Incorporating technology and cooperative learning to teach function transformations

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When teaching transformations of functions, teachers typically have students vary the coefficients of equations and examine the resulting changes in the graph. This approach, however, may lead students to memorise rules related to transformations (Hall & Giacin, 2013). Students need opportunities to think deeply about transformations beyond superficial observations about changes in the graphs. In this article, we describe an activity in which upper secondary school students used graphing calculators in cooperative learning groups to analyse and create motifs found in traditional Turkish rugs. The activity centres on transformations of functions. It encourages students to make connections between original and transformed graphs and to think deeply about the effect on the graph of changing a parameter.

The activity is mathematically rich and engaging since it relates culture with transforming functions. Function transformation is an important standard in mathematics curriculums around the globe. For instance, the national curriculums both in Turkey and Australia require students to analyse transformations of different families of functions at secondary school grade levels (ACARA, 2015; Ministry of National Education, 2013). Students may explore the connection between different representations of the transformed functions by using digital technologies.

Research suggests that technology is likely to enhance students' conceptual understanding of mathematics when it is used to encourage deep reasoning about mathematics and to promote cooperation among students (Johnson & Johnson, 2008; Roschelle, Pea, Hoadley, Gordin, & Means, 2000). When students perform mathematical tasks in cooperation with others, they have greater opportunities to make their thinking visible and to resolve misunderstandings. We designed the rug motifs task based on the key elements of cooperative learning, as discussed by Johnson and Johnson (2008).

Positive interdependence is at the heart of cooperative learning and exists when groups have a clear task, desired outcome, shared roles, as well as a group goal. In the rug motifs task, the students worked in groups of four. Their first task was to create the rug motif called Running Water (see Figure 1), by entering the suitable function into their graphing calculators, with a restricted domain.

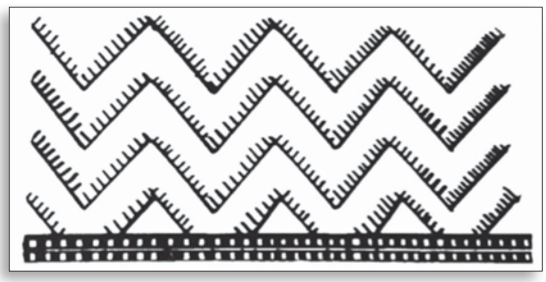

Figure 1: Running Water motif.

Each group member was assigned one part of the motif and was responsible for creating that part on the calculator screen. The group was then tasked with combining the individual functions in order to create the whole motif, and answer related questions. They completed other tasks in a similar manner. This role sharing promoted another element of cooperative learning: individual accountability.

Each group member was responsible for, and assessed on being able to correctly produce, their assigned part of the motif. Using graphing calculators helped to provide immediate feedback to individual group members, since they were able to compare their prediction against the actual motif. To further promote individual accountability, the students were told that the teacher would randomly select students to respond the questions and justify their answers.Promotive interaction, another element of cooperative learning, exists when group members challenge and facilitate each other's reasoning in order to achieve the group goal. Traditionally, promotive interaction exists face-to-face, while technology offers other ways of promotive interaction, such as electronic bulletin boards (Johnson & Johnson, 2008). In our activity, the group members combined their individual motifs to create a single whole motif on one calculator. This arrangement required students to challenge each other's thinking about functions and to reach an agreement for the problem. Both in the beginning and during the activity, the students were asked to support each other's learning, criticise ideas constructively, and make sure that all group members understood the completed tasks.

Transformation notation

We built the rug motifs task based on the function meaning of transformations as explained by Hall and Giacin (2013). Viewing transformations as functions means that a transformation maps points in the plane as inputs to other points as outputs (CCSSI, 2010). For instance, the function $T(x, y) = (x, y + 2)$ is a vertical translation and moves the input points two units upward.

When the input is a function, then the function is considered as the set of ordered pairs. In other words, $y = f(x)$ is considered as the set of all points $(x, f(x))$. Our activity focused on vertical and horizontal transformations of absolute value functions. We slightly revised the notation used by Hall and Giacin (2013) by adding subscripts v and h to $T(x, f(x))$. For example, a vertical transformation $T_v(x, f(x)) = (x, f(x)-3)$ takes $y = f(x)$ and maps it to $y = f(x) - 3$. For a horizontal transformation $T_h(x, f(x)) = (x-3, f(x))$, the transformed function is not explicit since the abscissa of the ordered pair is $x-3$. In this case, a substitution $x' = x - 3$ is used to rewrite $T_h(x, f(x)) = (x', f(x' + 3))$. Hence, $\mathrm{T}_h(x, f(x))$ takes $y = f(x)$ and maps it to $\mathrm{y} = f(x' + 3)$.

Motivation for the activity

The activity described in this article took place over two hours and was implemented in two 10th grade classrooms in Turkey. The first author taught the lessons; the second author prepared the students for the activity by teaching them the necessary graphing calculator skills, since the students had not previously used graphing calculators. She also introduced students with the transformation notations $T_v(x, y)$ and $T_h(x, y)$ in the context of transforming points in the Cartesian coordinate. The lessons were taught in Turkish, but for the purpose of this paper student conversations and activity worksheets have been translated into English.

The teacher started off the lesson by asking about the students' familiarity with the motifs found in traditional Turkish rugs. The students knew that the rug motifs had certain meanings, but their knowledge was very limited. The students were then shown pictures of rugs from different regions of Turkey, and they discussed the meaning of some of the motifs. For instance, a motif called 'hair band' denotes a desire for marriage. It was also shared that cultural materials such as rugs, quilts, or bags exist in different cultures and often they tell a story about these cultures. Then the teacher introduced the Running Water motif by showing Figures 1 and 2 to the students, and explained that they were to examine the Running Water motif using their graphing calculators. This motif emphasises the importance of water in life and symbolises life, purity, and virtue.

Creating the running water motif with graphing calculators

Figure 2: A rug made using the Running Water motif.

The teacher gave one worksheet (see Appendix) to each group to support the group work and cooperative learning. The first task was to create the whole motif in Figure 3, which is from the designer book used to make rugs (Ministry of National Education, Youth, and Sport, 1985). The students were told to ignore the short line segments at v-shapes while plotting the graph. Each group member selected one part of the motif (f_1, f_2, f_3, or f_4) and then tried to graph it by producing an appropriate equation considering the parent function $f(x) = |x|$. They were asked to plot each function on the interval $[-x_1, x_1]$.

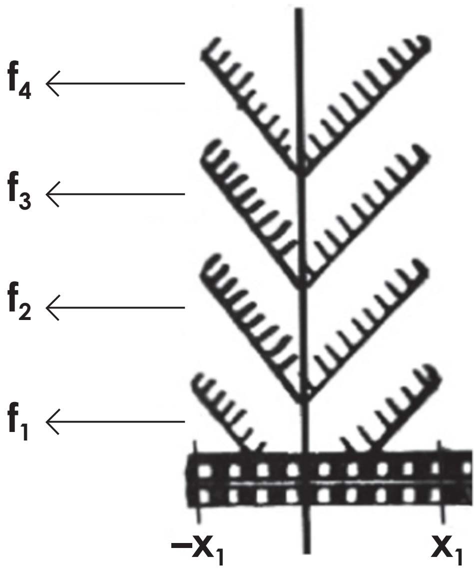

Figure 3: Vertical section of the motif.

We observed that some students first graphed $y = |x|$ and discovered that this function did not match with any part of the motif. Some groups discussed shifting the parent function as a potential solution. The following excerpt illustrates one group's discussion:

S1: Let's plot $|x|$ and try to shift it.
S2: Ok, I am plotting $|x|$ and you plot $|x + 2|$.
S4: No, no, it is not that. Look at my calculator, $|x + 2|$ is like this. This is not the same with the motif. This graph lies on the x-axis but we have to draw it on the y-axis.
S3: Ok, I think you have to plot $|x| + 2$, try and check it.
S4: Ok, this time the graph appears on the correct axis. Now we have to set equal distance between the graphs. If mine is $|x| + 2$, then what should yours be?

Note: S1, S2, S3, S4 refer to students. For example, S1 means student 1.

Students discussed the location of y-intercepts. Despite the first function's y-intercept not being visible in Figure 3, students debated about the slope of the lines and decided that the first function's y-intercept also lies on the y-axis. Most of the groups decided to work with the following y-intercepts: –1, 1, 3 and 5. After plotting each function on their individual calculators, the groups checked whether the motif was completed or not by entering all the equations into a single calculator. Then the students completed the table in Question 1–b (an example is provided in Figure 4), and answered the related questions (1–c, 1–d, 1–e).

x	$f(x)=\|x\|$	$f_1(x)=$ $\|x\|-1, -2<x<2$	$f_2(x)=$ $\|x\|+1, -2<x<2$	$f_3(x)=$ $\|x\|+3, -2<x<2$	$f_4(x)=$ $\|x\|+5, -2<x<2$
-2	2	1	3	5	7
-1	1	0	2	4	6
0	0	-1	1	3	5
1	1	0	2	4	6
2	2	1	3	5	7

Figure 4: The table from a group's worksheet.

During the whole class discussion, the teacher asked "What patterns do you observe when you add a constant to the equation of the parent function? Explain your answer by using the table, graph, or equation". One of the students answered "The constant that we add to the equation moves the graph of the function on the y-axis upwards or downwards. In the table the values changed as well (Figure 4). The constants change the range." The teacher asked a follow-up question: "Why does the graph move up or down when we add a constant to the equation of the parent function?" Another student explained "if we add a constant, it changes the y values; but if we add the constant to the inside of the absolute value, then we change the x values". The class discussion continued with locating specific points on the graph, table, and equations before and after a constant is added to the parent function in order to make connections between different representations. The discussion about specific points seemed to help students make sense of the transformation notation. All of the groups were able to represent the vertical transformation by using the transformation notation: $T_v\,(x, |x|) = (x\,, |x| \pm c)$, where c is a constant.

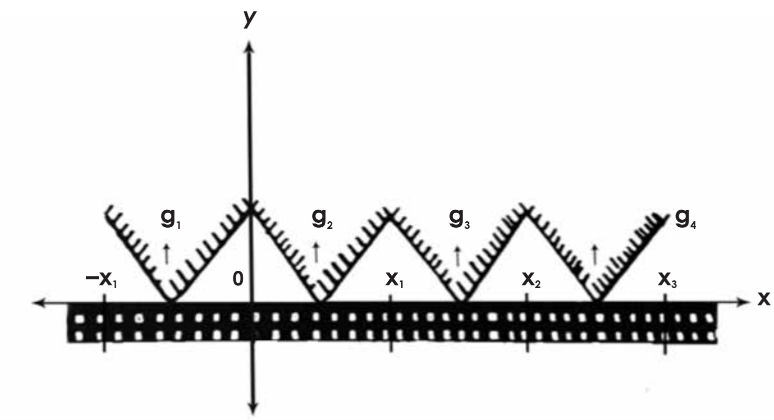

Figure 5: Horizontal section of the motif

Then the teacher explained the second task to the students (see Question 2), which was constructing the motif as seen in Figure 5. The students experienced adding a constant to x in the absolute value function in their trial and error efforts at the beginning of the first task and observed that it yielded a horizontal transformation. Therefore, for the second task they readily came up with the idea of adding a constant inside the absolute value.

In the second task, the effect of adding a constant to the equation is different from the effect in the first task. For example, in the previous task, adding a positive constant to the equation shifted the graph upward (towards positive y-axis). However, in the second task, adding a positive constant shifted the graph left. The following conversation in one group is about this relationship.

S3: The question asks us to explain how the change in the equation of the parent function affects the graph.
S4: In this case, we wrote the constant inside the absolute value.
S1: And also when we add a constant to the equation, the graph shifts to the left, actually it is quite interesting.
S2: No, no you are wrong!
S1: Yes it is right. See my function looks like this, I added '1' inside the absolute value as $|x + 1|$, then my graph appeared on the left side.
S2: I see. That's right.
Tchr: How you can explain this shift?
S3: Can I explain by using the idea of parabola? In the parabola we have the formula $f(x) = a\,(x - r)^2 + k$. k is zero in this example, so we have to speak about r. In this formula, when we add a number r, it is actually a subtraction, and when we subtract r, it becomes addition.
Tchr: And why does adding or subtracting r make that effect on the graph? (Silence)
Tchr: Ok, let's think about the x-intercepts (of the functions in Figure 5). Can you possibly justify your observation using the x-intercepts?
S2: On the right side (of the origin), we have to subtract a constant from x because only then we get zero (for the y-value). For example, 3 minus 3 is zero, then the formula for g_3 would be $|x - 3|$.
Tchr: Good, now elaborate on your ideas by writing on your worksheet.

Note: S1, S2, S3, S4 refer to students. For example, S1 means Student 1; Tchr refers to the Teacher.

During the whole class discussion, we observed that even though students understood the transformation of the function on the y-axis, they were confused about the transformation on the x-axis. For shifting the graph of the function to the right side of the x-axis, some students thought that they should add a constant, and not subtract it from x. To clarify this confusion, the teacher encouraged them to analyze the horizontal transformation graphically, write some original and shifted ordered pairs and think about the transformation as a function that assigned a new value of x, called x', with the same value of $f(x)$. The teacher wrote $(x, f(x))$ and $(x', f(x))$ on the board and then asked the class, "How do you get this new variable x' by shifting the function on the x-axis, let's say 3 units to the right?" One of the students said "We have to undo the operation that we applied to x. I mean if we shifted the graph on the x-axis 3 units to the right, we have to add 3 to x. Then $x' = x + 3$ and $x = x' - 3$." Accordingly, the teacher wrote the pair $(x', f(x'-3))$ up on the board. Another student explained the horizontal transformation using the table, saying "To get 0 as a y value, I need to subtract the unit as much as I have shifted the function to the right. For example, in our table we get zero when x is 5 for g_4 ($g_4 = |x - 5|$). This means that I shifted the function 5 units to the right (of the x-axis)." After sharing the reasons for horizontal transformation, one group explained that this transformation could be written as $T_h(x, |x|) = (x, |x \pm c|)$, where c is a constant.

The third task in the activity involved a question to find out whether the students could transform their knowledge to a new context. In this extension question, the motif is slightly modified and the parent function is changed to a linear function. Students were then asked to choose one of h_1, h_2, h_3, h_4, h_5, or h_6 in Figure 6 and plot that function on their graphing calculators according to the parent functions $y = 2x$ or $y = -2x$.

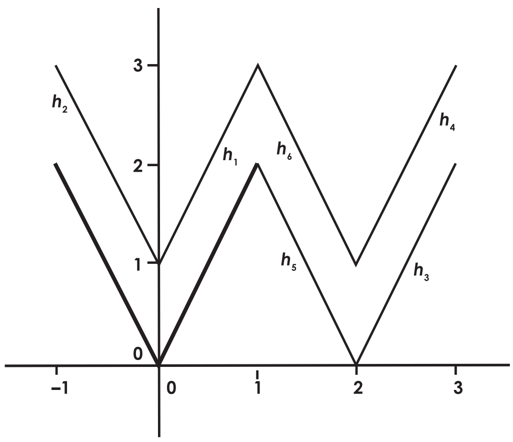

Figure 6: Modified Running Water motif.

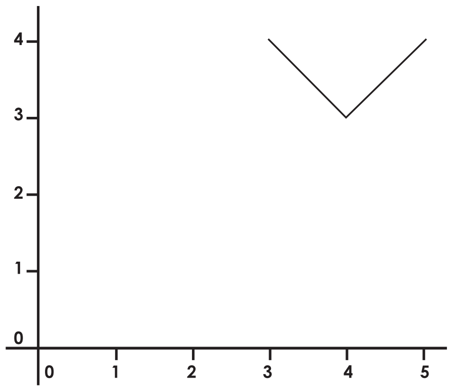

Figure 7: The graph used in the assessment question.

Students easily applied the horizontal and vertical transformation in this question. The following dialogue illustrates one group's solution process.

S1: The equation of this branch (the motif's right branch that starts at the origin) is $y = 2x$. So the equation for h1 should be $y = 2x + 1$ because I shifted it (the parent function) 1 unit upward.

S2: Ok, I can write the equation for this part (pointing to h3). It should be $y = 2x - 4$ (on the interval [2, 3]).

S3: Are you sure?

S2: Yes, look at this, I shifted $y = 2x$ two points to the right therefore I have to subtract 2 from x. (elaborating that $y = 2(x - 2) = 2x - 4$))

Note: S1, S2, S3, S4 refer to students. For example, S1 means Student 1.

At the end of the second lesson, the teacher asked them Question 4, which is an assessment question requiring students to determine which function, or functions, would create the graph shown in Figure 7. In this question, the groups either preferred the absolute value function and came up with $f(x) = |x-4| + 3$ on the interval [3, 5], or linear functions and came up with $y = x - 1$ on [4, 5] and $y = 7 - x$ on [3, 4].

Concluding remarks

The activity engaged students by employing an interesting context in which they could explore function transformations. The use of graphing calculators provided the students with immediate feedback as they tested different functions in order to construct the running water motif, and this also worked as a medium to support group interaction. Although the students had learnt about function transformations in the context of quadratic functions prior to this activity, group conversations indicated that initially they did not reason about how to apply the transformations to create the running water motif, rather they used trial and error method. The activity helped students to critique the incorrect solutions and construct arguments for applying the proper transformations. Overall, the conversations among students and the teacher show mathematically productive interactions in which the students did most of the talking and thinking.

Reflecting on the activity, it would seem to provide opportunities for the application of multiple solutions. For instance, for the assessment question, the students were able to use linear or absolute value functions to construct the graph. Another example, in Question 1, the intervals on the x and y-axis were not numbered on purpose, in order to encourage different solutions. Comparing different solutions may help students to acquire a deeper understanding of the generalisation of function transformations.

Another observation from our experience is that in some groups, most of the members were somewhat reluctant to use technology; whereas in some groups most members were technology friendly and eager to learn the features of graphing calculators. Teachers may form heterogeneous groups by mixing students who are reluctant to use technology with students who are more comfortable with using technology. This kind of enhanced group dynamic may help students to learn from one another in exploring new technologies.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA) (2015). *The Shape of the National Curriculum Paper: A Proposal for Discussion.* Retrieved 12 February 2015 from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Common Core State Standards Initiative (CCSSI). (2010). *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association and Center for Best Practices and the Council of Chief State School Officers. Retrieved 12 February 2015 from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Hall, B. & Giacin, R. (2013). Exploring Function Transformations Using the Common Core. *Mathematics Teacher 107*(2), 132–137.

Johnson, D.W. & Johnson, R.T. Cooperation and the use of technology. In J. M. Spector, Merrill, M. D., Merrienboer, J. V. & Driscoll, M. P. (Eds.), *Handbook of research on educational communications and technology* (3rd ed.). (2008). New York: Lawrence Eribaum Associates.

Ministry of National Education. (2013). *Secondary School Mathematics Standards.* Ankara: Turkish Education Board. Retrieved 12 February 2015 from http://ttkb.meb.gov.tr/www/guncellenen-ogretim-programlari/icerik/151

Ministry of National Education, Youth, and Sport. (1985). *National Handcrafts in Mut Region.* Ankara: National Education, Youth, and Sport Press.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics. Retrieved 12 February 2015 from http://www.nctm.org/standards/

Roschelle, J.M., Pea, R.D., Hoadley, C.M., Gordin, D.N., Barbara M. Means, B.M. (2000). Changing How and What Children Learn in Schools with Computer-based Technologies. *Children and Computer Technology, 10*(2), 76–101.