The trouble with zero

Robert Lewis
University of Tasmania
<rob.lewis@utas.edu.au>

Maybe you, like me, have always had it instilled in you that you do not divide by zero or nasty things may happen? Some confusion still surrounds the number zero, particularly in using it in division.

You might never have questioned why you cannot divide by zero, or been asked to explain why it's a mortal sin. Why would you? It's fairly obvious, but the fairly obvious is always difficult to explain. How can we explain this concept to children in a method that is easily grasped and learnt? The purpose of this short article is to demonstrate a few pedagogy methods to show children the trouble that zero can cause. Some concepts appear trivial at first glance, but look closer and one begins to think deeply about some of our long held truths about zero.

The history of the number zero is an interesting one. In early times, zero was not used as a number at all, but instead was used as a place holder to indicate the position of hundreds and tens. For example, in representing the number 5048 the number of hundreds in this number is nothing, that is, there are no 'hundreds' in the hundred column. Zero in this case is being used to represent the concept of absence.

Zero first started being used as an actual number around 458 AD in India by Hindu astronomer and mathematician Brahmagupta. He devised the methods by which zero is reached in calculations, for example 6 – 6 = 0, and also how zero is used in equations. Prior to this, zero was not recognised as a number at all but just a concept representing absence or nothing (Kaplan, 2000).

Let's look at what appears a fairly intuitive calculation:

$$\frac{0}{a} = 0$$

Any number divided into zero gives zero as the answer no matter what the value of a. Similarly, $\frac{a}{a} = 1$; but if this is true, then is $\frac{0}{0}$ equal to 1? We know that $\frac{1}{1} = 1$ so let's try replacing numbers in that equation with numbers that are getting closer and closer to zero:

$$\frac{0.0001}{0.0001} = 1$$

$$\frac{0.00000000001}{0.00000000001} = 1$$

$$\frac{0.0000000000000000000000001}{0.0000000000000000000000001} = 1$$

As we keep putting in numbers that are getting closer and closer to zero the answer is still one. Therefore it is reasonable to say as the numbers tend towards zero the answer is still 1, ergo $\frac{0}{0} = 1$ (in case you are throwing this article down in disgust, I know that's not really true, but stay with me).

You might disagree and argue that zero goes into zero no times, therefore $\frac{0}{0} = 0$ and since you know that $\frac{0}{a} = 0$, I think you may be on some solid ground. Let me try and prove it by replacing the variable 'a' with a number that is getting closer and closer to zero.

$$\frac{0}{0.0001} = 0$$

$$\frac{0}{0.00000000001} = 0$$

$$\frac{0}{0.0000000000000000000000001} = 0$$

Just as in the prior example, it is reasonable to say as the value of a gets closer and closer to zero, the answer is still zero, ergo $\frac{0}{0} = 0$. Wait a minute, I think. Maybe $\frac{0}{0}$ is infinity.

Let me explain with reference to a simple division:

$$\frac{15}{3} = 5$$

We could ask how many times can we take 3 from 15 before we cannot take any more threes away, and of course the answer would be five. Similarly, we could ask "How many times can we take zero away from zero before we cannot take any more zeroes away?". Conceptually you may say you can take zero from zero all day—up to an infinite amount of times.

So, what is the answer? Is zero divided by zero 1, 0, or infinity? Clearly it cannot be all three at the same time, therefore the answer is not exactly known or is unable to be determined. Mathematically, the answer to zero divided by zero is indeterminate. Okay, so zero divided by zero is indeterminate but what about:

$$\frac{a}{0} = \text{undefined}$$

Why is any number divided by zero undefined? I might argue that the answer should be infinity. After all, you can take an infinite amount of zeroes from any number—keep taking zeros away from 15 until the end of time and you will still have the number 15 unchanged.

Using an everyday example that students understand, I explain that I have a pizza that I want to divide up between six people. How many slices will I need to cut the pizza into—clearly the answer is six slices. Now, what if there are no people demanding a slice of my pizza—how many slices will I need to cut my pizza into? Ignoring the fact that the question doesn't make sense (how can no people demand a slice of my pizza) I may say 'I don't have to cut my pizza into any slices', therefore $\frac{a}{0} = 0$. But this solution also has problems. Look at our previous division problem of $\frac{15}{3} = 5$. The opposite of division is multiplication, therefore five multiplied by 3 is 15, mathematically you would expect if $\frac{15}{0} = 0$ then $0 * 0 = 15$. This obviously is not the case.

Perhaps you want to prove it mathematically. Looking at $\frac{15}{0}$ lets replace zero with real numbers getting closer and closer to zero and see what happens:

$$\frac{15}{1} = 15$$

$$\frac{15}{0.1} = 150$$

$$\frac{15}{0.01} = 1500$$

$$\frac{15}{0.000000001} = 15000000000$$

The answer keeps getting larger and larger, approaching very large numbers. This seems to support that $\frac{a}{0} = \infty$. So is it infinity, or is it zero, or does the question not even make sense? Mathematically, we say that the question is unanswerable or that the answer is undefined.

Calculations involving zero can at first glance appear trivial. It is the questions that come from 'left field' that have a tendency to challenge the thinking of any teacher. I hope that this article has gone some way to challenge the thinking where divisions using zero are used and also better prepare teachers to answer student questions using critical thinking.

References

Kaplan, R. (2000). *The nothing that is: A natural history of zero.* United Kingdom:Penguin Books Ltd.