Selecting Proportional Reasoning Tasks

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With careful consideration given to task selection, students can construct their own solution strategies to solve complex proportional reasoning tasks while the teacher's instructional goals are still met. Several aspects of the tasks should be considered including their numerical structure, context, difficulty level, and the strategies they are likely to elicit from students.

In the middle grades, it is extremely important for students to develop sound proportional reasoning skills as a foundation for future coursework in mathematics. Researchers consider proportional reasoning skills to involve more than applying the cross-multiplication algorithm. According to the National Council of Teachers of Mathematics (NCTM; 2000), "it involves recognizing quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship" (p. 217). In other words, proportional reasoning is not simply the application of a memorised procedure (e.g., cross-multiplication), but also involves a conceptual understanding of proportional relationships.

Teachers can help students develop this understanding by postponing the introduction of the cross-multiplication algorithm and engaging them in well-designed problem solving situations. When students solve contextual problems in their own ways, they are forced to make sense of the proportional relationship involved, and often the context helps cue students into it. Ely and Cohen (2010) offer that teachers should choose tasks that best meet their instructional goals. In order to do so, teachers need to consider the strategies and reasoning that a task will encourage. They also need to consider both the concepts and procedures they are working to engage students with. Teachers often choose tasks without fully analysing the characteristics of the tasks and their influence on students. de la Cruz (2008) found that teachers primarily chose tasks based on what appeared in their instructional planning resources (e.g., textbooks), with little further consideration. This paper shares a framework for consideration when choosing or developing tasks focused on proportional reasoning. It assumes that students are developing their own strategies for solving the tasks, prior to the introduction of the cross multiplication algorithm. The framework has two areas that need to be considered when selecting or creating tasks to support students to engage in proportional reasoning.

Context

The underlying context of a task should be considered during the task selection process. More specifically, the familiarity of the context and influence of context on strategy choices should be examined.

Proportion problems are often distinguished by their context (Lesh, Post & Behr, 1988; Tourniaire & Pulos, 1985). The context of a problem has been shown to influence students' solution strategies and success rates. First, all proportion problems can be characterised as either missing value problems or comparison problems. The first requires the solver to find the missing value when given three others. A comparison problem involves comparing two given ratios which may or may not be proportionally related. Comparison problems are generally more difficult than missing value problems (Singh, 2000) and do not help students to develop an initial understanding of proportional relationships. For this reason, comparison problems should be postponed until students have a working understanding of proportionality.

Proportion problems can be further distinguished based on their context. Some common contexts are rates, similarity, mixtures, and part-part-whole. Examples of these common contexts for proportion problems are presented in Table 1.

Table1. Common contexts for proportional reasoning problems.

Common contexts	Examples
Rates	A printing press takes exactly 12 minutes to print 14 dictionaries. How many dictionaries can it print in 30 minutes?
Similarity	You gave your grandmother a 4 in by 6 in picture but she would like to enlarge it to match the other photos hanging on her wall. If she enlarges the length from 6 in to 8 in, what would the width of the enlarged photo be?
Mixture	If Suzie uses a lemonade recipe that calls for 1 cup of lemon juice for every 2 cups of water, how many cups of lemon juice would she need to make lemonade if she was using 8 cups of water?
Part-part-whole	Ms Levi's class has 12 girls and 18 boys. If there is the same ratio of girls to boys in the school as there is in Ms. Levi's class and there are 360 children in the school, how many boys are there?

When selecting tasks, the teacher should consider the affect their chosen context is likely to have on their students' success (at developing their own strategies and making sense of proportionality) and on the strategies their students are likely to use when solving them. Middle school students who are developing an initial understanding of proportionality have the most success with rates, compared to the other common contexts for proportion problems (Miller & Fey, 2000), particularly when the measures were familiar associations (e.g., miles to hours, dollars to ounces) (Kaput & West, 1994). Similarity (Lamon, 1993; Miller & Fey, 2000) and mixture (Tourniaire, 1986) problems were typically the most difficult contexts for students to comprehend the proportional relationships.

Additionally, rates are likely to encourage students to use a unit rate strategy, particularly when money is involved, while the other common contexts are more likely to encourage a factor of change strategy. Finally, familiarity with the context also influenced the difficulty level of a problem. When the solver is less familiar with the context, the effects of numerical complexity are more pronounced (Heller, Ahlegren, Post, Behr & Lesh, 1989). In addition to context, when selecting as task, teachers should also consider the numerical structure within the task and its influence on strategies and approachability.

Numerical structure

The numerical structure of a problem refers to the types of numbers (e.g., whole/fractional amounts, small/large numbers), as well as the relationships between the numbers involved in the problem. Most simplistically, will the numerical computation be straightforward or complex and/or will the answer involve whole or fractional amounts? Will the proportional relationship be apparent?

Research on proportional reasoning tasks suggest the numerical structure of a problem affects not only the difficulty level of a task, but also the strategies students are likely to employ when solving it. Lesh, Behr, and Post (1987) and Fernandez, Llinares, Van Dooren, De Bock and Verschaffel (2011) studied the effects of numerical structure of proportional reasoning tasks. They found that varying the size of the numbers or the numerical relationship between the quantities greatly affected student performance.

In addition to context, proportion problems can be characterised according to the types of relationships that occur between the quantities in the problem (Bezuk, 1986; 1988). There are four different types of such relationships: (a) the factor of change across the ratios is an integer, (b) the factor of change within the given ratio is an integer, (c) both factors of change are integers, and (d) neither factor of change is an integer value. Table 2 provides examples of tasks for each of these four numerical structures.

Table 2. Examples of subcategories of missing value problems according to numerical structure.

Subcategories	Examples	Numerical structure
(a) The factor of change across ratios is an integer	If 10 pieces of gum costs 34 cents, how much will 5 pieces of gum cost?	$\frac{34\text{c}}{10\text{ pieces}} = \frac{?}{5\text{ pieces}}$ × 2
(b) The factor of change within the given ratio is an integer	If 10 pieces of gum costs 50 cents, how much will 15 pieces of gum cost?	× 5 $\frac{50\text{c}}{10\text{ pieces}} = \frac{?}{15\text{ pieces}}$
(c) Both factors of change are integers	If 10 pieces of gum costs 50 cents, how much will 5 pieces of gum cost?	× 5 $\frac{50\text{c}}{10\text{ pieces}} = \frac{?}{5\text{ pieces}}$ × 2
(d) Neither factor of change is an integer	If 10 pieces of gum costs 34 cents, how much will 15 pieces of gum cost?	$\frac{34\text{c}}{10\text{ pieces}} = \frac{?}{15\text{ pieces}}$

Researchers have found that middle school students had significantly higher success rates when solving proportion tasks of type b (Karplus, 1983). Problems of type d were found to be significantly more difficult than the other three types. Furthermore, students are likely to use a unit rate strategy on problems of type b and a factor of change strategy on problems of type a. When no integer multiples exist (type d) students often rely on either cross multiplication or an incorrect additive strategy (Cramer et al., 1993; Misailidou & Williams, 2003; Singh, 2000). Fernandez and colleagues (2011) also found that in addition to having more success with problems involving integer relationships, high school students more frequently used additive reasoning when non-integer relationships were involved than when integer relationships were present.

Depending on the teacher's instructional goals and the students' background knowledge, certain numerical structures and contexts should be selected to encourage the development of those goals.

Selecting tasks

Considering the underlying philosophy that a deeper understanding is reached if the students are active in their learning process, teachers should engage their students in problem solving situations where they must come up with their own ways to solve the problems. Certain characteristics of the problems are more likely to lead to specific outcomes, as discussed above.

If the goal is to develop an initial understanding of proportionality, and the multiplicative relationship among ratios, then teachers should use numerical structures of types a, b, or c, where the numbers are "nice" enough that the multiplicative relationship between the quantities is noticeable. Additionally, they should choose contexts that students are familiar with, so the students can make sense of the proportionality through the context.

If the goal is to encourage students to develop a unit rate strategy, then teachers should use a rate context, again, that the students are familiar with. It is also important to create a numerical structure of type b for two reasons: (1) Students are more likely to create a unit rate if that unit rate is an integer and (2) Students are more likely to use a unit rate strategy if the other rate of change is not an integer (which would encourage a factor of change strategy). For example, I would use problems similar to the following: When training for the Olympics, Usain Bolt ran 7 km in 35 minutes. If he ran at the same speed, how long would it take him to run 5 km?

In conclusion, through careful task choices, teachers can facilitate a student-centred lesson where the students construct strategies for solving proportional problems on their own while accomplishing the teacher's instructional goals. In order to do so, teachers must focus on the numerical structure and context of the task and how they may influence students' thinking.

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