Listening to Children in Order to Capture Potential Learning

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Jill Cheeseman and Deborah Michels let us eavesdrop on a young child's attempt to solve a four-digit subtraction. In the process they describe the use of an empty number line as a tool for describing the calculation process. If you haven't tried using an empty number line, then the authors outline some compelling reasons for using empty number lines in the classroom.

The story that we relate happened in a Year 4 classroom which is a part of the *Encouraging persistence maintaining challenge* research project[1].

This project is founded on a belief that while it is possible for everyone to learn mathematics, it takes concentration and effort over an extended period of time to build the connections between topics and mathematical ideas, and to be able to transfer learning to practical contexts and new topics. The type of actions that are associated with learning mathematics include connecting, representing, identifying, describing, interpreting, sorting, applying, designing, planning, checking, imagining, explaining, justifying, comparing, contrasting, inferring, deducing and proving. All of these require persistence (Sullivan et al. 2011). One of the research questions we are investigating is, "When teachers pose challenging tasks, how do students respond, and are there interventions that can influence that response positively?" The story of James (a pseudonym for a student) helps to describe ways in which

1 *The Encouraging persistence maintaining challenge* project is funded through an ARC Discovery Project funded project (DP110101027) and is a collaboration between Peter Sullivan, Jill Cheeseman, Doug Clarke, Deborah Michels, Jim Middleton, Angela Mornane and Anne Roche of Monash University, Australian Catholic University and Arizona State University. The views expressed are those of the authors. We acknowledge the generous participation of the project schools.

students respond to challenging tasks and how teachers' actions can support their thinking.

The participants in our project, including James' teacher, expect their students to persist in mathematics lessons and, as members of the research team, they have been experimenting with challenging tasks. We hypothesise that challenging tasks allow students opportunities to:

- plan their approach, especially sequencing more than one step;
- process multiple pieces of information, with an expectation that they make connections between those pieces, and see concepts in new ways;
- engage with important mathematical ideas;
- choose their own strategies, goals, and level of accessing the task;
- spend time on the task;
- explain their strategies and justify their thinking to the teacher and other students; and
- extend their knowledge and thinking in new ways (Sullivan, Cheeseman, Michels, Mornane, Clarke, Roche & Middleton, 2010).

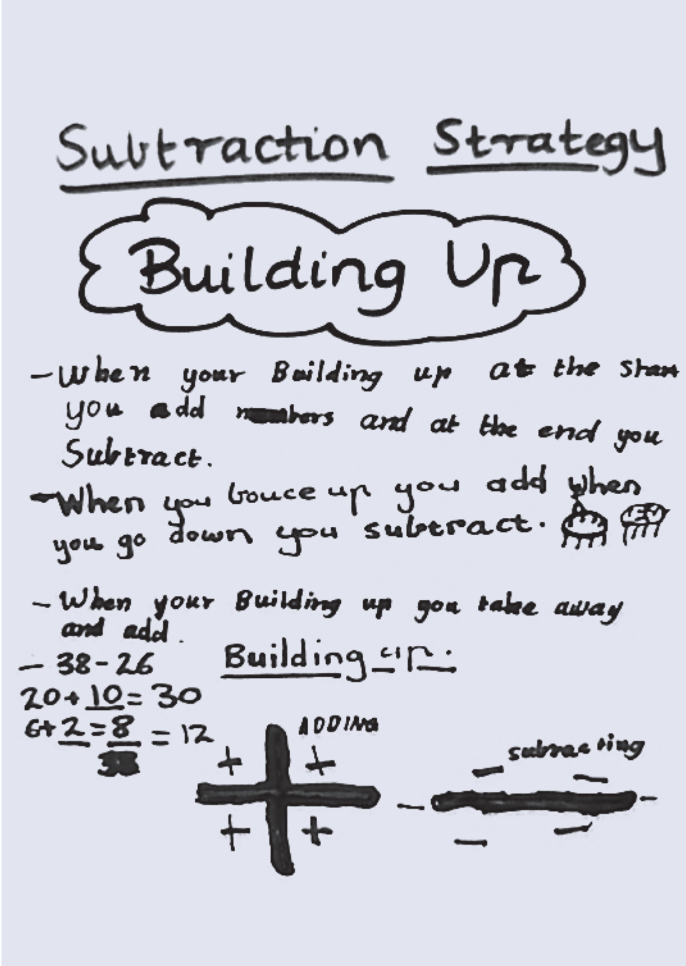

Figure 1. The chart made by students.

At the time of this event, the mathematical focus in Year 4 was on strategic thinking for addition and subtraction. The incident involving James concerned the complementary addition aspect of subtraction, where subtraction is solved by counting up, "which is an amazingly powerful way of solving subtraction" (Van De Walle, 2007, p. 223).

In an earlier class discussion the students had been taught to label this thinking as a "building up idea". Some students had made a classroom poster for the wall (see Figure 1) in which they were trying to explain how to solve a subtraction by counting up. The same idea has been referred to as the "inverse-of-addition" structure; for example, the subtraction 542 – 275 can be interpreted as "What do you add to 275 to get 542?" (Haylock, 2010, p. 93).

A challenging task (first designed by Brian Tickle in 2003) was presented to the students of Year 4.

In James' classroom, the students were asked to find the example that was most challenging for them and to try to work it out. As observers, we (the authors) wrote the narrative of the lessons we were invited to observe. We listened to the interactions between the teacher and students, between student and student, and occasionally asked a student to talk to us about what they were doing and why.

On this occasion, Deb (one of the researchers) spoke to James about what he was trying to do. The example James selected was 9357 – 4689. He reported to Deb that he had attempted the problem using each of the strategies his class had discussed but had had no success in finding a solution. He was determined to solve this particular problem but was at a loss as to how to proceed. This is when he asked for help. The reconstruction of James' thinking was based on Deb's notes and an audio record of their conversation.

YOU DECIDE

Work on this sheet with a partner. There are 30 number problems on this page. Write a tick (✓) next to any problem you think you could work out without a calculator or pencil and paper.

Take turns to explain your strategy to your partner. Share your strategies with the rest of the class.

1.	14 – 5 =	11.	4573 – 0 =	21.	923 – 876 =
2.	78 – 40 =	12.	100 – 47 =	22.	$1.00 – 65c =
3.	100 – 6 =	13.	986 – 700 =	23.	787 – 86 =
4.	83 – 27 =	14.	976 – 976 =	24.	1206 – 7 =
5.	5000 – 1 =	15.	9357 – 4689 =	25.	6748 – 99 =
6.	700 – 100 =	16.	400 – 263 =	26.	32 – 7 =
7.	1002 – 4 =	17.	1000 – 13 =	27.	60 – 24 =
8.	57 – 19 =	18.	40 – 17 =	28.	8241 – 7663 =
9.	42 – 2 =	19.	101 – 6 =	29.	100 – 98 =
10.	800 – 150 =	20.	7777 – 1111 =	30.	48 – 32 =

Figure 2. Task *You Decide.*

Deb's story of James

When James sought my assistance, the class was nearing the end of its time on task. James reported that he had already attempted to solve the problem using all the strategies posted in the classroom but he had not been successful. I needed to probe into his recent attempts at using a variety of different strategies in order to appreciate what understandings and capabilities he already had.

The first strategy he reported to have attempted was "take tens, then ones." James concluded that that this strategy was not appropriate because the numbers were too big. "We can't take tens and ones. It's kind of like take thousands…" He recognised the inefficiency of taking tens and ones from numbers in the thousands and understood that it would be efficient to start by taking away thousands, but did not actually follow through on his idea, perhaps because he did not recognise it as a valid strategy, as it was not posted as part of the collection of class strategies.

The next strategy James reported attempting was "add them up." James had broken down each number into its place value units and then attempted to build up to each value individually, starting with the thousands.

4000 + 600 + 80 + 9
9000 + 300 + 50 + 7

He started by adding 5000 to 4000 to get to 9000 but proceeded to run into trouble with the next three place values as the initial values were greater than the end values. He declared that it would not work because 600 plus another number cannot equal 300. This illustrated his limited understanding of the idea of regrouping or decomposition which is crucial for understanding the vertical subtraction procedure.

Then he tried vertical subtraction (a formal subtraction algorithm) despite the fact that it was not one of their class mathematics strategies nor was it a technique he had been formally taught at school. When that failed to work for him, he decided to make the numbers easier to work with by removing the thousands digits, leaving him with the problem: 357 – 689. This left him with the same difficulty he experienced when trying to add up: "300 take away 600: you can't do that."

The last strategy James attempted was the use of a number line. He drew a line and labelled two points at opposite ends. He labelled the lesser point 4000 and the greater point 9357. From there he showed me how

he had begun to draw in intervals, counting backwards from 9357. He quickly realised that this was a very lengthy and inefficient process and would not work given the constraints of time and the available space on the number line. This was the point at which James sought my assistance.

From listening to James explain his choices and reasoning, I realised that he had a good grasp of basic place-value ideas. I did not know whether or not to explore how James had got stuck using the inverse methods of adding up and subtracting the decomposed numbers because I was intrigued by his excellent number sense and his ability to decompose numbers and compute quickly, easily and flexibly, and yet he had not employed those abilities with these two methods. He was somehow constrained when confronted with "impossibilities" that both methods (inverse methods, really) illuminated. It never seemed to occur to him that when confronted with "300 – 600" and "600 + ? = 300" he could decompose the numbers further when moving into other place values.

I made a decision to use the number line strategy for two reasons. The first reason was to offer James a flexible tool to record his mental strategy. The second reason was to allow James the opportunity to flexibly add up chunks of numbers that were comfortable for him. I told James that we could use the number line in a different way to what he had already done. I asked James, "What would be the greatest number you could add to 4000 in your head to bring you closer to 9357?" This question was a way not only to get a good idea of the magnitude of numbers James was comfortable working with mentally, but also a way to scaffold him to the usage of the empty number line. James answered that he would "want to get to 8000 by adding 4000" because "if we add 5000 we would go too far". I then added the subtrahend as a point on the number line and asked, "What if we started with 4689 from the problem and added 4000 to it?" He mentally calculated with ease, bringing us to 8689, which I recorded as an interval. He then offered, "We can do hundreds now." After trialling a complex number and finding it too difficult to calculate mentally, he adjusted his next chunk to 400, bringing us to 9089. It was here that he began to struggle because he was keen on getting to 9357 as quickly as possible, thus wanting to stick with larger numbers that were too unwieldy to calculate mentally. I offered the possibility of adding on 1 to bring us to 9090 and James seemed quite pleased with the idea. "That's easier!" From there, he added on 10 to get to 9100 and then added 257 to reach our finally destination of 9357. By then, the class had been called to the front of the room together to report their thinking.

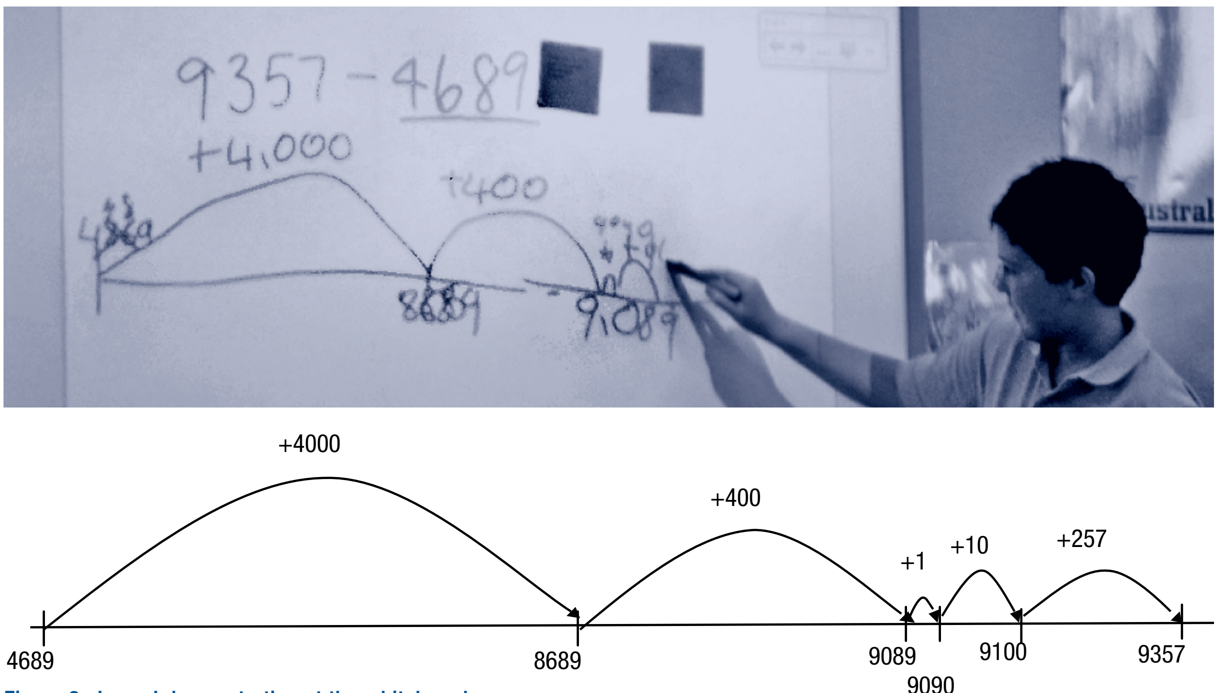

Figure 3. James' demonstration at the whiteboard.

Reporting session at the conclusion of the lesson

Several students were chosen to report their thinking to the class. James was recommended as having a way of showing a 'building up' strategy to solve a very tricky subtraction. He confidently stepped up to the whiteboard to describe his method. Figure 3 shows a slightly messy record as he inadvertently transposed his figures and wrote 4869 for 4689 and corrected them when he realised his mistake. However, that was his only error. He drew his solution and explained his reasoning using the empty number line as a way to support 'adding up' from 4689 to 9357.

He explained to his classmates that "to find the answer you just need to add up all of the jumps along the number line". So he did it in the way that was easiest for him: 257 + 1 = 258; 258 + 10 = 268; 268 + 400 = 668; 668 + 4000 = 4668.

What was clear from James' explanation was that the empty number line had helped to support his thinking as he found the difference between 9357 and 4689 using a 'building up' approach. It is a powerful mathematical model for use in such situations.

The potential of the 'empty number line' model

Empty number lines have great potential for primary-school students as a model that scaffolds mental addition and subtraction calculations. They model linear counting situations, support students' intuitive thinking, and lead towards more sophisticated thinking strategies. They appear in the curriculum recommendations of several countries around the world as reported in Cheeseman (2005, 2010) and Bobis and Bobis (2005). The so-called 'empty number line' was originally proposed as a central model for addition and subtraction by Treffers (Treffers, 1991; Treffers & De Moor, 1990) in The Netherlands in the 1980s. Results of research on Dutch children using empty number lines have been impressive (Gravemeijer, 1994). At the end of Grade 2, success of students on the difficult subtraction problems in the National Arithmetic Test confirmed that the empty number line was a powerful tool for instruction (Klein, Beishuizen & Treffers, 1998). Subsequently, the use of empty number lines has been advocated by curriculum documents in The Netherlands (van den Heuvel-Panhuizen, 2000), England, New Zealand and Australia (New South Wales Board of Studies, 2002).

What is it?

There are no marks or numbers on this model. Students only mark the numbers they need for their calculation. It is worth noting that the marks on the empty number line are not intended to be proportional, they are used as a way of recording thinking as it happens. The empty number line is a sort of sketch of the steps of thinking that happens when using mental computation.

Why use it?

The empty number line is a model that fits the counting aspect of number. Freudenthal (1973) described four aspects of number: counting number, numerosity number ("manyness"), measurement number, and reckoning number. In the James' case, he was reckoning number.

An empty number line supports informal solution procedures in that it is not restrictive and allows students to express and communicate their own solutions in a variety of ways. Marking numbers on the number line scaffolds a way of thinking. It shows partial results, the way an operation has been carried out, and what remains to be done. The empty number line also fosters the development of more sophisticated thinking strategies. The most basic strategy would involve counting by ones. Next, structured counting sequences may be used by counting in groups of tens and ones. The empty number line also supports strategies for skilful calculating such as 'compensating'; for example, in solving 76 – 49, first subtract 50 then add. Eventually, students' strategies become so sophisticated that they no longer need empty number lines to scaffold mental computation. The model becomes superfluous because children use it

to sketch their thinking when they need the support, and as their abstract ideas become stronger, they no longer need it. Perhaps one of the greatest strengths of the model is its 'planned obsolescence'.

With an increasing emphasis on developing students' mental computation and building mathematical concepts from students' intuitive thinking, there seems to be a place for the empty number line in our 'toolkit' in the primary school.

Concluding comments

We learned from James some characteristics of students who persist with difficult mathematics and insights into the advantage of having a knowledgeable teacher. Persistent students really want to succeed with mathematical problems. They seem to relish challenge and try everything they know to find a solution. If they are 'stuck' after that, they ask for help. In this case, James was fortunate that Deb had the time and the inclination to listen carefully while he described the approaches he had already tried. Further, Deb was able to identify an appropriate model for his particular problem and match it to his skill set. She then connected what he had tried with a slightly new twist. He was eager to learn and she was ready to teach him. It could be said that she was able to offer the right mathematical model to support his thinking at just the right time.

We have three suggestions for classroom teachers of mathematics:

- offer children challenging tasks and expect persistence;
- listen carefully to children to be aware of the learning potential children bring to the task;
- be knowledgeable about a range of mathematical models that support children's thinking.

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