Investigative approaches to teaching mathematics and ‘getting through the curriculum’

The example of pendulums

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The benefits of rich tasks, project-based learning, and other inquiry-based approaches in terms of student understanding and engagement with mathematics are well documented (e.g., Fielding-Wells, Dole & Makar, 2014; Lam, Cheng & Ma, 2009; Sullivan & Lilburn, 2004). Such pedagogies are consistent with the development of mathematical proficiencies as described in the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority [ACARA], 2013) and many teachers are keen to implement them. Basing mathematics teaching around projects and ill-structured problems, however, can be daunting for teachers who lack confidence in the mathematics content that they are teaching. In addition, Toolin (2004) reported concern among teachers and parents that students in schools using project-based learning might not have access to the whole curriculum.

This paper describes an investigation of the relationship between the length of a pendulum and its period (time for one complete swing) conducted as part of a professional learning program with ten teachers in a Year 9–12 school attempting to teach the entire curriculum using project-based learning[1]. As described by Beswick, Callingham and Muir (2012) none of teachers at the school had studied mathematics beyond secondary school. Nevertheless, they were reasonably adept at identifying at least some of the mathematics that could be taught using a particular context although they struggled to know how to engage students with the mathematics (Beswick et al., 2012). They were also concerned about whether it would be possible to cover the entire mathematics curriculum using only projects.

The pendulum investigation was intended to illustrate the extent of the mathematics that could potentially be taught through a single investigation and to build teachers’ confidence in exploring mathematics with their students. Content from the *Australian Curriculum: Mathematics* that was ultimately touched upon is shown in Table 1. The investigation


Figure 1. A pendulum in use.

1 For more about project based learning and the educational design used in the school see www.bigpicture.org.au.

took place over two after school sessions that included other activities as well, a few weeks apart. In the following sections the investigation is described and opportunities to discuss mathematical ideas are highlighted.

Getting started

As soon as the problem of finding a relationship between the length and period of a pendulum had been introduced, one of the teachers challenged the presenters with the question, "Why should I care?", explaining that this is what many of his students would say. A discussion ensued about contexts in which pendulums are found—e.g., in grandfather clocks, children's swings (Figure 1), and metronomes—and how understanding the affect of the length of the pendulum on the speed at which it swings is important in each case. The teachers agreed that "Why should I care?" was an important question for them to consider as they attempted to engage students in exploring mathematical aspects of their projects. The acceptability of an answer in terms of the content being in the curriculum and, therefore, needing to be learned was more controversial, with some teachers claiming that obtaining qualifications and doing work that you do not particularly want to do is part of 'real life' and hence a legitimate justification for engaging in some school activities.

The discussion then moved to establishing the meaning of 'period' in this context, after which the eight participating teachers discussed how the problem could be explored. Collecting some data about the periods of pendulums with different lengths seemed an obvious way to go (as well as an approach that they believed would be engaging for students). Further discussion focussed on issues including how many different lengths should be used, and how the period should be measured. It was decided that teachers would work in pairs to collect data for three different pendulum lengths and use a stopwatch to measure the period. The issue of measurement error was raised in relation to the time measure (but not the length) and it was agreed that they would measure the time for ten complete swings (because the division to find the time for one would be easy), and that each measure of the period should be repeated three times and the average taken as the period. The need to use the same bob throughout the investigation was also discussed.

At this stage of the investigation it would have been possible to have explored the issue of measurement error in greater detail. In particular it would be interesting to explore why they were concerned about error in the time measure but not the length measure, and to talk in more detail about the choice to measure the period for three different pendulum lengths in terms of the number of points required to identify the shape of graph. More detailed discussion of the range of variables that might influence the period and how each might be controlled could also have been productive.

Collecting and plotting the data

Each pair of teachers selected a bob for their pendulum from a range of objects provided that would be relatively easy to tie to a piece of string to make a pendulum. They then found a suitable place from which to suspend their pendulums—for example, in doorways—and set about collecting the data. None of the teachers made the error of suspending their pendulum above a doorway so that when it swung into the adjoining room it had the

Table 1. Aspects of Australian Curriculum: Mathematics covered in the pendulum investigation

Year	Content descriptions	
	Addressed	**Could be pursued**
7	**Statistics and probability** Identify and investigate issues involving numerical data collected from primary and secondary sources (ACMSP169) Construct and compare a range of data displays including stem-and-leaf plots and dot plots (ACMSP170) Calculate mean, median, mode and range for sets of data. Interpret these statistics in the context of data (ACMSP171) Describe and interpret data displays using median, mean and range (ACMSP172)	**Number and algebra** Investigate and use square roots of perfect square numbers (ACMNA150)
8	**Number and algebra** Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point (ACMNA178) Solve simple linear equations (ACMNA179) Investigate, interpret and analyse graphs from authentic data (ACMNA180) Plot linear relationships on the Cartesian plane with and without the use of digital technologies (ACMNA193) **Statistics and probability** Investigate techniques for collecting data, including census, sampling and observation (ACMSP284) Investigate the effect of individual data values , including outliers, on the mean and median (ACMSP207)	**Number and algebra** Introduce the concept of variables as a way of representing numbers using letters (ACMNA175) Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176) Solve linear equations using algebraic and graphical techniques. Verify solutions by substitution (ACMNA194)
9	**Number and algebra** Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems (ACMNA208) Find the midpoint and gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software (ACMNA294) Sketch linear graphs using the coordinates of two points and solve linear equations (ACMNA215) Graph simple non-linear relations with and without the use of digital technologies and solve simple related equations (ACMNA296)	**Measurement and geometry** Investigate Pythagoras' Theorem and its application to solving simple problems involving right angled triangles (ACMMG222)
10	**Statistics and probability** Use scatter plots to investigate and comment on relationships between two numerical variables (ACMSP251)	**Number and algebra** Substitute values into formulas to determine an unknown (ACMNA234) Solve linear inequalities and graph their solutions on a number line (ACMNA236) Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate (ACMNA239) Solve linear equations involving simple algebraic fractions (ACMNA240)
10A	**Statistics and probability** Use information technologies to investigate bivariate numerical data sets. Where appropriate use a straight line to describe the relationship allowing for variation (ACMSP279)	**Number and algebra** Define rational and irrational numbers and perform operations with surds and fractional indices (ACMNA264) Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations (ACMNA267)

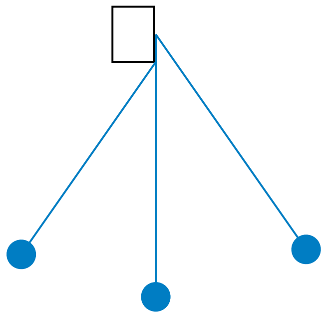

Figure 2. Pendulum swinging through a doorway such that the length varies.

effect of shifting the fulcrum during each swing and consequently varying the pendulum length during each swing. Such an arrangement is shown in cross section in Figure 2. If students set up a pendulum as shown in Figure 2 the teacher would be faced with the choice as to whether or not to intervene. These data could be used as an opportunity to discuss the impact of errors of this kind on the outcomes of experiments.

Each pair of teachers entered their data into a table, labelled with their names, which had been prepared in *TinkerPlots* (Konold & Miller, 2005). The table was displayed on the wall so that all of the data were public. Although they had not used *TinkerPlots* before, entering the data was straight forward. It was agreed that everyone could add a data point to show that when the pendulum length was zero, the period would also be zero. Once all of the data were entered graphs of period and length were created for each data set. Two of these are shown in Figure 3.

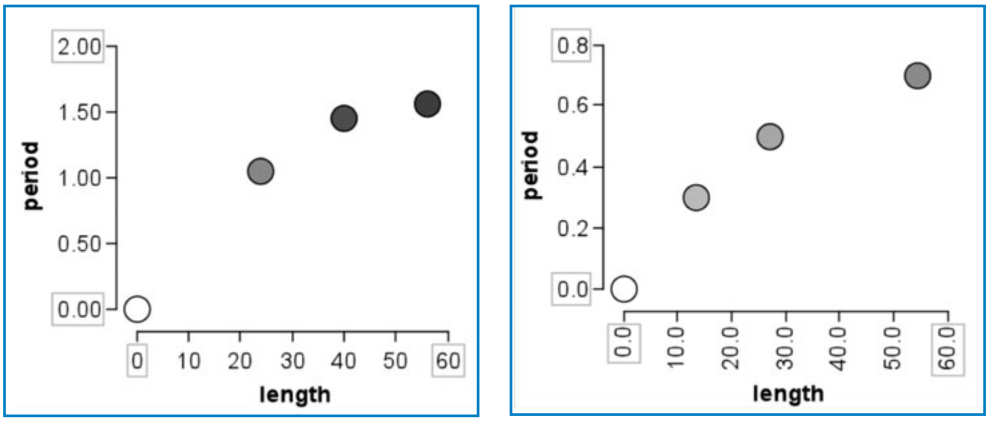

Figure 3. Two graphs of period versus length.

Overall the graphs suggested a nonlinear relationship between length and period. The inclusion of the point for zero length made this much clearer and so, had it not been suggested already, this is the point where the discussion arises naturally. Alternatively, collecting additional data, particularly for relatively short lengths, could have been proposed. This would, of course, lead to shorter periods and hence increased potential for errors in measuring the time for ten complete swings. Matters of presentation such as including the units on the labels of the axes were not raised as these were considered peripheral to the intent of the activity and, because all of the teachers had used centimetres and seconds, there was no communication issue to be addressed in this regard.

Exploring the relationship

At this stage it would have been possible for the pairs of teachers to use their own graphs to interpolate or extrapolate in order to predict the periods of pendulums of un-measured lengths. Ideas around lines of best fit and the assumptions and risks inherent in interpolating and extrapolating could also have been explored, as could the sufficiency or otherwise of relying on just three data points. We chose, however, to pursue the algebraic relationship between length and period because this was an area in which the teachers were particularly lacking in confidence.

Between the two sessions an additional column, showing the square root of the length was added to all but one data table. At the start of the second session the tables were displayed and the process for adding the square root of length column, including the use of a formula, was demonstrated in *TinkerPlots* using the remaining data table. The reason for adding this column was explained in terms of the relative ease of finding an equation for the relationship between two variables if they are proportional – recognisable from a linear graph that passes through the origin. This reasoning was possible because the teachers had previously participated in several sessions on aspects of proportional reasoning. Because the relationship between period and length was not linear and the graphs curved as they did, the relationship between the period and the square root of the length might be linear so we could check this in *TinkerPlots*. The graphs of period against square root of length were readily created by dragging the header of the square root column onto the horizontal axes of the existing period vs. length plots. The teachers could see that these graphs were approximately linear. The two that correspond to the graphs shown in Figure 3 are shown in Figure 4. Both show believably linear relationships.

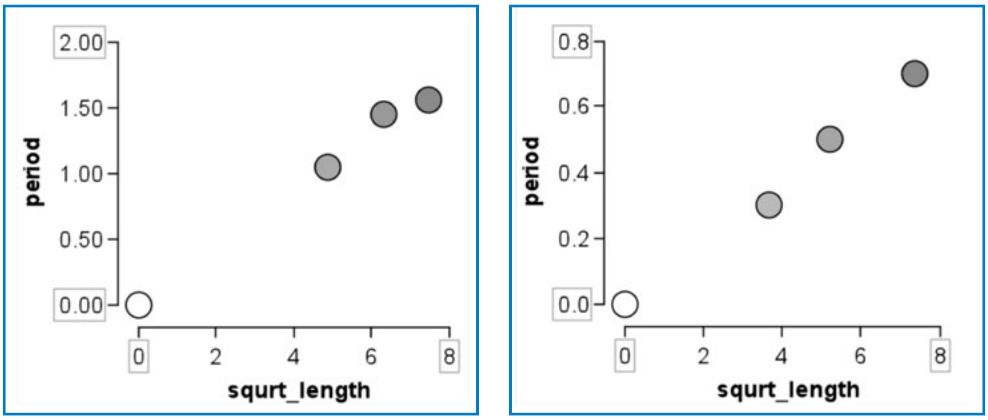

Figure 4. Two graphs or period versus square root of length.

Calculating the square roots of the lengths was a step introduced by the authors without mathematical justification at that stage. Later, links were made between this move and the simple pendulum equation. In addition, its relevance to the Tasmanian senior secondary subject, *Mathematics Applied*, was highlighted. *Mathematics Applied* includes a unit on algebraic modelling that covers transformations of a range of non-linear functions including various exponential functions. The situation also illustrates the need that sometimes arises to ask students to do something without knowing why, relying on them having sufficient trust in the process and ultimate end to remain engaged. Nevertheless, we believe it was important that the mathematical reason was evident not too long afterward. The need for this step also highlighted the importance of mathematical knowledge at crucial junctures in an investigation.

Combining all four datasets resulted in the graphs shown in Figure 5 for period versus length and period versus square root of length. The three circled points in each case relate to the data shown in the right-hand side of each of Figures 3 and 4 that were obtained from one particular pair of teachers. Without these three points the shapes of the graphs are more clearly as expected. *TinkerPlots* has a 'Hide selected cases' function which allows the cases such as these to be hidden and unhidden to clearly demonstrate the effect on the graph. When hidden, the graphs are as expected—i.e., a

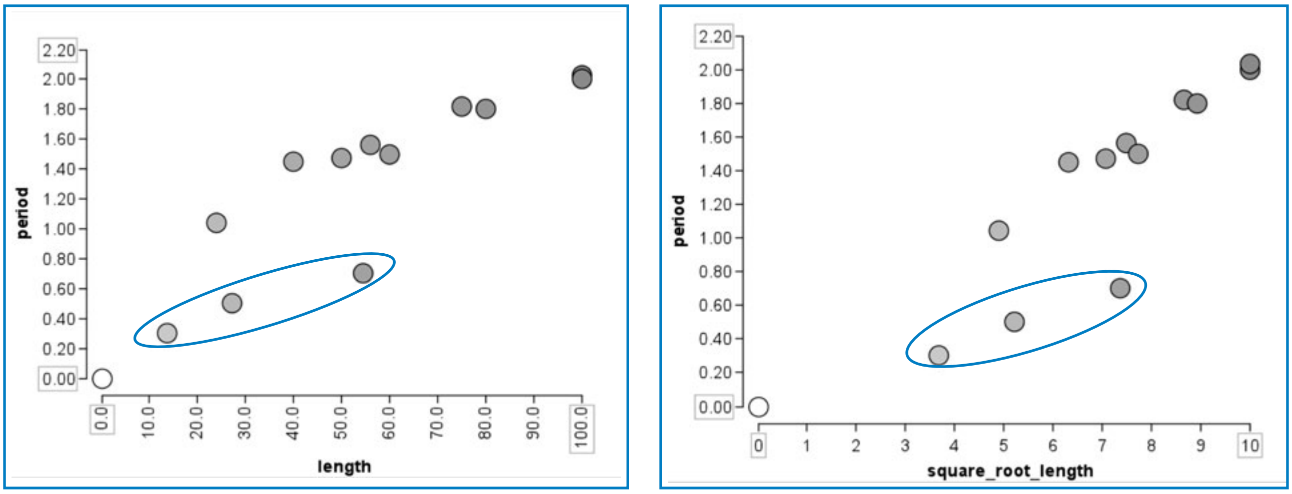

Figure 5. Graphs of period versus length and period versus square root of length for the combined data set.

flattening curve for period versus length on the left and a straight line for period versus square root of length on the right.

Exploring the relationship in greater depth

Figure 6 shows the plots of the combined data sets without the three problematic data points (circled in Figure 5). At this stage it would be possible to calculate the equation of the line of best fit for period versus square root length in the form

period = constant × square root of length

where the constant is the slope of the best fit line. As shown in Figure 6 the slope (rise/run) is approximately $\frac{2.05}{10} = 0.205$. This means that for these data, the period is equal to approximately 0.205 × the square root of the length, or $T = 0.205\sqrt{L}$ where T is the period in seconds and L is the length in centimetres.

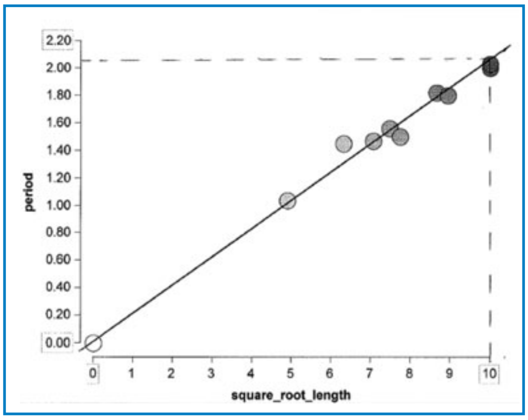

Figure 6. Line of best fit for period versus square root length graph.

An Internet search for "pendulum equation" led to the discovery that the equation for calculating the period T in seconds of a simple pendulum for small amplitude swings is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in metres, and g is the acceleration due to gravity (~9.8 ms^{-2}). There is a lot to unpack here. First, the teachers had measured the periods of their pendulums in seconds but the lengths were not in metres; rather they were in centimetres. Effectively, our length measures were 100 times what they should be to make the equation work. Alternatively, we could adjust the equation to reflect lengths in centimetres. It would be

$$T = 2\pi\sqrt{\frac{L}{100g}}$$

Taking the square root of 100 allows the equation to be written as

$$T = \frac{2\pi}{10\sqrt{g}}\sqrt{L}$$

We know $\sqrt{g}$ = 9.8 ≈ 3.13 so the equation becomes

$$T = \frac{2\pi}{31.3}\sqrt{L}$$

which is the same as $T = 0.201\sqrt{L}$, where T is in seconds and L is in centimetres. This is very close to the experimental value that was obtained. The algebraic manipulation required here to establish the reasonableness of the experimentally derived equation by comparison with the standard form of the simple pendulum equation would only be appropriate for upper secondary students. As was the case for the teachers in this activity, it is likely that many students would need careful assistance to work through the steps meaningfully.

The second issue is why did three of the results (the ones circled in Figure 5 and excluded from Figure 6) not fit the line? One way to examine this would be to consider the data table that corresponds to Figure 6. The data table is shown in Figure 7 with the three cases hidden in Figure 6 enclosed in the rectangle. An additional column, headed calculated_period has also been added. The figures in that column are the results of the formula, $T = 0.201\sqrt{L}$. Comparing the period and calculated period for these three cases shows that in each case the period is about half the value that it 'should' be. It seems likely that the pair of teachers who obtained these results timed only half a full period. When doubled and included, the graph of period versus square root of length (Figure 8) is the result. The three cases are highlighted and fit the line reasonably well.

Third, there is the issue of why the data points do not fit the line (or equation) exactly. This could lead to a more detailed discussion of measurement errors and also to a closer look at the condition on the simple pendulum equation,

$$T = 2\pi\sqrt{\frac{L}{g}}$$

that it applies to small amplitude oscillations. Depending on their interest and expertise senior students could explore more precise simple pendulum formulae.

Combined data — Options

	length	period	square_root_length	calcuated_period
1	56.0	1.56	7.48331	1.50197
2	40.0	1.45	6.32456	1.2694
3	24.0	1.04	4.89898	0.983269
4	54.5	0.70	7.38241	1.48172
5	27.2	0.50	5.21536	1.04677
6	13.6	0.30	3.68782	0.740178
7	100.0	2.03	10	2.00709
8	75.0	1.82	8.66025	1.73819
9	50.0	1.47	7.07107	1.41923
10	100.0	2.00	10	2.00709
11	80.0	1.80	8.94427	1.7952
12	60.0	1.50	7.74597	1.55469
13	0.0	0.00	0	0

Figure 7. Combined data table including points hidden in Figure 6.

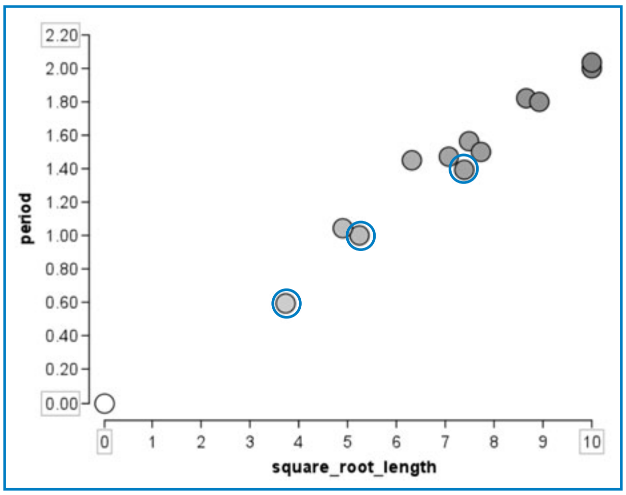

Figure 8. Graph of complete data with adjustment for misfitting data points.

Some teaching decisions

Many of the issues that arose with misfitting data could have been avoided if we had been more prescriptive in our instructions. Specifically, we could have stressed the need to use small amplitudes, the importance of setting up the pendulum free of obstacles that would vary its length (avoiding the situation shown in Figure 2), asked the teachers to provide their length measurements in metres, and checked that all groups were measuring complete periods rather than half periods—intervening as needed to make sure the data were the highest quality possible. Instead, after discussing a few parameters that the teachers raised, we allowed the groups to make their own decisions, choosing to deal with issues if and as they arose. This was possible because we were relatively confident with the mathematics (and physics) involved and hence confident that issues could be worked through and used as learning experiences for the teachers. In this context we were also interested in modelling to the teachers how to deal with 'unexpected' results. This was particularly relevant to the out of area teachers with whom we were working who would, to some extent, be learning with their students. In the project-based learning context of the particular school students were encouraged to draw upon expertise external to the school, including but not limited to the internet. Teachers were happy to model this process in their own learning and recognised the authors as among available external resources.

Links with the Australian curriculum

One objective of the activity was to illustrate for teachers the extent of curriculum content that could be touched on in a single investigation. To this end we asked pairs of teachers to look at one or more of the Year 7, Year 8, Year 9, Year 10 and Year 10A Australian curricula, and the Tasmanian Certificate of Education *Mathematics Applied* and *Essential Skills Numeracy* curricula, and identify aspects of these that were addressed in the investigation or that we could have pursued further. The results of this, for the *Australian Curriculum*, are shown in Table 1. Some of the content descriptors are only partially covered but the teachers were quite surprised that so much was touched upon.

There is clearly a great deal of Science also involved in this investigation that could be pursued and would be very relevant in project based learning contexts such as in the school with which we were working.

Finding out more

The following website was one of the more helpful results of the Internet search. It allows students to input any two of the three variables in

$$T = 2\pi\sqrt{\frac{L}{g}}$$

and calculate the third. Go to:
http://hyperphysics.phy-astr.gsu.edu/hbase/pend.html

Conclusion

The pendulum investigation provided an example of a single investigation, much smaller in scope than the typical projects that students in the school undertook, that included scope for a great deal of mathematics that crossed year levels and content strands of the *Australian Curriculum: Mathematics*. As a result of the investigation we and the teachers who participated were more confident that the whole of the mathematics curriculum could be covered through project based learning. Of course, the suitability of this particular investigation would depend upon the existing knowledge, experience, and interest of students which would also determine the extent of teacher support required. In the school in which we were working the students were given considerable freedom to design their own learning experiences and, consistent with this, were expected to display commensurate resourcefulness and persistence in pursuing their learning.

For teachers in more traditional contexts the investigation illustrates how an investigative task of this kind can incorporate many aspects of curriculum content from mathematics and beyond. Thinking about the content of the Australian curriculum in integrated ways reduces the apparent crowding of the curriculum by highlighting the interconnectedness of ideas. It also illustrates how students with an enormous range of mathematical sophistication can engage meaningfully in a common task and achieve worthwhile learning.

Acknowledgement

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