counter, different from counters removed before.] How many are left?
Meghan: Nineteen.

Notice that both tasks involve the removal of one counter from a set of counters, and both tasks involve removing different counters from the original set. From her responses, it appears that Meghan understood an important mathematical idea. That is, it does not matter which single counter is removed, the number of counters remaining is still one less that the initial number of counters.

Now, consider Meghan's response to the third set of mathematical tasks (Transcript 5), in which one counter is removed from a set of 37 counters, and the removal is then repeated with different counters from the set of 37 counters each time.

Transcript 5

Teacher: Why don't you tell me how many counters there are? This is a pile of ten, that's a pile of ten, that's another pile of ten, and that's not a pile of ten.
Meghan: 10, 20, 30, 31, 32, 33, 34, 35, 36, 37.
Teacher: Thirty-seven. Agree?
Meghan: Uh-hmm.
Teacher: Now how many are left? [Removes one from set of 7 ones.]
Meghan: Thirty-six.
Teacher: Are you sure?
Meghan: [Counting on from the 3 groups of 10.] 30, 31, 32, 33, 34, 36.
Teacher: There are 36?
Meghan: Uh-hmm.
Teacher: Now how many are there? [Puts removed counter back.]
Meghan: Thirty-seven.
Teacher: Now how many are left? [Removes one counter from set of ten.]
Meghan: I can't do that. No, I can't do that.
Teacher: How many do you think are left? There were 37. How many do you think are left? I removed one.
Meghan: [Lengthy pause.]
Teacher: Not sure? Okay.

While it appeared from the first two tasks that Meghan understood that one less than a total is the same regardless of which one is removed, she did not extend her understanding to the situation with 37 counters. That is, while she understood that the removal of counters from the ones is one less than the original total, she was uncertain about how many are left when the counter was removed from a ten in this situation. Our instruction should be designed to assist her in considering the application of this particular idea to other situations with various numbers of counters.

For these students, we suggest having them explain and consider other situations where they could apply an idea, and to what extent they can generalise their use of a particular idea. For example, we could ask Meghan to explain with a prompt such as: "So whenever you remove one object from 10 objects, the result is nine objects. Would this work with 20 objects, that we would always have 19 objects if one object is removed? If we had 25 objects and removed any one object, would it always be 24 objects? Does it matter which one is removed or would the result be the same no matter which is removed?" As such, we are encouraging Meghan to be explicit about how this idea could be applied to other related situations.

Selecting and designing tasks that assess and promote generalisation

The questions and prompts described above demonstrate the difficulties students may have with generalisation. However, it also highlights the importance of task design and selection to allow teachers to identify the specific difficulties students may have had generalising. To help

us think about task design and selection in our work, we considered two different characteristics or dimensions of mathematical tasks. The two dimensions included: (a) the form of mathematical representation and (b) the mathematical domain. In particular, we used these two dimensions to select and develop sets of tasks that differed across one dimension, while remaining similar across the other dimension. With these sets of tasks we aimed to determine specific difficulties students were experiencing, and to extend and develop their mathematical understanding with common features across tasks.

The term 'form of mathematical representation' refers to one of five primary representations. These representations include: oral language, written symbols, diagrams, manipulatives, and contextual situations (Lesh et al., 1987). For any mathematical idea, tasks can vary across the five forms of representations (see Figure 3) or vary within a single form of representation (see Figure 4).

When selecting and developing tasks that vary in the type of representation used, we considered whether or not a student noticed common features across representations and applied the same mathematical idea across various representations. For example, recall the tasks within Transcripts 1 and 2 involved the same quantities and operations, but differed in the representation used. These tasks helped us understand whether or not the student attended to and was aware that both tasks represented 27 and 10 more. As such, these tasks varied in the form of representation, but remained constant in mathematical domain.

It is important to note that sometimes students have a deeper understanding of mathematical ideas for one representation over another, but may not have connected their understanding to other representations.

For example, some students quickly recognised 10 more or less with a model (demonstrating an understanding of the meaning of place value), but did not connect this understanding to written symbols (e.g., 25 + 10). Other times, students calculated with written symbols (e.g., 25 – 1), but did not apply their reasoning to various representations (e.g., that any one object can be removed from 25 objects and the result is 24 objects). One of our goals was to help student see the important similarities across the different representations.

Representation	Example
Oral language	**Say:** "What is one less than 24?"
Written symbols	**Show:** 24 – 1 = __.
Diagrams (with symbols provided)	**Show:** Consecutive numbers 1–24 on a piece of paper. Beneath each number provide an image of a circle, but omit one circle from the end.
Manipulative models	**Show:** A line of 24 counters. **Say:** "How many counters are in the line?" **Do:** Remove one counter from the end of the line. **Ask:** "How many counters are there now?"
Contextual situation	**Show/read:** Kelley has 24 apples. She gives one apple to her friend. Now, Kelley has one less apple. How many apples does Kelley have?

Figure 3. Various representational forms focused on the same mathematical idea.

Representation	Example One	Example Two	Example Three
Manipulative models	**Show:** A line of 24 counters. **Say:** "How many counters are in the line?" **Do:** Remove one counter from the end of the line. **Ask**: "How many counters are there now?"	**Show:** Twenty-four Unifix cubes (i.e., two ten sticks and four ones). **Say:** "How many cubes are there?" **Do:** Remove one cube from the end of the ones. **Ask:** "How many cubes are there now?"	**Show:** Three ten-frames with 24 counters. **Say:** "How many counters are there?" **Do:** Remove one counter from the last ten-frame. **Ask:** "How many counters are there now?"

Figure 4. Three different manipulatives that focus on the same mathematical idea.

The mathematical domain dimension refers to the range of application of a particular mathematical idea, where the term domain refers to "any mathematical object or system for which a mathematical relationship is defined" (Lannin et al., 2011, p. 23). For example, students can consider which instances the mathematical idea applies. Similarly, students can consider whether the mathematical idea applies to "'all rectangles', 'any whole number greater than four', 'any situation that involves proportional reasoning', or 'the geometric world of Euclidean geometry'" (Lannin et al., 2011, p. 23).

As we selected tasks that differed in the mathematical domain, we aimed to determine whether or not the student could apply his or her understanding across the applicable domains. For example, Transcripts 3,4, and 5 involved the same representation, but differed in initial quantities, which helped us to understand whether or not the student understood the same mathematical idea applied across various numbers of objects. As such, these tasks varied in mathematical domain, but remained constant in form of mathematical representation (see Figure 3).

An important part of selecting tasks that differ in mathematical domain involves helping students recognise when they should not apply a particular idea. As such, it is important to provide instances for which the mathematical idea does not apply in order to help students understand the boundaries of a particular mathematical idea. For example, we could examine whether a student recognises the idea that anytime objects are rearranged (but no objects are added or removed), the total number of objects remains the same, by including situations where objects are added or removed to determine whether the student recognises that the total number of objects is no longer the same.

Conclusion

One critical instructional challenge involves helping students see similarities between two or more instances of a situation, and then generalise beyond a specific instance of a situation. This is particularly difficult for students who struggle in mathematics. Too often students see a mathematical idea as situated within an example, such as a particular representation or domain. As Dienes (1961) pointed out, "if [the] situation is even slightly altered it will not be treated as the same kind of situation" (p. 292). To facilitate generalising, we encouraged students to consider mathematical ideas beyond a specific instance of a situation. We encourage teachers to select and design mathematical tasks that allow them to assess generalisation, and to facilitate discussions around those mathematical tasks that help students to consider the situations to which a particular mathematical idea should and should not be applied.

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