Reversible reasoning and the working backwards problem solving strategy

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Introduction

Making sense of mathematical concepts and solving mathematical problems may demand different forms of reasoning. These could be either domain-based, such as algebraic, geometric or statistical reasoning, while others are more general such as inductive/deductive reasoning. This article aims at giving visibility to a particular form of reasoning which Piaget referred to as reversibility of thought or equivalently reversible reasoning (Inhelder & Piaget, 1958). Reversible reasoning essentially involves reasoning from a given result to the source producing the result. Using examples from the school mathematics curriculum, this article illustrates how this mode of reasoning may be involved in the solution of mathematical problems. Further, it provides a strategy to foster reversible reasoning by formulating tasks in a primal and dual mode. Simultaneously, the article explains why such a mode of reasoning is essential in developing flexibility in thinking.

Working backwards and reversible reasoning

Firstly, a brief overview of the two main ideas—namely working backwards and reversible reasoning—is presented to set the context of the discussion. Among the range of problem solving strategies in mathematics, working backwards is a particularly useful method in situations when the end result of a problem is known and one has to find the initial quantity. As clearly distinguished by Polya (1945/1988) in 'working forwards', we start from the given initial situation (initial state) to the desired final goal (goal state), from data to unknown. In 'working backwards', "we start from what is required and assume what is sought as already found", or "from what antecedent the desired result could be derived" (p. 227). In working back-wards, it is often required to reverse the operations as in finding the inverse of a function (e.g., $y = 4x + 1$). Not only are the operations reversed but the sequence of the operations is also reversed as is schematically illustrated in Figure 1.

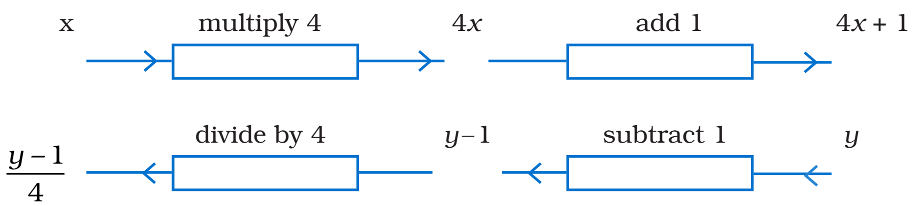

Figure 1. Diagrammatic representation of a function and its inverse.

Working backwards is thus analogous to an unwinding or undoing action. In playing the situation backward, one has to identify the goal state and the operations and imagine reversing the operations. To work backward, often one has to reverse a mental or physical action to return from the result of a process to the start of the process. Inhelder and Piaget (1958) described this mode of reasoning as reversibility of thought. To make this idea more concrete, consider the following problem: Jim has 5 marbles. He has 8 fewer marbles than Connie. How many marbles does Connie have? (Carpenter & Moser, 1983, p.16). To solve this question, the problem solver has to realise that if Jim has 8 fewer marbles than Connie then the latter has 8 more marbles than Jim, that is, the problem solver has to reverse their thought process. Inhelder and Piaget (1958) called this form of reversible reasoning *negation* or *inversion*. Similarly, if one knows that quantity A is greater than B, then one has to reverse their thought to deduce that quantity B is less than quantity A.

Consider another common problem from school mathematics concerned with percentage increase or decrease. For instance, after a 10% reduction, the price of a laptop went down to $600. What was the original price of the laptop? To solve this problem, one has to make the inference that the current price ($600) represents 90% of the original price. This inference rests on the realisation that the original price of the laptop (an unknown quantity) is equivalent to 100%. However, since there is a 10% reduction, the current price is 90% of the original price. In solving the laptop problem, one is not inverting a particular operation or order of a process to find the original price of the laptop but rather one is compensating for the reduction (by considering 100% –10% = 90%). Thus this type of inference allows us to go back to find the unknown starting quantity (the original price of the laptop). Inhelder and Piaget (1958) called this form of reversible reasoning as compensation. Hackenberg (2010) describes compensation in terms of creating a state equivalent to the original state or “a transformation back and forth between equivalent states” (p. 425) to find the starting unknown quantity.

To make this mode of reasoning more explicit, one must distinguish between two types of problems: primal and dual. In the primal problem, the source and relation are specified and the aim is to find the result. In the dual problem the result and the relation are specified and the aim is to find the source. Figure 2 provides a diagrammatic illustration of the primal-dual pair.

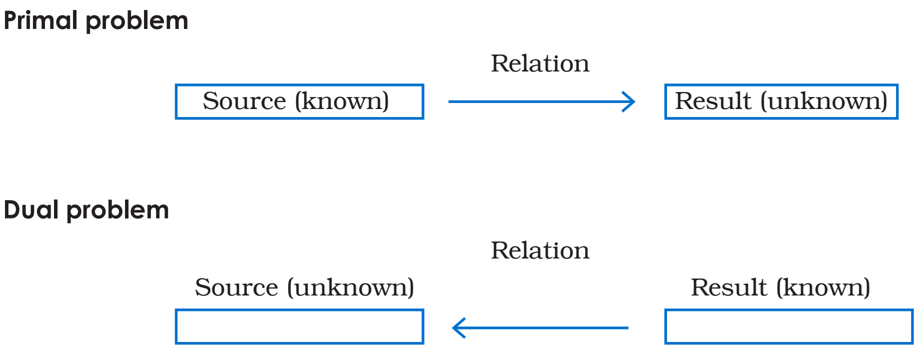

Figure 2. Structure of a primal and dual problem.

Example problems

Five example problems from different areas of mathematics are provided to illustrate how this mode of reasoning may be involved. These are: fractions, percentage, algebra, geometry, and data. The same problem is reformulated from the primal to the dual form.

Example 1: Fractions

- **Primal problem:** A parking lot can contain a maximum of 45 cars. How many cars can $\frac{2}{3}$ of the parking lot contain?

- **Dual problem:** There are 30 cars in a parking lot. This is $\frac{2}{3}$ of the number of cars that the parking lot can contain. How many cars can the parking lot contain?

In the primal problem, one has to find a part from a given whole. In the dual problem, the part is given and one has to construct the whole. Thus, if $\frac{2}{3}$ of the number of cars represents 30, this means that $\frac{1}{2}$ of $\frac{2}{3}$, that is, $\frac{1}{3}$ represents 15 cars. Therefore, one whole, which can be made from 3 one third units, corresponds to 3 × 15 cars = 45 cars. Clearly, the dual problem creates more context for deep mathematical thinking.

Example 2: Percentage

- **Primal problem:** Megan contributed \$32 to a charity fund. Jessica contributed 175% of what Megan contributed. How much did Jessica contribute?

- **Dual problem:** Jessica contributed \$56 to a charity fund. This represents 175% of the money that Megan contributed. How much money did Megan contribute?

The primal problem involves a multiplying action, 175% of \$32 = \$56. However, the dual problem requires the realisation that 175% of an unknown quantity (Megan's share) represent \$56. Thus, to construct 100%, one has to find a way to 'get rid' of the extra 75%. One way to proceed is to interpret 175% as $1\frac{3}{4}$ or $\frac{7}{4}$ (that is, 7 quarters or 7 one-fourth units). Thus, $\frac{7}{4}$ of the unknown quantity represents \$56 and therefore $\frac{1}{4}$ of the unknown quantity represents $\frac{56}{7}$ = \$8 and 4 one-fourth units represents \$8 x 4 = \$32. Another approach could be to reason as follows: 175% represents \$56; 1% represents $\frac{\$56}{175}$; and 100% represents $\frac{\$56}{175} \times 100 = \$32$

One of the intuitive responses for the dual problem is illustrated in Figure 3. The Year 7 student took $1\frac{3}{4}$ of the given quantity, that is $1\frac{3}{4}$ of 56, rather than finding the quantity which when multiplied by $1\frac{3}{4}$ yields \$56. First, she divided 56 by 4 to obtain 14 (representing $\frac{1}{4}$ of 56). Then she multiplied 14 by 3 to get 42 (representing $\frac{3}{4}$ of 56). Eventually, she added 56 (representing 1 whole) to 42 (representing $\frac{3}{4}$) to obtain 98. Discussing the primal and dual problems with students in conjunction can potentially lead students to appreciate the subtle difference between these two related problems. Further, the dual problem may prompt students to think in terms of an unknown quantity.

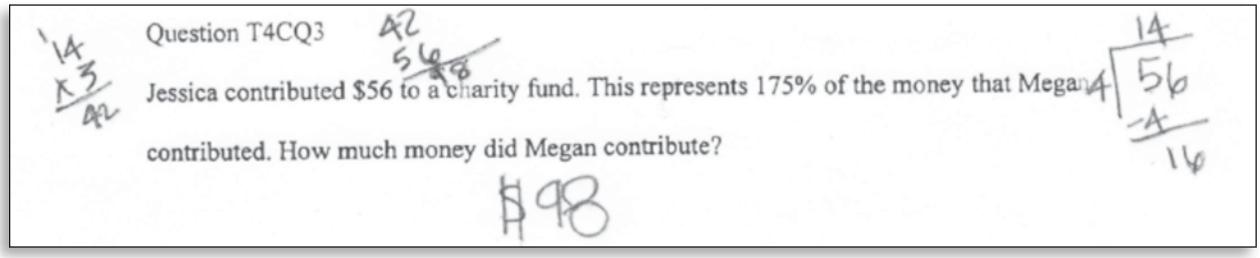

Figure 3: Example of an intuitive response for the dual problem

Example 3: Algebra

- **Primal problem:** Evaluate the expression $100 - \frac{57}{x}$ given that $x = 3$.

- **Dual problem:** Find the value of x in $100 - \frac{57}{x} = 81$.

The primal problem merely requires the substitution of $x = 3$ in the expression and evaluating the expression to produce the result. Thus, it requires working in a forward direction. The dual problem can be solved in a number of ways. One possible method is as follows: '100 minus what equals 81?' [19] and '57 divides by what makes 19' [3]. This requires reversible reasoning in the sense that one has to work back to think that if 100 minus a number equals 81 then 100 minus 81 should be the number. Similarly, if 57 divides by a number gives 19, then 57 divides by 19 produces the unknown number. Another way of solving this problem is to multiply the left hand side and right hand side of the equation by x. Conceptually, this is based on the realisation that multiplying the two sides of the equation by x does not change the equation. Thus, $100x - 57 = 81x$. Similarly, one can subtract $81x$ on either side of the equation without changing it to obtain $19x - 57 = 0$. In the next step, 57 is added on both sides of the equation to obtain $19x = 57$. Finally, both sides of the equation is divided by 19 to obtain $x = 3$. By performing similar operations on either side of the equation, the problem solver is creating a system equivalent to the original state to work back to determine the source ($x = 3$) that created the result (81). Here, the automaticity and procedural swiftness with which the problem solver manipulates the equation algebraically may suppress the visibility of reversible reasoning. In fact, the importance of reversibility in algebraic thinking has also been pointed out by Driscoll (1999):

> Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them). In effect, it is the capacity not only to use a process to get to a goal, but able to understand the process well enough to work backward from the answer to the starting point. (p.1)

Example 4: Geometry

- **Primal problem:** Find the gradient of the ramp in Figure 4.

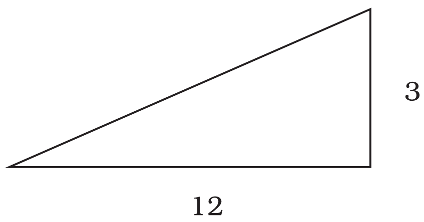

Figure 4. Ramp of base 12 units and height 3 units.

- **Dual problem:** Create a ramp with the same steepness as the one shown above but which has a base length of 13cm (Lobato & Thanheiser, 2002). The primal problem involves working in a forward direction ($3 \div 12$). The dual problem, however, requires one to think of a number which when divided by 13 gives $\frac{1}{4}$. Thus, one has to multiply 13 by $\frac{1}{4}$ to obtain the height.

Example 5: Data

- **Primal problem:** Four objects have weights 3kg, 4kg, 6kg, and 7 kg. What is the average weight of the four objects?

- **Dual problem:** The average weight of 4 objects is 5kg. Three of the objects have respective weights 3kg, 4kg, and 6 kg. What is the weight of the fourth object? (Zazkis & Hazzan, 1999, p.433).

The primal problem involves the addition of four numbers and the division of the sum. In the dual problem, the result of the addition of the four numbers and the subsequent division is known (i.e., the average is known) and one has to find the fourth unknown number.

The importance of creating opportunities for fostering reversible reasoning

Discussing the primal and dual tasks in conjunction allows students to construct relations bi-directionally. For instance, in example 1, students are prompted to articulate the multiplicative relationship between the part and the whole. From this perspective, the primal-dual pair generates situations that allow students to develop meaningful concepts as well as exploring them in depth. The primal-dual pair may also be regarded as generating conflicts to their instrumentally-packed experiences.

The role of tasks and their consequences about how students engage in sense-making have been pointed out by Stein, Grover, and Henningsen (1996) who emphasised the importance of providing problematic situations to students rather than algorithmic routines. They assert that carefully chosen mathematical tasks can productively help students construct mathematical meaning and may promote deep learning. The level and kind of thinking that students engage in, determine what they will learn. Dual problems can be regarded as a specific type of task with much thought-revealing power. The dual formulation of problems may be used as a strategy for rewording problems.

In line with current constructivist efforts to foster the conceptual understanding of mathematical ideas, this paper offers one way in which teachers may engage students in thinking about the interrelationships between problem parameters by requiring them to work bi-directionally from the source to the result and vice versa. This type of mathematical articulation is consistent with the *Australian Curriculum: Mathematics* (ACARA, 2014) which emphasises 'reasoning' as one of its proficiency standards and underlines the necessity for students to develop capacity for logical thought. As illustrated in the examples, the conversion of a primal problem in terms of its corresponding dual counterpart can be a substantial resource in reformulating common mathematical problems so that they become richer tasks. Engaging students in such situations may provide teachers valuable information about students' extant conceptions and enhance their repertoire of knowledge to support their reasoning and problem solving skills as well as their intellectual and dispositional growth.

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